Example 10 on page 1042 shows how integration and a trigonometric model can be used to find the average temperature during a four-hour period.
14.1 Radian Measure of Angles

- Find coterminal angles.
- Convert from degree to radian measure and from radian to degree measure.
- Use formulas relating to triangles.

Angles and Degree Measure

As shown in Figure 14.1, an angle has three parts: an initial ray, a terminal ray, and a vertex. An angle is in standard position when its initial ray coincides with the positive x-axis and its vertex is at the origin.

FIGURE 14.1

Figure 14.2 shows the degree measures of several common angles. Note that \( \theta \) (the lowercase Greek letter theta) is used to represent an angle and its measure. Angles whose measures are between 0° and 90° are acute, and angles whose measures are between 90° and 180° are obtuse. An angle whose measure is 90° is a right angle, and an angle whose measure is 180° is a straight angle.

FIGURE 14.2

Positive angles are measured counterclockwise beginning with the initial ray. Negative angles are measured clockwise. For instance, Figure 14.3 shows an angle whose measure is \(-45^\circ\).

Merely knowing where an angle’s initial and terminal rays are located does not allow you to assign a measure to the angle. To measure an angle, you must know how the terminal ray was revolved. For example, Figure 14.3 shows that the angle measuring \(-45^\circ\) has the same terminal ray as the angle measuring 315°. Such angles are called coterminal.

Denis Pepin/www.shutterstock.com
Although it may seem strange to consider angle measures that are larger than 360°, such angles have very useful applications in trigonometry. An angle that is larger than 360° is one whose terminal ray has revolved more than one full revolution counterclockwise. Figure 14.4 shows two angles measuring more than 360°. In a similar way, you can generate an angle whose measure is less than −360° by revolving a terminal ray more than one full revolution clockwise.

**Example 1** Finding Coterminal Angles

For each angle, find a coterminal angle \( \theta \) such that \( 0^\circ \leq \theta < 360^\circ \).

- **a.** \( 450^\circ \)
- **b.** \( 750^\circ \)
- **c.** \(-160^\circ \)
- **d.** \(-390^\circ \)

**SOLUTION**

- **a.** To find an angle coterminal to \( 450^\circ \), subtract 360°, as shown in Figure 14.5(a).
  \( \theta = 450^\circ - 360^\circ = 90^\circ \)

- **b.** To find an angle that is coterminal to \( 750^\circ \), subtract 2(360°), as shown in Figure 14.5(b).
  \( \theta = 750^\circ - 2(360^\circ) = 750^\circ - 720^\circ = 30^\circ \)

- **c.** To find an angle coterminal to \(-160^\circ \), add 360°, as shown in Figure 14.5(c).
  \( \theta = -160^\circ + 360^\circ = 200^\circ \)

- **d.** To find an angle that is coterminal to \(-390^\circ \), add 2(360°), as shown in Figure 14.5(d).
  \( \theta = -390^\circ + 2(360^\circ) = -390^\circ + 720^\circ = 330^\circ \)

**Checkpoint 1**

For each angle, find a coterminal angle \( \theta \) such that \( 0^\circ \leq \theta < 360^\circ \).

- **a.** \(-210^\circ \)
- **b.** \(-330^\circ \)
- **c.** \( 495^\circ \)
- **d.** \( 390^\circ \)
Radian Measure

A second way to measure angles is in terms of radians. To assign a radian measure to an angle \( \theta \), consider \( \theta \) to be the central angle of a circular sector of radius 1, as shown in Figure 14.6. The radian measure of \( \theta \) is then defined to be the length of the arc of the sector. Recall that the circumference of a circle is given by

\[
\text{Circumference} = (2\pi)(\text{radius}).
\]

So, the circumference of a circle of radius 1 is simply \( 2\pi \), and you can conclude that the radian measure of an angle measuring \( 360^\circ \) is \( 2\pi \). In other words,

\[360^\circ = 2\pi \text{ radians}\]

or

\[180^\circ = \pi \text{ radians}.
\]

Figure 14.7 gives the radian measures of several common angles.

\[
\begin{align*}
30^\circ &= \frac{\pi}{6} \\
45^\circ &= \frac{\pi}{4} \\
60^\circ &= \frac{\pi}{3} \\
90^\circ &= \frac{\pi}{2} \\
180^\circ &= \pi \\
360^\circ &= 2\pi
\end{align*}
\]

Radian Measures of Several Common Angles

FIGURE 14.7

It is important for you to be able to convert back and forth between the degree and radian measures of an angle. You should remember the conversions for the common angles shown in Figure 14.7. For other conversions, you can use the conversion rule below.

**Angle Measure Conversion Rule**

The degree measure and radian measure of an angle are related by the equation

\[180^\circ = \pi \text{ radians}.
\]

Conversions between degrees and radians can be done as follows.

1. To convert degrees to radians, multiply degrees by \( \frac{\pi \text{ radians}}{180^\circ} \).

2. To convert radians to degrees, multiply radians by \( \frac{180^\circ}{\pi \text{ radians}} \).
Example 2 Converting from Degrees to Radians

Convert each degree measure to radian measure.

a. 135°  b. 40°  c. 540°  d. −270°

**SOLUTION** To convert from degree measure to radian measure, multiply the degree measure by \( \frac{\pi \text{ radians}}{180 \text{ degrees}} \).

a. \( 135° = (135 \text{ degrees}) \left( \frac{\pi \text{ radians}}{180 \text{ degrees}} \right) = \frac{3\pi}{4} \text{ radians} \)

b. \( 40° = (40 \text{ degrees}) \left( \frac{\pi \text{ radians}}{180 \text{ degrees}} \right) = \frac{2\pi}{9} \text{ radian} \)

c. \( 540° = (540 \text{ degrees}) \left( \frac{\pi \text{ radians}}{180 \text{ degrees}} \right) = 3\pi \text{ radians} \)

d. \( −270° = (−270 \text{ degrees}) \left( \frac{\pi \text{ radians}}{180 \text{ degrees}} \right) = −\frac{3\pi}{2} \text{ radians} \)

**Checkpoint 2**

Convert each degree measure to radian measure.

a. 225°  b. −45°  c. 240°  d. 150°

Although it is common to list radian measure in multiples of \( \pi \), this is not necessary. For instance, when the degree measure of an angle is 79.3°, the radian measure is \( 79.3° = (79.3 \text{ degrees}) \left( \frac{\pi \text{ radians}}{180 \text{ degrees}} \right) = 1.384 \text{ radians} \).

Example 3 Converting from Radians to Degrees

Convert each radian measure to degree measure.

a. \( −\frac{\pi}{2} \)  b. \( \frac{7\pi}{4} \)  c. \( \frac{11\pi}{6} \)  d. \( \frac{9\pi}{2} \)

**SOLUTION** To convert from radian measure to degree measure, multiply the radian measure by \( \frac{180°}{\pi \text{ radians}} \).

a. \( −\frac{\pi}{2} \text{ radians} = \left( −\frac{\pi}{2} \text{ radians} \right) \left( \frac{180 \text{ degrees}}{\pi \text{ radians}} \right) = −90° \)

b. \( \frac{7\pi}{4} \text{ radians} = \left( \frac{7\pi}{4} \text{ radians} \right) \left( \frac{180 \text{ degrees}}{\pi \text{ radians}} \right) = 315° \)

c. \( \frac{11\pi}{6} \text{ radians} = \left( \frac{11\pi}{6} \text{ radians} \right) \left( \frac{180 \text{ degrees}}{\pi \text{ radians}} \right) = 330° \)

d. \( \frac{9\pi}{2} \text{ radians} = \left( \frac{9\pi}{2} \text{ radians} \right) \left( \frac{180 \text{ degrees}}{\pi \text{ radians}} \right) = 810° \)

**Checkpoint 3**

Convert each radian measure to degree measure.

a. \( \frac{5\pi}{3} \)  b. \( \frac{7\pi}{6} \)  c. \( \frac{3\pi}{2} \)  d. \( −\frac{3\pi}{4} \)
Triangles

A Summary of Rules About Triangles

1. The sum of the angles of a triangle is $180^\circ$.
2. The sum of the two acute angles of a right triangle is $90^\circ$.
3. Pythagorean Theorem The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse, as shown in Figure 14.8.
4. Similar Triangles If two triangles are similar (have the same angle measures), then the ratios of the corresponding sides are equal, as shown in Figure 14.9.
5. The area of a triangle is equal to one-half the base times the height. That is, $A = \frac{1}{2}bh$.
6. Each angle of an equilateral triangle measures $60^\circ$.
7. Each acute angle of an isosceles right triangle measures $45^\circ$.
8. The altitude of an equilateral triangle bisects its base.

Example 4 Finding the Area of a Triangle

Find the area of an equilateral triangle with one-foot sides.

SOLUTION To use the formula $A = \frac{1}{2}bh$, you must first find the height of the triangle, as shown in Figure 14.10. To do this, apply the Pythagorean Theorem to the shaded portion of the triangle.

$$h^2 + \left(\frac{1}{2}\right)^2 = 1^2 \quad \text{Pythagorean Theorem}$$

$$h^2 = \frac{3}{4} \quad \text{Simplify.}$$

$$h = \frac{\sqrt{3}}{2} \quad \text{Solve for } h.$$

So, the area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2} \left(1\right) \left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4} \text{ square foot.}$$

Checkpoint 4

Find the area of an isosceles right triangle with a hypotenuse of $\sqrt{2}$ feet.

SUMMARIZE (Section 14.1)

1. Explain how to convert from degree measure to radian measure (page 1000). For an example of converting from degrees to radians, see Example 2.
2. Explain how to convert from radian measure to degree measure (page 1000). For an example of converting from radians to degrees, see Example 3.
3. State the formula for the area of a triangle (page 1002). For an example of finding the area of a triangle, see Example 4.
SKILLS WARM UP 14.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Section 1.2 and 1.3.

In Exercises 1 and 2, find the area of the triangle.
1. Base: 10 cm; height: 7 cm

In Exercises 3–6, let \( a \) and \( b \) represent the lengths of the legs, and let \( c \) represent the length of the hypotenuse, of a right triangle. Solve for the missing side length.
3. \( a = 5, b = 12 \)
4. \( a = 3, c = 5 \)
5. \( a = 8, c = 17 \)
6. \( b = 8, c = 10 \)

In Exercises 7–10, let \( a, b, \) and \( c \) represent the side lengths of a triangle. Use the given information to determine whether the figure is a right triangle, an isosceles triangle, or an equilateral triangle.
7. \( a = 4, b = 4, c = 4 \)
8. \( a = 3, b = 3, c = 4 \)
9. \( a = 12, b = 16, c = 20 \)
10. \( a = 1, b = 1, c = \sqrt{2} \)

Exercises 14.1

Finding Coterminal Angles In Exercises 1–6, determine two coterminal angles (one positive and one negative) for each angle. Give the answers in degrees. See Example 1.
1. \( \theta = 45^\circ \)
2. \( \theta = -41^\circ \)
3. \( \theta = -120^\circ \)
4. \( \theta = 740^\circ \)
5. \( \theta = -420^\circ \)
6. \( \theta = 300^\circ \)

Converting from Degrees to Radians In Exercises 11–22, express the angle in radian measure as a multiple of \( \pi \). Use a calculator to verify your result. See Example 2.
11. \( 30^\circ \)
12. \( 60^\circ \)
13. \( 270^\circ \)
14. \( 210^\circ \)
15. \( 675^\circ \)
16. \( 120^\circ \)
17. \( -24^\circ \)
18. \( -585^\circ \)
19. \( -144^\circ \)
20. \( -315^\circ \)
21. \( 330^\circ \)
22. \( 405^\circ \)

Converting from Radians to Degrees In Exercises 23–32, express the angle in degree measure. Use a calculator to verify your result. See Example 3.
23. \( \frac{5\pi}{2} \)
24. \( \frac{5\pi}{4} \)
25. \( \frac{7\pi}{3} \)
26. \( \frac{\pi}{9} \)
27. \( -\frac{\pi}{12} \)
28. \( -\frac{7\pi}{12} \)
29. \( \frac{4\pi}{15} \)
30. \( -\frac{8\pi}{9} \)
31. \( \frac{19\pi}{6} \)
32. \( \frac{8\pi}{3} \)
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Analyzing Triangles  In Exercises 33–40, solve the triangle for the indicated side and/or angle.

33. \[ \triangle \text{with sides } 5, 5\sqrt{3}, 10 \]

34. \[ \triangle \text{with sides } a, 288, 60^\circ \]

35. \[ \triangle \text{with sides } 8, 4, 2 \]

36. \[ \triangle \text{with sides } 4, 4, 4 \]

37. \[ \triangle \text{with sides } 5, 5, 4 \]

38. \[ \triangle \text{with sides } 5, 4, 2 \]

39. \[ \triangle \text{with sides } 2, 2\sqrt{3}, 2 \]

40. \[ \triangle \text{with sides } 2.5, 2.5, 2 \]

Finding the Area of an Equilateral Triangle  In Exercises 41–44, find the area of the equilateral triangle with sides of length \( s \). See Example 4.

41. \( s = 4 \) in.

42. \( s = 8 \) m

43. \( s = 5 \) ft

44. \( s = 12 \) cm

45. Height  A person 6 feet tall standing 16 feet from a streetlight casts a shadow 8 feet long (see figure). What is the height of the streetlight?

46. Length  A guy wire is stretched from a broadcasting tower at a point 200 feet above the ground to an anchor 125 feet from the base (see figure). How long is the wire?

47. Using the Arc Length Formula  Complete the table using the formula for arc length.

<table>
<thead>
<tr>
<th>( r )</th>
<th>8 ft</th>
<th>15 in.</th>
<th>85 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s )</td>
<td>12 ft</td>
<td>96 in.</td>
<td>8642 mi</td>
</tr>
</tbody>
</table>
| \( \theta \) | 1.6 | \( \frac{3\pi}{4} \) | 4 | \( \frac{2\pi}{3} \)

48. Distance  A tractor tire that is 5 feet in diameter is partially filled with a liquid ballast for additional traction. To check the air pressure, the tractor operator rotates the tire until the valve stem is at the top so that the liquid will not enter the gauge. On a given occasion, the operator notes that the tire must be rotated 80° to have the stem in the proper position (see figure).

(a) What is the radius of the tractor tire?
(b) Find the radian measure of this rotation.
(c) How far must the tractor be moved to get the valve stem in the proper position?
49. **Clock**  The minute hand on a clock is 3\(\frac{1}{2}\) inches long (see figure). Through what distance does the tip of the minute hand move in 25 minutes?

![Figure for 49](image)

50. **Instrumentation**  The pointer of a voltmeter is 6 centimeters in length (see figure). Find the angle (in radians and degrees) through which the pointer rotates when it moves 2.5 centimeters on the scale.

51. **Speed of Revolution**  A compact disc can have an angular speed of up to 3142 radians per minute. 
(a) At this angular speed, how many revolutions per minute would the CD make?
(b) How long would it take the CD to make 10,000 revolutions?

52. **HOW DO YOU SEE IT?**  Determine which angles in the figure are coterminal angles with Angle A. Explain your reasoning.

![Figure for 50](image)

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**Area of a Sector of a Circle**  In Exercises 53 and 54, use the following information. A sector of a circle is the region bounded by two radii of the circle and their intercepted arc (see figure).

For a circle of radius \(r\), the area \(A\) of a sector of the circle with central angle \(\theta\) (in radians) is given by \(A = \frac{1}{2}r^2\theta\).

53. **Sprinkler System**  A sprinkler system on a farm is set to spray water over a distance of 70 feet and rotates through an angle of 120°. Find the area of the region.

54. **Windshield Wiper**  A car’s rear windshield wiper rotates 125°. The wiper mechanism has a total length of 25 inches and wipes the windshield over a distance of 14 inches. Find the area covered by the wiper.

**True or False?**  In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

55. An angle whose measure is 75° is obtuse.
56. \(\theta = -35^\circ\) is coterminal to 325°.
57. A right triangle can have one angle whose measure is 89°.
58. An angle whose measure is \(\pi\) radians is a straight angle.
14.2 The Trigonometric Functions

- Understand the definitions of the trigonometric functions.
- Understand the trigonometric identities.
- Evaluate trigonometric functions and solve right triangles.
- Solve trigonometric equations.

The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one case the trigonometric functions are defined as ratios of two sides of a right triangle. In the other case these functions are defined in terms of a point on the terminal side of an arbitrary angle. The first approach is the one generally used in surveying, navigation, and astronomy, where a typical problem involves a triangle, three of whose six parts (sides and angles) are known and three of which are to be determined. The second approach is the one normally used in science and economics, where the periodic nature of the trigonometric functions is emphasized. In the definitions below, the six trigonometric functions

\[ \text{sine, cosecant, cosine, secant, tangent, and cotangent} \]

are defined from both viewpoints. These six functions are normally abbreviated sin, csc, cos, sec, tan, and cot, respectively.

### Definitions of the Trigonometric Functions

**Right Triangle Definition:** \(0 < \theta < \frac{\pi}{2}\) (See Figure 14.11.)

\[
\begin{align*}
sin \theta &= \frac{\text{opp}}{\text{hyp}} \\
csc \theta &= \frac{\text{hyp}}{\text{opp}} \\
cos \theta &= \frac{\text{adj}}{\text{hyp}} \\
sec \theta &= \frac{\text{hyp}}{\text{adj}} \\
tan \theta &= \frac{\text{opp}}{\text{adj}} \\
cot \theta &= \frac{\text{adj}}{\text{opp}}
\end{align*}
\]

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle.

- opp = the length of the side opposite \(\theta\)
- adj = the length of the side adjacent to \(\theta\)
- hyp = the length of the hypotenuse

**Circular Function Definition:** Let \(\theta\) be an angle in standard position with \((x, y)\) a point on the terminal ray of \(\theta\) and \(r = \sqrt{x^2 + y^2} \neq 0\). (See Figure 14.12.)

\[
\begin{align*}
sin \theta &= \frac{y}{r} \\
csc \theta &= \frac{r}{y} \\
cos \theta &= \frac{x}{r} \\
sec \theta &= \frac{r}{x} \\
tan \theta &= \frac{y}{x} \\
cot \theta &= \frac{x}{y}
\end{align*}
\]
Trigonometric Identities

In the circular function definition of the six trigonometric functions, the value of $r$ is always positive. From this, it follows that the signs of the trigonometric functions are determined from the signs of $x$ and $y$, as shown in Figure 14.13.

The trigonometric reciprocal identities below are also direct consequences of the definitions.

Furthermore, because $\sin^2 \theta + \cos^2 \theta = 1$, you can obtain the Pythagorean Identity $\sin^2 \theta + \cos^2 \theta = 1$. Other trigonometric identities are listed below. In the list, $\phi$ is the lowercase Greek letter phi.

### Trigonometric Identities

#### Pythagorean Identities

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\tan^2 \theta + 1 = \sec^2 \theta$
- $\cot^2 \theta + 1 = \csc^2 \theta$

#### Reduction Formulas

- $\sin(-\theta) = -\sin \theta$
- $\cos(-\theta) = \cos \theta$
- $\tan(-\theta) = -\tan \theta$

#### Sum or Difference of Two Angles

- $\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$
- $\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi$
- $\tan(\theta \pm \phi) = \frac{\tan \theta \pm \tan \phi}{1 \mp \tan \theta \tan \phi}$

#### Double Angle

- $\sin 2\theta = 2 \sin \theta \cos \theta$
- $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta$

#### Half Angle

- $\sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos \theta)$
- $\cos^2 \frac{\theta}{2} = \frac{1}{2}(1 + \cos \theta)$

Although an angle can be measured in either degrees or radians, radian measure is preferred in calculus. So, all angles in the remainder of this chapter are assumed to be measured in radians unless otherwise indicated. In other words, $\sin 3$ means the sine of 3 radians, and $\sin 3^\circ$ means the sine of 3 degrees.
Evaluating Trigonometric Functions

There are two common methods of evaluating trigonometric functions: decimal approximations using a calculator and exact evaluations using trigonometric identities and formulas from geometry. The next three examples illustrate the second method.

Example 1 Evaluating Trigonometric Functions

Let be a point on the terminal side of as shown in Figure 14.14. Find the sine, cosine, and tangent of

\[
\text{SOLUTION}
\]

Referring to Figure 14.14, you can see that and

So, the values of the sine, cosine, and tangent of are as shown.

\[
\sin \theta = \frac{y}{r} = \frac{4}{5} \\
\cos \theta = \frac{x}{r} = \frac{-3}{5} \\
\tan \theta = \frac{y}{x} = \frac{-4}{3}
\]

So, the values of the sine, cosine, and tangent of \( \theta \) are as shown.

\[
\sin \theta = \frac{y}{r} = \frac{4}{5} \\
\cos \theta = \frac{x}{r} = \frac{-3}{5} \\
\tan \theta = \frac{y}{x} = \frac{-4}{3}
\]

Checkpoint 1

Let \((\sqrt{3}, 1)\) be a point on the terminal side of \( \theta \), as shown in the figure. Find the sine, cosine, and tangent of \( \theta \).

\[
\text{The sines, cosines, and tangents of several common angles are listed in the table below. You should remember, or be able to derive, these values.}
\]

<table>
<thead>
<tr>
<th>Trigonometric Values of Common Angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta ) (degrees)</td>
</tr>
<tr>
<td>( \sin \theta )</td>
</tr>
<tr>
<td>( \cos \theta )</td>
</tr>
<tr>
<td>( \tan \theta )</td>
</tr>
</tbody>
</table>
To extend the use of the values in the table on the preceding page, you can use the concept of a reference angle, as shown in Figure 14.15, together with the appropriate quadrant sign. The reference angle \( \theta' \) for an angle \( \theta \) is the smallest positive angle between the terminal side of \( \theta \) and the x-axis. For instance, the reference angle for \( 135^\circ \) is \( 45^\circ \) and the reference angle for \( 210^\circ \) is \( 30^\circ \).

![FIGURE 14.15](image_url)

To find the value of a trigonometric function of any angle \( \theta \), first determine the function value for the associated reference angle \( \theta' \). Then, depending on the quadrant in which \( \theta \) lies, prefix the appropriate sign to the function value.

**Example 2** Evaluating Trigonometric Functions

Evaluate each trigonometric function.

a. \( \sin \frac{3\pi}{4} \)  
b. \( \tan 330^\circ \)  
c. \( \cos \frac{7\pi}{6} \)

**SOLUTION**

a. Because the reference angle for \( 3\pi/4 \) is \( \pi/4 \) and the sine is positive in the second quadrant, you can write

\[
\sin \frac{3\pi}{4} = \sin \frac{\pi}{4} \quad \text{Reference angle}
\]

\[
= \frac{\sqrt{2}}{2} \quad \text{See Figure 14.16(a).}
\]

b. Because the reference angle for \( 330^\circ \) is \( 30^\circ \) and the tangent is negative in the fourth quadrant, you can write

\[
\tan 330^\circ = -\tan 30^\circ \quad \text{Reference angle}
\]

\[
= -\frac{\sqrt{3}}{3} \quad \text{See Figure 14.16(b).}
\]

c. Because the reference angle for \( 7\pi/6 \) is \( \pi/6 \) and the cosine is negative in the third quadrant, you can write

\[
\cos \frac{7\pi}{6} = -\cos \frac{\pi}{6} \quad \text{Reference angle}
\]

\[
= -\frac{\sqrt{3}}{2} \quad \text{See Figure 14.16(c).}
\]

**Checkpoint 2**

Evaluate each trigonometric function.

a. \( \sin \frac{5\pi}{6} \)  
b. \( \cos 135^\circ \)  
c. \( \tan \frac{5\pi}{3} \)
Example 3 Evaluating Trigonometric Functions

Evaluate each trigonometric function.

a. \( \sin \left( \frac{\pi}{3} \right) \)  
b. \( \sec 60^\circ \)

c. \( \cos 15^\circ \)  
d. \( \sin 2\pi \)

e. \( \cot 0^\circ \)  
f. \( \tan \frac{9\pi}{4} \)

SOLUTION

a. By the reduction formula \( \sin(-\theta) = -\sin \theta \),
\[
\sin \left( -\frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}.
\]

b. By the reciprocal identity \( \sec \theta = 1/\cos \theta \),
\[
\sec 60^\circ = \frac{1}{\cos 60^\circ} = \frac{1}{1/2} = 2.
\]

c. By the difference formula \( \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi \),
\[
\cos 15^\circ = \cos(45^\circ - 30^\circ) \\
= (\cos 45^\circ)(\cos 30^\circ) + (\sin 45^\circ)(\sin 30^\circ) \\
= \left( \frac{\sqrt{2}}{2} \right) \left( \frac{\sqrt{3}}{2} \right) + \left( \frac{\sqrt{2}}{2} \right) \left( \frac{1}{2} \right) \\
= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
= \frac{\sqrt{6} + \sqrt{2}}{4}.
\]

d. Because the reference angle for \( 2\pi \) is 0,
\[
\sin 2\pi = \sin 0 = 0.
\]

e. Using the reciprocal identity
\[
\cot \theta = \frac{1}{\tan \theta}
\]
and the fact that \( \tan 0^\circ = 0 \), you can conclude that \( \cot 0^\circ \) is undefined.

f. Because the reference angle for \( 9\pi/4 \) is \( \pi/4 \) and the tangent is positive in the first quadrant,
\[
\tan \frac{9\pi}{4} = \tan \frac{\pi}{4} = 1.
\]

Checkpoint 3

Evaluate each trigonometric function.

a. \( \sin \left( -\frac{\pi}{6} \right) \)  
b. \( \csc 45^\circ \)

c. \( \cos 75^\circ \)  
d. \( \cos 2\pi \)

e. \( \sec 0^\circ \)  
f. \( \cot \frac{13\pi}{4} \)
Example 4  Solving a Right Triangle

A surveyor is standing 50 feet from the base of a large tree, as shown in Figure 14.17. The surveyor measures the angle of elevation to the top of the tree as $71.5^\circ$. How tall is the tree?

**SOLUTION**  From Figure 14.17, you can see that

$$ \tan 71.5^\circ = \frac{y}{x} $$

where $x = 50$ and $y$ is the height of the tree. So, the height of the tree is

$$ y = (x)(\tan 71.5^\circ) \approx (50)(2.98868) \approx 149.4 \text{ feet.} $$

Checkpoint 4

Find the height of a building that casts a 75-foot shadow when the angle of elevation of the sun is $35^\circ$.

Example 5  Calculating Peripheral Vision

To measure the extent of your peripheral vision, stand 1 foot from the corner of a room, facing the corner. Have a friend move an object along the wall until you can just barely see it. When the object is 2 feet from the corner, as shown in Figure 14.18, what is the total angle of your peripheral vision?

**SOLUTION**  Let $\alpha$ represent the total angle of your peripheral vision. As shown in Figure 14.19, you can model the physical situation with an isosceles right triangle whose legs are $\sqrt{2}$ feet and whose hypotenuse is 2 feet. In the triangle, the angle $\theta$ is given by

$$ \tan \theta = \frac{y}{x} = \frac{\sqrt{2}}{\sqrt{2} - 1} \approx 3.414. $$

Using the inverse tangent function of a calculator in degree mode and the following keystrokes

\[
\text{TAN}^{-1} 3.4141 \text{ ENTER}
\]

you can determine that $\theta \approx 73.7^\circ$. So, $\alpha/2 \approx 180^\circ - 73.7^\circ = 106.3^\circ$, which implies that $\alpha \approx 212.6^\circ$. In other words, the total angle of your peripheral vision is about $212.6^\circ$.

Checkpoint 5

When the object in Example 5 is 4 feet from the corner, find the total angle of your peripheral vision.
Solving Trigonometric Equations

An important part of the study of trigonometry is learning how to solve trigonometric equations. For example, consider the equation

\[ \sin \theta = 0. \]

You know that \( \theta = 0 \) is one solution. Also, in Example 3(d), you saw that \( \theta = 2\pi \) is another solution. But these are not the only solutions. In fact, this equation has infinitely many solutions. Any one of the values of \( \theta \) shown below will work.

\[ \ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots \]

To simplify the situation, the search for solutions can be restricted to the interval

\[ 0 \leq \theta \leq 2\pi \]

as shown in Example 6.

**Example 6**  
Solving Trigonometric Equations

Solve for \( \theta \) in each equation. Assume \( 0 \leq \theta \leq 2\pi \).

**a.** \( \sin \theta = -\frac{\sqrt{3}}{2} \)  
**b.** \( \cos \theta = 1 \)  
**c.** \( \tan \theta = 1 \)

**SOLUTION**

**a.** To solve the equation \( \sin \theta = -\frac{\sqrt{3}}{2} \), first remember that

\[ \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}. \]

Because the sine is negative in the third and fourth quadrants, it follows that you are seeking values of \( \theta \) in these quadrants that have a reference angle of \( \pi/3 \). The two angles fitting these criteria are

\[ \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \]

and

\[ \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \]

as indicated in Figure 14.20.

**b.** To solve \( \cos \theta = 1 \), remember that \( \cos 0 = 1 \) and note that in the interval \([0, 2\pi]\), the only angles whose reference angles are 0 are 0, \( \pi \), and 2\( \pi \). Of these, 0 and 2\( \pi \) have cosines of 1. (The cosine of \( \pi \) is \(-1\).) So, the equation has two solutions:

\[ \theta = 0 \quad \text{and} \quad \theta = 2\pi. \]

**c.** Because \( \tan \pi/4 = 1 \) and the tangent is positive in the first and third quadrants, it follows that the two solutions are

\[ \theta = \frac{\pi}{4} \quad \text{and} \quad \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}. \]

**Checkpoint 6**

Solve for \( \theta \) in each equation. Assume \( 0 \leq \theta \leq 2\pi \).

**a.** \( \cos \theta = \frac{\sqrt{3}}{2} \)  
**b.** \( \tan \theta = -\sqrt{3} \)  
**c.** \( \sin \theta = -\frac{1}{2} \)
Example 7  Solving a Trigonometric Equation

Solve the equation for $\theta$.

$$\cos 2\theta = 2 - 3 \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

**SOLUTION** You can use the double-angle identity $\cos 2\theta = 1 - 2 \sin^2 \theta$ to rewrite the original equation, as shown.

$$\begin{align*}
\cos 2\theta &= 2 - 3 \sin \theta \\
1 - 2 \sin^2 \theta &= 2 - 3 \sin \theta \\
0 &= 2 \sin^2 \theta - 3 \sin \theta + 1 \\
0 &= (2 \sin \theta - 1)(\sin \theta - 1)
\end{align*}$$

Next, set each factor equal to zero. For $2 \sin \theta - 1 = 0$, you have $\sin \theta = \frac{1}{2}$, which has solutions of

$$\theta = \frac{\pi}{6} \quad \text{and} \quad \theta = \frac{5\pi}{6}.$$ 

For $\sin \theta - 1 = 0$, you have $\sin \theta = 1$, which has a solution of

$$\theta = \frac{\pi}{2}.$$ 

So, for $0 \leq \theta \leq 2\pi$, the three solutions are

$$\theta = \frac{\pi}{6}, \quad \frac{\pi}{2}, \quad \text{and} \quad \frac{5\pi}{6}.$$ 

**Checkpoint 7**

Solve the equation for $\theta$.

$$\sin 2\theta + \sin \theta = 0, \quad 0 \leq \theta \leq 2\pi$$

**STUDY TIP**

In Example 7, note that the expression $2 \sin^2 \theta - 3 \sin \theta + 1$ is a quadratic in $\sin \theta$, and as such can be factored. For instance, when you let $x = \sin \theta$, the quadratic factors as $2x^2 - 3x + 1 = (2x - 1)(x - 1)$.

**SUMMARIZE** (Section 14.2)

1. State the circular function definition of the six trigonometric functions (page 1006). For an example of evaluating trigonometric functions, see Example 1.
2. Explain what is meant by a reference angle (page 1009). For an example of evaluating trigonometric functions using reference angles, see Example 2.
3. State the reduction formulas (page 1007). For an example of evaluating trigonometric functions using a reduction formula, see Example 3(a).
4. Describe a real-life example of how a trigonometric function can be used to find the height of a tree (page 1011, Example 4).
5. State the double-angle identities (page 1007). For an example of solving a trigonometric equation using a double-angle identity, see Example 7.
The following warm-up exercises involve skills that were covered in a previous course or in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 1.1, 1.3, and 14.1.

In Exercises 1–4, convert the angle to radian measure.
1. \(315^\circ\)
2. \(-300^\circ\)
3. \(-225^\circ\)
4. \(390^\circ\)

In Exercises 5–8, solve for \(x\).
5. \(x^2 - x = 0\)
6. \(2x^2 + x = 0\)
7. \(x^2 - 2x = 3\)
8. \(x^2 - 5x = -6\)

In Exercises 9–12, solve for \(t\).
9. \(\frac{2\pi}{24}(t - 4) = \frac{\pi}{2}\)
10. \(\frac{2\pi}{12}(t - 2) = \frac{\pi}{4}\)
11. \(\frac{2\pi}{365}(t - 10) = \frac{\pi}{4}\)
12. \(\frac{2\pi}{12}(t - 4) = \frac{\pi}{2}\)

### Evaluating Trigonometric Functions

In Exercises 1–6, determine all six trigonometric functions of the angle \(\theta\). See Example 1.

1. \(\theta\)
2. \(\theta\)
3. \(\theta\)
4. \(\theta\)
5. \(\theta\)
6. \(\theta\)

### Finding Trigonometric Functions

In Exercises 7–12, sketch a right triangle corresponding to the trigonometric function of the angle \(\theta\) and find the other five trigonometric functions of \(\theta\).

7. \(\sin \theta = \frac{1}{2}\)
8. \(\cot \theta = 5\)
9. \(\sec \theta = 2\)
10. \(\cos \theta = \frac{3}{4}\)
11. \(\tan \theta = 3\)
12. \(\csc \theta = 4.25\)

### Determining a Quadrant

In Exercises 13–18, determine the quadrant in which \(\theta\) lies.

13. \(\sin \theta < 0, \cos \theta > 0\)
14. \(\sin \theta > 0, \cos \theta < 0\)
15. \(\sin \theta > 0, \sec \theta > 0\)
16. \(\cot \theta < 0, \cos \theta > 0\)
17. \(\csc \theta > 0, \tan \theta < 0\)
18. \(\cos \theta > 0, \tan \theta < 0\)

### Evaluating Trigonometric Functions

In Exercises 19–32, evaluate the six trigonometric functions of the angle without using a calculator. See Examples 2 and 3.

19. \(60^\circ\)
20. \(-\frac{2\pi}{3}\)
21. \(\frac{\pi}{4}\)
22. \(\frac{5\pi}{4}\)
23. \(-\frac{3\pi}{2}\)
24. \(150^\circ\)
25. \(225^\circ\)
26. \(\frac{4\pi}{3}\)
27. \(300^\circ\)
28. \(210^\circ\)
29. \(750^\circ\)
30. \(510^\circ\)
31. \(\frac{10\pi}{3}\)
32. \(\frac{17\pi}{3}\)

### Evaluating Trigonometric Functions

In Exercises 33–42, use a calculator to evaluate the trigonometric function to four decimal places.

33. \(\sin 10^\circ\)
34. \(\csc 10^\circ\)
35. \(\tan \frac{\pi}{9}\)
36. \(\cot \frac{10\pi}{9}\)
37. \(\cos (-110^\circ)\)
38. \(\cos 250^\circ\)
39. \(\tan 240^\circ\)
40. \(\cot 210^\circ\)
41. \(\sin (-0.65)\)
42. \(\tan 4.5\)
Solving a Right Triangle  In Exercises 43–48, solve for \( x, y, \) or \( r \) as indicated. See Example 4.

43. Solve for \( y \).

\[
\begin{array}{c}
\text{30}\degree \\
100
\end{array}
\]

44. Solve for \( x \).

\[
\begin{array}{c}
\text{60}\degree \\
10
\end{array}
\]

45. Solve for \( x \).

\[
\begin{array}{c}
\text{60}\degree \\
25
\end{array}
\]

46. Solve for \( r \).

\[
\begin{array}{c}
\text{45}\degree \\
20
\end{array}
\]

47. Solve for \( r \).

\[
\begin{array}{c}
\text{40}\degree \\
10
\end{array}
\]

48. Solve for \( x \).

\[
\begin{array}{c}
\text{20}\degree \\
50
\end{array}
\]

71. Length  A 20-foot ladder leaning against the side of a house makes a \( 75\degree \) angle with the ground (see figure). How far up the side of the house does the ladder reach?

72. Width of a River  A biologist wants to know the width \( w \) of a river in order to set instruments to study the pollutants in the water. From point \( A \), the biologist walks downstream 100 feet and sights to point \( C \). From this sighting it is determined that \( \theta = 50\degree \) (see figure). How wide is the river?

73. Distance  An airplane flying at an altitude of 6 miles is on a flight path that passes directly over an observer (see figure). Let \( \theta \) be the angle of elevation from the observer to the plane. Find the distance \( d \) from the observer to the plane when (a) \( \theta = 30\degree \), (b) \( \theta = 60\degree \), and (c) \( \theta = 90\degree \).

74. Skateboard Ramp  A skateboard ramp with a height of 4 feet has an angle of elevation of \( 18\degree \) (see figure). How long is the skateboard ramp?
75. Empire State Building  You are standing 45 meters from the base of the Empire State Building. You estimate that the angle of elevation to the top of the 86th floor is 82°. The total height of the building is another 123 meters above the 86th floor.
(a) What is the approximate height of the building?
(b) One of your friends is on the 86th floor. What is the distance between you and your friend?

76. Height  A 25-meter line is used to tether a helium-filled balloon. Because of a breeze, the line makes an angle of approximately 75° with the ground.
(a) Draw the right triangle that gives a visual representation of the problem. Show the known side lengths and angles of the triangle and use a variable to indicate the height of the balloon.
(b) Use a trigonometric function to write an equation involving the unknown quantity.
(c) What is the height of the balloon?

77. Loading Ramp  A ramp 17 1/2 feet in length rises to a loading platform that is 3 1/2 feet off the ground (see figure). Find the angle (in degrees) that the ramp makes with the ground.

78. Height  The height of a building is 180 feet. Find the angle of elevation (in degrees) to the top of the building from a point 100 feet from the base of the building (see figure).

79. Height of a Mountain  In traveling across flat land, you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9°. Approximate the height of the mountain.

80. HOW DO YOU SEE IT?  Consider an angle in standard position with \( r = 12 \) centimeters, as shown in the figure. Describe the changes in the values of \( x, y, \sin \theta, \cos \theta, \) and \( \tan \theta \) as \( \theta \) increases from 0° to 90°.

81. Medicine  The temperature \( T \) (in degrees Fahrenheit) of a patient \( t \) hours after arriving at the emergency room of a hospital at 10:00 P.M. is given by
\[
T(t) = 98.6 + 4 \cos \frac{\pi t}{36} \quad 0 \leq t \leq 18.
\]
Find the patient’s temperature at (a) 10:00 P.M., (b) 4:00 A.M., and (c) 10:00 A.M. (d) At what time do you expect the patient’s temperature to return to normal? Explain your reasoning.

82. Sales  A company that produces a window and door insulating kit forecasts monthly sales over the next 2 years to be
\[
S = 23.1 + 0.442t + 4.3 \sin \frac{\pi t}{6}
\]
where \( S \) is measured in thousands of units and \( t \) is the time in months, with \( t = 1 \) corresponding to January 2011. Find the monthly sales for (a) February 2011, (b) February 2012, (c) September 2011, and (d) September 2012.

Graphing Functions  In Exercises 83 and 84, use a graphing utility or a spreadsheet to complete the table. Then graph the function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

83. \( f(x) = \frac{2}{5} x + 2 \sin \frac{\pi x}{5} \)
84. \( f(x) = \frac{1}{2} (5 - x) + 3 \cos \frac{\pi x}{5} \)

True or False? In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.
85. \( \sin 10° \csc 10° = 1 \)
86. \( \sin 60° = \sin 30° \)
87. \( \sin^2 45° - \cos^2 45° = 1 \)
88. Because \( \sin(-t) = -\sin t \), it can be said that the sine of a negative angle is a negative number.
14.3 Graphs of Trigonometric Functions

- Sketch graphs of trigonometric functions.
- Evaluate limits of trigonometric functions.
- Use trigonometric functions to model real-life situations.

Graphs of Trigonometric Functions

When you are sketching the graph of a trigonometric function, it is common to use \( x \) (rather than \( \theta \)) as the independent variable. For instance, you can sketch the graph of

\[ f(x) = \sin x \]

by constructing a table of values, plotting the resulting points, and connecting them with a smooth curve, as shown in Figure 14.21. Some examples of values are shown in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( \pi/6 )</th>
<th>( \pi/4 )</th>
<th>( \pi/3 )</th>
<th>( \pi/2 )</th>
<th>( 2\pi/3 )</th>
<th>( 3\pi/4 )</th>
<th>( 5\pi/6 )</th>
<th>( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin x )</td>
<td>0.00</td>
<td>0.50</td>
<td>0.71</td>
<td>0.87</td>
<td>1.00</td>
<td>0.87</td>
<td>0.71</td>
<td>0.50</td>
<td>0.00</td>
</tr>
</tbody>
</table>

In Figure 14.21, note that the maximum value of \( \sin x \) is 1 and the minimum value is \(-1\). The amplitude of the sine function (or the cosine function) is defined to be half of the difference between its maximum and minimum values. So, the amplitude of \( f(x) = \sin x \) is 1.

![Amplitude = 1](image)

**FIGURE 14.21**

The periodic nature of the sine function becomes evident when you observe that as \( x \) increases beyond \( 2\pi \), the graph repeats itself over and over, continuously oscillating about the \( x \)-axis. The period of the function is the distance (on the \( x \)-axis) between successive cycles, as shown in Figure 14.22. So, the period of \( f(x) = \sin x \) is \( 2\pi \).

![Period: 2\pi](image)

**FIGURE 14.22**

In Exercise 73 on page 1025, you will use a trigonometric function to model the air flow of a person’s respiratory cycle.
Figure 14.23 shows the graphs of at least one cycle of all six trigonometric functions.

Familiarity with the graphs of the six basic trigonometric functions allows you to sketch graphs of more general functions such as

\[ y = a \sin bx \quad \text{and} \quad y = a \cos bx. \]

Note that the function \( y = a \sin bx \) oscillates between \(-a\) and \(a\) and so has an amplitude of \(|a|\).

**Amplitude of \( y = a \sin bx \)**

Furthermore, because \( bx = 0 \) when \( x = 0 \) and \( bx = 2\pi \) when \( x = 2\pi/b \), it follows that the function \( y = a \sin bx \) has a period of

\[ \frac{2\pi}{|b|}. \]

**Period of \( y = a \sin bx \)**

When graphing general functions such as

\[ f(x) = a \sin[b(x - c)] + d \quad \text{or} \quad g(x) = a \cos[b(x - c)] + d \]

note how the constants \( a, b, c, \) and \( d \) affect the graph. You already know that \( a \) is the amplitude and \( b \) is the period of the graph. The constants \( c \) and \( d \) determine the horizontal shift and vertical shift of the graph, respectively. Two examples are shown in Figure 14.24. In the first graph, notice that relative to the graph of \( y = \sin x \), the graph of \( f \) is shifted \( \pi/2 \) units to the right, stretched vertically by a factor of 2, and shifted up one unit. In the second graph, notice that relative to the graph of \( y = \cos x \), the graph of \( g \) is shifted \( \pi/2 \) units to the right, stretched horizontally by a factor of \( 1/2 \), stretched vertically by a factor of 3, and shifted down two units.
Example 1  **Graphing a Trigonometric Function**

Sketch the graph of \( f(x) = 4 \sin x \).

**SOLUTION**  The graph of \( f(x) = 4 \sin x \) has the characteristics below.

Amplitude: 4
Period: \( 2\pi \)

Three cycles of the graph are shown in Figure 14.25, starting with the point (0, 0).

✓ **Checkpoint 1**

Sketch the graph of \( g(x) = 2 \cos x \).

---

Example 2  **Graphing a Trigonometric Function**

Sketch the graph of \( f(x) = 3 \cos 2x \).

**SOLUTION**  The graph of \( f(x) = 3 \cos 2x \) has the characteristics below.

Amplitude: 3
Period: \( \frac{2\pi}{2} = \pi \)

Almost three cycles of the graph are shown in Figure 14.26, starting with the maximum point (0, 3).

✓ **Checkpoint 2**

Sketch the graph of \( g(x) = 2 \sin 4x \).

---

Example 3  **Graphing a Trigonometric Function**

Sketch the graph of \( f(x) = -2 \tan 3x \).

**SOLUTION**  The graph of this function has a period of \( \frac{\pi}{3} \). The vertical asymptotes of this tangent function occur at

\[ x = \ldots, -\frac{\pi}{6}, -\frac{\pi}{6}, -\frac{\pi}{2}, -\frac{5\pi}{6}, \ldots \]

Period = \( \frac{\pi}{3} \)

Several cycles of the graph are shown in Figure 14.27, starting with the vertical asymptote \( x = -\pi/6 \).

✓ **Checkpoint 3**

Sketch the graph of \( g(x) = \tan 4x \).
Limits of Trigonometric Functions

The sine and cosine functions are continuous over the entire real number line. So, you can use direct substitution to evaluate a limit such as

$$\lim_{x \to 0} \sin x = \sin 0 = 0.$$ 

When direct substitution with a trigonometric limit yields an indeterminate form, such as

$$\frac{0}{0}$$

you can rely on technology to help evaluate the limit. The next example examines the limit of a function that you will encounter again in Section 14.4.

**Example 4 Evaluating a Trigonometric Limit**

Use a calculator to evaluate the function

$$f(x) = \frac{\sin x}{x}$$

at several $x$-values near $x = 0$. Then use the result to estimate

$$\lim_{x \to 0} \frac{\sin x}{x}.$$ 

Use a graphing utility (set in radian mode) to confirm your result.

**SOLUTION**

The table shows several values of the function at $x$-values near zero. (Note that the function is undefined when $x = 0$.)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-0.20$</th>
<th>$-0.15$</th>
<th>$-0.10$</th>
<th>$-0.05$</th>
<th>$0.05$</th>
<th>$0.10$</th>
<th>$0.15$</th>
<th>$0.20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin x / x$</td>
<td>0.9933</td>
<td>0.9963</td>
<td>0.9983</td>
<td>0.9996</td>
<td>0.9996</td>
<td>0.9983</td>
<td>0.9963</td>
<td>0.9933</td>
</tr>
</tbody>
</table>

From the table, it appears that the limit is 1. That is,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$ 

Figure 14.28 shows the graph of

$$f(x) = \frac{\sin x}{x}.$$ 

From this graph, it appears that $f(x)$ gets closer and closer to 1 as $x$ approaches zero (from either side).

**Checkpoint 4**

Use a calculator to evaluate the function

$$f(x) = \frac{1 - \cos x}{x}$$

at several $x$-values near $x = 0$. Then use the result to estimate

$$\lim_{x \to 0} \frac{1 - \cos x}{x}.$$
Applications

There are many examples of periodic phenomena in both business and biology. Many businesses have cyclical sales patterns, and plant growth is affected by the day-night cycle. The next example describes the cyclical pattern followed by many types of predator-prey populations, such as coyotes and rabbits.

**Example 5  Modeling Predator-Prey Cycles**

The population $P$ of a predator at time $t$ (in months) is modeled by

$$ P = 10,000 + 3000 \sin \frac{2\pi t}{24}, \quad t \geq 0 $$

and the population $p$ of its primary food source (its prey) is modeled by

$$ p = 15,000 + 5000 \cos \frac{2\pi t}{24}, \quad t \geq 0. $$

Graph both models on the same set of axes and explain the oscillations in the size of each population.

**SOLUTION** Each function has a period of 24 months. The predator population has an amplitude of 3000 and oscillates about the line $y = 10,000$. The prey population has an amplitude of 5000 and oscillates about the line $y = 15,000$. The graphs of the two models are shown in Figure 14.29. The cycles of this predator-prey population are explained by the diagram below.

**Checkpoint 5**

Repeat Example 5 for the following models.

$$ P = 12,000 + 2500 \sin \frac{2\pi t}{12}, \quad t \geq 0 \quad \text{Predator population} $$

$$ p = 18,000 + 6000 \sin \frac{2\pi t}{12}, \quad t \geq 0 \quad \text{Prey population} $$
Example 6  Modeling Biorhythms

A theory that attempts to explain the ups and downs of everyday life states that each person has three cycles, which begin at birth. These three cycles can be modeled by sine waves

\[ P(t) = \sin \frac{2\pi t}{23}, \quad t \geq 0 \]

\[ E(t) = \sin \frac{2\pi t}{28}, \quad t \geq 0 \]

\[ I(t) = \sin \frac{2\pi t}{33}, \quad t \geq 0 \]

where \( t \) is the number of days since birth. Describe the biorhythms during the month of September 2011, for a person who was born on July 20, 1991.

**SOLUTION**  Figure 14.30 shows the person’s biorhythms during the month of September 2011. Note that September 1, 2011 was the 7348th day of the person’s life.

![Figure 14.30](image)

Checkpoint 6

Use a graphing utility to describe the biorhythms of the person in Example 6 during the month of January 2011. Assume that January 1, 2011 is the 7105th day of the person’s life.

SUMMARIZE  (Section 14.3)

1. Describe the graph of \( y = \sin x \) (page 1018). For an example of graphing a sine function, see Example 1.
2. Describe the graph of \( y = \cos x \) (page 1018). For an example of graphing a cosine function, see Example 2.
3. Describe the graph of \( y = \tan x \) (page 1018). For an example of graphing a tangent function, see Example 3.
4. Describe a real-life example of how the graphs of trigonometric functions can be used to analyze biorhythms (page 1022, Example 6).
SKILLS WARM UP 14.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 7.1 and 14.2.

In Exercises 1 and 2, find the limit.
1. \( \lim_{x \to 2} (x^2 + 4x + 2) \)  
2. \( \lim_{x \to 3} (x^3 - 2x^2 + 1) \)

In Exercises 3–10, evaluate the trigonometric function without using a calculator.
3. \( \cos \frac{\pi}{2} \)  
4. \( \sin \pi \)  
5. \( \tan \frac{5\pi}{4} \)  
6. \( \cot \frac{2\pi}{3} \)  
7. \( \sin \frac{11\pi}{6} \)  
8. \( \cos \frac{5\pi}{6} \)  
9. \( \cos \frac{5\pi}{3} \)  
10. \( \sin \frac{4\pi}{3} \)

In Exercises 11–18, use a calculator to evaluate the trigonometric function to four decimal places.
11. \( \cos 15^\circ \)  
12. \( \sin 220^\circ \)  
13. \( \sin 275^\circ \)  
14. \( \cos 310^\circ \)  
15. \( \sin 103^\circ \)  
16. \( \cos 72^\circ \)  
17. \( \tan 327^\circ \)  
18. \( \tan 140^\circ \)

Exercises 14.3

Finding the Period and Amplitude  
In Exercises 1–14, find the period and amplitude of the trigonometric function.
1. \( y = 2 \sin 2x \)  
2. \( y = 3 \cos 3x \)
3. \( y = \frac{3}{2} \cos \frac{x}{2} \)  
4. \( y = -2 \sin \frac{x}{3} \)
5. \( y = \frac{1}{2} \cos \pi x \)  
6. \( y = \frac{5}{2} \cos \frac{\pi x}{2} \)
7. \( y = -\sin 3x \)  
8. \( y = -\frac{2}{3} \cos x \)
9. \( y = -\frac{3}{2} \sin 6x \)  
10. \( y = \frac{1}{4} \sin 8x \)
11. \( y = \frac{1}{2} \sin \frac{2x}{3} \)  
12. \( y = \frac{5}{4} \cos \frac{x}{3} \)
13. \( y = 3 \sin 4\pi x \)  
14. \( y = \frac{2}{3} \cos \frac{\pi x}{10} \)

Finding the Period  
In Exercises 15–20, find the period of the trigonometric function.
15. \( y = 3 \tan x \)  
16. \( y = 7 \tan 2\pi x \)
17. \( y = 3 \sec 5x \)  
18. \( y = \csc 4x \)
19. \( y = \cot \frac{\pi x}{6} \)  
20. \( y = 5 \tan \frac{2\pi x}{3} \)
Matching In Exercises 21–26, match the trigonometric function with the correct graph and give the period of the function. [The graphs are labeled (a)–(f).]

21. \( y = \sec 2x \)
22. \( y = \frac{1}{2} \csc 2x \)
23. \( y = \cot \frac{\pi x}{2} \)
24. \( y = -\sec x \)
25. \( y = 2 \csc \frac{x}{2} \)
26. \( y = \tan \frac{x}{2} \)

Graphing Trigonometric Functions In Exercises 27–40, sketch the graph of the trigonometric function by hand. Use a graphing utility to verify your sketch. See Examples 1, 2, and 3.

27. \( y = \sin \frac{x}{2} \)
28. \( y = 4 \sin \frac{x}{3} \)
29. \( y = 2 \cos \frac{\pi x}{3} \)
30. \( y = \frac{3}{2} \cos \frac{2x}{3} \)
31. \( y = -2 \sin 6x \)
32. \( y = -3 \cos 4x \)
33. \( y = \cos 2\pi x \)
34. \( y = \frac{3}{2} \sin \frac{\pi x}{4} \)
35. \( y = 2 \tan x \)
36. \( y = 2 \cot x \)
37. \( y = \frac{1}{2} \tan \frac{\pi x}{2} \)
38. \( y = -\csc \frac{x}{3} \)
39. \( y = 2 \sec 4x \)
40. \( y = -\tan 3x \)

Graphical Reasoning In Exercises 61–64, find \( a \) and \( d \) for \( f(x) = a \cos x + d \) such that the graph of \( f \) matches the figure.

61.

62.

63.

64.
Phase Shift In Exercises 65–68, match the function with the correct graph. [The graphs are labeled (a)–(d).]

65. \( y = \sin x \)
66. \( y = \sin \left( x - \frac{\pi}{2} \right) \)
67. \( y = \sin (x - \pi) \)
68. \( y = \sin \left( x - \frac{3\pi}{2} \right) \)

69. **Biology: Predator-Prey Cycle** The population \( P \) of a predator at time \( t \) (in months) is modeled by

\[
P = 8000 + 2500 \sin \frac{2\pi t}{24}
\]

and the population \( p \) of its prey is modeled by

\[
p = 12,000 + 4000 \cos \frac{2\pi t}{24}
\]

(a) Use a graphing utility to graph both models in the same viewing window.

(b) Explain the oscillations in the size of each population.

70. **Biology: Predator-Prey Cycle** The population \( P \) of a predator at time \( t \) (in months) is modeled by

\[
P = 5700 + 1200 \sin \frac{2\pi t}{24}
\]

and the population \( p \) of its prey is modeled by

\[
p = 9800 + 2750 \cos \frac{2\pi t}{24}
\]

(a) Use a graphing utility to graph both models in the same viewing window.

(b) Explain the oscillations in the size of each population.

71. **Biorhythms** For a person born on July 20, 1991, use the biorhythm cycles given in Example 6 to calculate this person’s three energy levels on December 31, 2015. Assume this is the 8930th day of the person’s life.

72. **Biorhythms** Use your birthday and the biorhythm cycles given in Example 6 to calculate your three energy levels on December 31, 2015.

73. **Health** For a person at rest, the velocity \( v \) (in liters per second) of air flow into and out of the lungs during a respiratory cycle is given by

\[
v = 0.9 \sin \frac{\pi t}{3}
\]

where \( t \) is the time (in seconds). Inhalation occurs when \( v > 0 \), and exhalation occurs when \( v < 0 \).

(a) Find the time for one full respiratory cycle.

(b) Find the number of cycles per minute.

(c) Use a graphing utility to graph the velocity function.

74. **Health** After a person exercises for a few minutes, the velocity \( v \) (in liters per second) of air flow into and out of the lungs during a respiratory cycle is given by

\[
v = 1.75 \sin \frac{\pi t}{2}
\]

where \( t \) is the time (in seconds). Inhalation occurs when \( v > 0 \), and exhalation occurs when \( v < 0 \).

(a) Find the time for one full respiratory cycle.

(b) Find the number of cycles per minute.

(c) Use a graphing utility to graph the velocity function.

75. **Music** When tuning a piano, a technician strikes a tuning fork for the A above middle C and sets up wave motion that can be approximated by

\[
y = 0.001 \sin 880 \pi t
\]

where \( t \) is the time (in seconds).

(a) What is the period \( p \) of this function?

(b) What is the frequency \( f \) of this note \( (f = 1/p) \)?

(c) Use a graphing utility to graph this function.

76. **Health** The function

\[
P = 100 - 20 \cos(5\pi t/3)
\]

approximates the blood pressure \( P \) (in millimeters of mercury) at time \( t \) (in seconds) for a person at rest.

(a) Find the period of the function.

(b) Find the number of heartbeats per minute.

(c) Use a graphing utility to graph the pressure function.
77. **Construction Workers** The number $W$ (in thousands) of construction workers employed in the United States during 2010 can be modeled by

$$W = 5488 + 347.6 \sin(0.45t + 4.153)$$

where $t$ is the time (in months), with $t = 1$ corresponding to January. *(Source: U.S. Bureau of Labor Statistics)*

(a) Use a graphing utility to graph $W$.

(b) Did the number of construction workers exceed 5.5 million in 2010? If so, during which month(s)?

78. **Sales** The snowmobile sales $S$ (in units) at a dealership are modeled by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where $t$ is the time (in months), with $t = 1$ corresponding to January.

(a) Use a graphing utility to graph $S$.

(b) Will the sales exceed 75 units during any month? If so, during which month(s)?

79. **Physics** Use the graphs below to answer each question.

(a) Which graph (A or B) has a longer wavelength, or period?

(b) Which graph (A or B) has a greater amplitude?

(c) The frequency of a graph is the number of oscillations or cycles that occur during a given period of time. Which graph (A or B) has a greater frequency?

(d) Based on the definition of frequency in part (c), how are frequency and period related?

*(Source: Adapted from Shipman/Wilson/Todd, An Introduction to Physical Science, Eleventh Edition)*

80. **Think About It** Consider the function given by $y = \cos bx$ on the interval $(0, 2\pi)$.

(a) Sketch the graph of $y = \cos bx$ for $b = \frac{1}{2}, 2$, and 3.

(b) How does the value of $b$ affect the graph?

(c) How many complete cycles occur between 0 and $2\pi$ for each value of $b$?

81. **Think About It** Consider the functions given by $f(x) = 2 \sin x$ and $g(x) = 0.5 \csc x$ on the interval $(0, \pi)$.

(a) Graph $f$ and $g$ in the same coordinate plane.

(b) Approximate the interval in which $f > g$.

(c) Describe the behavior of each of the functions as $x$ approaches $\pi$. How is the behavior of $g$ related to the behavior of $f$ as $x$ approaches $\pi$?

82. **HOW DO YOU SEE IT?** The normal monthly high temperatures for Erie, Pennsylvania are approximated by

$$H(t) = 56.94 - 20.86 \cos \frac{\pi t}{6} - 11.58 \sin \frac{\pi t}{6}$$

and the normal monthly low temperatures for Erie, Pennsylvania are approximated by

$$L(t) = 41.80 - 17.13 \cos \frac{\pi t}{6} - 13.39 \sin \frac{\pi t}{6}$$

where $t$ is the time (in months), with $t = 1$ corresponding to January. *(Source: National Climatic Data Center)*

(a) During what part of the year is the difference between the normal high and low temperatures greatest? When is it smallest?

(b) The sun is the farthest north in the sky around June 21, but the graph shows the highest temperatures at a later date. Approximate the lag time of the temperatures relative to the position of the sun.

**True or False?** In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

83. The amplitude of $f(x) = -3 \cos 2x$ is $-3$.

84. The period of $f(x) = 5 \cot \left(\frac{-4x}{3}\right)$ is $\frac{3\pi}{2}$.

85. $\lim_{x \to 0} \frac{\sin 5x}{3x} = \frac{5}{3}$

86. One solution of $\tan x = 1$ is $\frac{5\pi}{4}$. 
Quiz Yourself

Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, express the angle in radian measure as a multiple of \( \pi \). Use a calculator to verify your result.

1. \( 15^\circ \)  
2. \( 105^\circ \)  
3. \( -80^\circ \)  
4. \( 35^\circ \)

In Exercises 5–8, express the angle in degree measure. Use a calculator to verify your result.

5. \( \frac{2\pi}{3} \)  
6. \( \frac{4\pi}{15} \)  
7. \( -\frac{4\pi}{3} \)  
8. \( \frac{11\pi}{12} \)

In Exercises 9–14, evaluate the trigonometric function without using a calculator.

9. \( \sin\left(-\frac{\pi}{4}\right) \)  
10. \( \cos 240^\circ \)  
11. \( \tan \frac{5\pi}{6} \)  
12. \( \cot 45^\circ \)  
13. \( \sec(-60^\circ) \)  
14. \( \csc \frac{2\pi}{3} \)

In Exercises 15–17, solve the equation for \( \theta \). Assume \( 0 \leq \theta \leq 2\pi \).

15. \( \tan \theta - 1 = 0 \)  
16. \( \cos^2 \theta - 2 \cos \theta + 1 = 0 \)  
17. \( \sin^2 \theta = 3 \cos^2 \theta \)

In Exercises 18–20, find the indicated side and/or angle.

18.  
19.  
20.  

21. A map maker needs to determine the distance \( d \) across a small lake. The distance from point A to point B is 500 feet and the angle \( \theta \) is 35° (see figure). What is \( d \)?

22. \( y = -3 \sin \frac{3x}{4} \)  
23. \( y = -2 \cos 4x \)  
24. \( y = \tan \frac{\pi x}{3} \)

25. A company that produces snowboards forecasts monthly sales for 1 year to be

\[ S = 53.5 + 40.5 \cos \frac{\pi t}{6} \]

where \( S \) is the sales (in thousands of dollars) and \( t \) is the time (in months), with \( t = 1 \) corresponding to January.

(a) Use a graphing utility to graph \( S \).

(b) Use the graph to determine the months of maximum and minimum sales.
14.4 Derivatives of Trigonometric Functions

- Find derivatives of trigonometric functions.
- Find the relative extrema of trigonometric functions.
- Use derivatives of trigonometric functions to answer questions about real-life situations.

Derivatives of Trigonometric Functions

In Example 4 and Checkpoint 4 in the preceding section, you looked at two important trigonometric limits:

\[ \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \]

and

\[ \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0. \]

These two limits are used in the development of the derivative of the sine function.

\[
\frac{d}{dx}(\sin x) = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}
\]

\[ = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \]

\[ = \lim_{\Delta x \to 0} \left[ \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \right] \]

\[ = \lim_{\Delta x \to 0} \left[ (\cos x) \left( \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left( \frac{1 - \cos \Delta x}{\Delta x} \right) \right] \]

\[ = (\cos x)(1) - (\sin x)(0) \]

\[ = \cos x \]

This differentiation rule is illustrated graphically in Figure 14.31. Note that the slope of the sine curve determines the value of the cosine curve. If \( u \) is a function of \( x \), then the Chain Rule version of this differentiation rule is

\[ \frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}. \]

The Chain Rule versions of the differentiation rules for all six trigonometric functions are listed below. To help you remember these differentiation rules, note that each trigonometric function that begins with a “c” has a negative sign in its derivative.

Derivatives of the Six Basic Trigonometric Functions

\[
\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} \quad \frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}
\]

\[
\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx} \quad \frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}
\]

\[
\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx} \quad \frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}
\]
Example 1  Differentiating Trigonometric Functions

Differentiate each function.

\[ \text{a. } y = \sin 2x \quad \text{b. } y = \cos(x - 1) \quad \text{c. } y = \tan 3x \]

SOLUTION

\[ \text{a. Letting } u = 2x, \text{ you obtain } u' = 2, \text{ and the derivative is } \]
\[ \frac{dy}{dx} = \cos u \frac{du}{dx} = \cos 2x \frac{d}{dx}[2x] = (\cos 2x)(2) = 2 \cos 2x. \]

\[ \text{b. Letting } u = x - 1, \text{ you can see that } u' = 1. \text{ So, the derivative is } \]
\[ \frac{dy}{dx} = -\sin u \frac{du}{dx} = -\sin(x - 1) \frac{d}{dx}[x - 1] = -\sin(x - 1)(1) = -\sin(x - 1). \]

\[ \text{c. Letting } u = 3x, \text{ you have } u' = 3, \text{ which implies that } \]
\[ \frac{dy}{dx} = \sec^2 u \frac{du}{dx} = \sec^2 3x \frac{d}{dx}[3x] = (\sec^2 3x)(3) = 3 \sec^2 3x. \]

Example 2  Differentiating a Trigonometric Function

Differentiate the function

\[ f(x) = \cos 3x^2. \]

SOLUTION  Letting \( u = 3x^2, \) you obtain

\[ f'(x) = -\sin u \frac{du}{dx} \quad \text{Apply Cosine Differentiation Rule.} \]
\[ = -\sin 3x^2 \frac{d}{dx}[3x^2] \quad \text{Substitute } 3x^2 \text{ for } u. \]
\[ = -6x \sin 3x^2. \quad \text{Simplify.} \]

Example 3  Differentiating a Trigonometric Function

Differentiate the function

\[ f(x) = \tan^4 x. \]

SOLUTION  Begin by rewriting the function.

\[ f(x) = \tan^4 x \quad \text{Write original function.} \]
\[ = (\tan x)^4 \quad \text{Rewrite.} \]
\[ f'(x) = 4(\tan x)^3 \frac{d}{dx}[\tan x] \quad \text{Apply Power Rule.} \]
\[ = 4 \tan^3 x \sec^2 x \quad \text{Apply Tangent Differentiation Rule.} \]

Checkpoints 1, 2, and 3

Differentiate each function.

\[ \text{a. } y = \cos 4x \quad \text{b. } y = \sin(x^2 - 1) \quad \text{c. } y = \tan \frac{x}{2} \]
\[ \text{d. } y = 2 \cos x^3 \quad \text{e. } y = \sin^3 x \]
**Example 4** Differentiating a Trigonometric Function

Differentiate \( y = \csc \frac{x}{2} \).

**SOLUTION**

\[
y = \csc \frac{x}{2}
\]

\[
\frac{dy}{dx} = -\csc \frac{x}{2} \cot \frac{x}{2} \left( \frac{1}{2} \right) \frac{d}{dx} \left( \frac{x}{2} \right)
\]

\[
= -\frac{1}{2} \csc \frac{x}{2} \cot \frac{x}{2}
\]

**Checkpoint 4**

Differentiate each function.

a. \( y = \sec 4x \)  
   b. \( y = \cot x^2 \)

**STUDY TIP**

Notice that all of the differentiation rules that you learned in earlier chapters in the text can be applied to trigonometric functions. For instance, Example 5 uses the General Power Rule and Example 6 uses the Product Rule.

**Example 5** Differentiating a Trigonometric Function

Differentiate \( f(t) = \sqrt{\sin 4t} \).

**SOLUTION** Begin by rewriting the function in rational exponent form. Then apply the General Power Rule to find the derivative.

\[
f(t) = (\sin 4t)^{1/2}
\]

\[
f'(t) = \frac{1}{2} (\sin 4t)^{-1/2} \frac{d}{dt} (\sin 4t)
\]

\[
= \frac{1}{2} (\sin 4t)^{-1/2} (4 \cos 4t)
\]

\[
= \frac{2 \cos 4t}{\sqrt{\sin 4t}}
\]

**Checkpoint 5**

Differentiate each function.

a. \( f(x) = \sqrt{\cos 2x} \)  
   b. \( f(x) = \sqrt[6]{3x} \)

**Example 6** Differentiating a Trigonometric Function

Differentiate \( y = x \sin x \).

**SOLUTION** Using the Product Rule, you can write

\[
y = x \sin x
\]

\[
\frac{dy}{dx} = \frac{d}{dx} [x \sin x] + \sin x \frac{d}{dx} [x]
\]

\[
= x \cos x + \sin x.
\]

**Checkpoint 6**

Differentiate each function.

a. \( y = x^2 \cos x \)  
   b. \( y = t \sin 2t \)
Relative Extrema of Trigonometric Functions

Recall that the critical numbers of a function \( y = f(x) \) are the \( x \)-values for which \( f'(x) = 0 \) or \( f'(x) \) is undefined.

**Example 7** Finding Relative Extrema

Find the relative extrema of \( y = \frac{x}{2} - \sin x \) in the interval \((0, 2\pi)\).

**SOLUTION** To find the relative extrema of the function, begin by finding its critical numbers. The derivative of \( y \) is

\[
\frac{dy}{dx} = \frac{1}{2} - \cos x.
\]

By setting the derivative equal to zero, you obtain \( \cos x = \frac{1}{2} \). So, in the interval \((0, 2\pi)\), the critical numbers are \( x = \pi/3 \) and \( x = 5\pi/3 \). Using the First-Derivative Test, you can conclude that \( \pi/3 \) yields a relative minimum and \( 5\pi/3 \) yields a relative maximum, as shown in Figure 14.32.

**Checkpoint 7** Find the relative extrema of \( y = \frac{x}{2} - \cos x \) in the interval \((0, 2\pi)\).

**Example 8** Finding Relative Extrema

Find the relative extrema of \( f(x) = 2 \sin x - \cos 2x \) in the interval \((0, 2\pi)\).

**SOLUTION**

\[
\begin{align*}
f(x) &= 2 \sin x - \cos 2x & \text{Write original function.} \\
f'(x) &= 2 \cos x + 2 \sin 2x & \text{Differentiate.} \\
0 &= 2 \cos x + 2 \sin 2x & \text{Set derivative equal to 0.} \\
0 &= 2 \cos x + 4 \cos x \sin x & \text{Identity: } \sin 2x = 2 \cos x \sin x \\
0 &= 2(\cos x)(1 + 2 \sin x) & \text{Factor.}
\end{align*}
\]

From this, you can see that the critical numbers occur when \( \cos x = 0 \) and when \( \sin x = -\frac{1}{2} \). So, in the interval \((0, 2\pi)\), the critical numbers are

\[
x = \frac{\pi}{2}, \frac{7\pi}{6} \quad \frac{3\pi}{2} \quad \frac{11\pi}{6}.
\]

Using the First-Derivative Test, you can determine that \((\pi/2, 3)\) and \((3\pi/2, -1)\) are relative maxima, and \((7\pi/6, -\frac{1}{2})\) and \((11\pi/6, -\frac{1}{2})\) are relative minima, as shown in Figure 14.33.

**Checkpoint 8** Find the relative extrema of \( y = \frac{1}{2} \sin 2x + \cos x \) on the interval \((0, 2\pi)\).
Applications

Example 9  Modeling Seasonal Sales

A fertilizer manufacturer finds that the sales of one of its fertilizer brands follow a seasonal pattern that can be modeled by

\[ F = 100,000 \left[ 1 + \sin \frac{2\pi(t - 60)}{365} \right], \quad t \geq 0 \]

where \( F \) is the amount sold (in pounds) and \( t \) is the time (in days), with \( t = 1 \) corresponding to January 1. On which day of the year is the maximum amount of fertilizer sold?

**SOLUTION**  The derivative of the model is

\[ \frac{dF}{dt} = 100,000 \left( \frac{2\pi}{365} \right) \cos \frac{2\pi(t - 60)}{365}. \]

Setting this derivative equal to zero produces

\[ \cos \frac{2\pi(t - 60)}{365} = 0. \]

Because cosine is zero at \( \pi/2 \) and \( 3\pi/2 \), you can find the critical numbers as shown.

\[
\begin{align*}
\frac{2\pi(t - 60)}{365} &= \frac{\pi}{2} & \frac{2\pi(t - 60)}{365} &= \frac{3\pi}{2} \\
2\pi(t - 60) &= \frac{365\pi}{2} & 2\pi(t - 60) &= \frac{(365)(3\pi)}{2} \\
t - 60 &= \frac{365}{4} & t - 60 &= \frac{3(365)}{4} \\
t &= \frac{365}{4} + 60 & t &= \frac{3(365)}{4} + 60 \\
&= 151 & t &= 334
\end{align*}
\]

The 151st day of the year is May 31 and the 334th day of the year is November 30. From Figure 14.34, you can see that, according to the model, the maximum sales occur on May 31.

**FIGURE 14.34**

Checkpoint 9  
Using the model from Example 9, find the rate at which sales are changing when \( t = 59 \). 

---

**TECH TUTOR**

Because of the difficulty of solving some trigonometric equations, it can be difficult to find the critical numbers of a trigonometric function. For instance, consider the function

\[ f(x) = 2 \sin x - \cos 3x. \]

Setting the derivative of this function equal to zero produces

\[ 2 \cos x + 3 \sin 3x = 0. \]

This equation is difficult to solve analytically. So, it is difficult to find the relative extrema of \( f \) analytically. With a graphing utility, however, you can estimate the relative extrema graphically using the \textit{zoom} and \textit{trace} features. You can also try other approximation techniques, such as Newton’s Method, which is discussed in Section 15.8.
Example 10  Modeling Temperature Change

The temperature \( T \) (in degrees Fahrenheit) during a given 24-hour period can be modeled by

\[
T = 70 + 15 \sin \frac{\pi(t - 8)}{12}, \quad t \geq 0
\]

where \( t \) is the time (in hours), with \( t = 0 \) corresponding to midnight, as shown in Figure 14.35. Find the rate at which the temperature is changing at 6 A.M.

Solution  The rate of change of the temperature is given by the derivative

\[
\frac{dT}{dt} = 15 \pi \left( \frac{\cos \frac{\pi(t - 8)}{12}}{12} \right)\cos \frac{\pi(t - 8)}{12}.
\]

Because 6 A.M. corresponds to \( t = 6 \), the rate of change at 6 A.M. is

\[
15 \pi \left( \frac{\cos \left( \frac{2\pi}{12} \right)}{12} \right) = 5 \pi \left( \frac{\cos \left( -\frac{\pi}{6} \right)}{4} \right)
\]

\[
= \frac{5\pi}{4} \left( \frac{\sqrt{3}}{2} \right)
\]

\[= 3.4^\circ \text{ per hour}.\]

Checkpoint 10

In Example 10, find the rate at which the temperature is changing at 8 P.M.

SUMMARIZE  (Section 14.4)

1. State the Sine, Tangent, and Secant Differentiation Rules (page 1028). For examples of using these rules, see Examples 1, 3, 5, and 6.

2. State the Cosine, Cotangent, and Cosecant Differentiation Rules (page 1028). For examples of using these rules, see Examples 1, 2, and 4.

3. Explain how to find the relative extrema of a trigonometric function (page 1031). For examples of finding the relative extrema of trigonometric functions, see Examples 7 and 8.

4. Describe a real-life example of how the derivative of a trigonometric function can be used to analyze the rate of change of temperature (page 1033, Example 10).
**SKILLS WARM UP 14.4**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 7.4, 7.6, 7.7, 8.5, and 14.2.

In Exercises 1–4, find the derivative of the function.

1. \( f(x) = 3x^3 - 2x^2 + 4x - 7 \)
2. \( g(x) = (x^3 + 4)^4 \)
3. \( f(x) = (x - 1)(x^2 + 2x + 3) \)
4. \( g(x) = \frac{2x}{x^2 + 5} \)

In Exercises 5 and 6, find the relative extrema of the function.

5. \( f(x) = x^2 + 4x + 1 \)
6. \( f(x) = \frac{1}{3}x^3 - 4x + 2 \)

In Exercises 7–10, solve the equation for \( x \). Assume \( 0 \leq x \leq 2\pi \).

7. \( \sin x = \frac{\sqrt{3}}{2} \)
8. \( \cos x = -\frac{1}{2} \)
9. \( \cos \frac{x}{2} = 0 \)
10. \( \sin \frac{x}{2} = -\frac{\sqrt{2}}{2} \)

**Exercises 14.4**


**Differentiating Trigonometric Functions** In Exercises 1–28, find the derivative of the trigonometric function. See Examples 1, 2, 3, 4, 5, and 6.

1. \( y = \sin \frac{x}{3} \)
2. \( f(x) = \cos 2x \)
3. \( y = \tan 4x \)
4. \( y = \sin(3x + 1) \)
5. \( f(t) = \tan 5t \)
6. \( g(x) = 3 \cos x^4 \)
7. \( y = \sin^2 x \)
8. \( y = \cos^4 x \)
9. \( y = \sec \pi x \)
10. \( y = \frac{1}{2} \csc 2x \)
11. \( y = \cot(2x + 1) \)
12. \( f(t) = \cos \frac{2}{t} \)
13. \( y = \sqrt{\tan 2x} \)
14. \( y = \sqrt[3]{\sin 6x} \)
15. \( f(t) = t^2 \cos t \)
16. \( y = (x + 3) \csc x \)
17. \( g(t) = \frac{\cos t}{t} \)
18. \( f(x) = \frac{\sin x}{x} \)
19. \( y = e^x \sec x \)
20. \( y = e^{-2x} \cot x \)
21. \( y = \cos 3x + \sin^3 x \)
22. \( y = \csc^2 x - \sec 3x \)
23. \( y = x \sin \frac{1}{x} \)
24. \( y = x \cos \frac{1}{x} \)
25. \( y = 2 \tan^2 4x \)
26. \( y = -\sin^4 2x \)
27. \( y = e^{2x} \sin 2x \)
28. \( y = e^{-x} \cos x^2 \)

**Finding an Equation of a Tangent Line** In Exercises 29–48, find an equation of the tangent line to the graph of the function at the given point.

<table>
<thead>
<tr>
<th>Function</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>31. ( y = \cos^2 x - \sin^2 x )</td>
<td>32. ( y = \frac{x}{2} + \frac{\sin 2x}{2} )</td>
</tr>
<tr>
<td>33. ( y = \sin^2 x - \cos 2x )</td>
<td>34. ( y = 3 \sin x - 2 \sin^3 x )</td>
</tr>
<tr>
<td>35. ( y = \tan x - x )</td>
<td>36. ( y = \cot x + x )</td>
</tr>
<tr>
<td>37. ( y = \frac{\sin^3 x}{3} - \frac{\sin^2 x}{5} )</td>
<td>38. ( y = \frac{\sec^2 x - \sec^3 x}{2} )</td>
</tr>
<tr>
<td>39. ( y = \ln(\sin^2 x) )</td>
<td>40. ( y = \frac{1}{2} \ln(\cos^2 x) )</td>
</tr>
</tbody>
</table>

**Differentiating Trigonometric Functions** In Exercises 29–40, find the derivative of the function and simplify your answer by using the trigonometric identities listed in Section 14.2.

29. \( y = \cos^2 x \)
30. \( y = \frac{1}{4} \sin^2 2x \)
31. \( y = \frac{x}{2} + \frac{\sin 2x}{4} \)
32. \( y = \frac{x}{2} + \frac{\sin 2x}{4} \)
33. \( y = \sin^2 x - \cos 2x \)
34. \( y = 3 \sin x - 2 \sin^3 x \)
35. \( y = \tan x - x \)
36. \( y = \cot x + x \)
37. \( y = \frac{\sin^3 x}{3} - \frac{\sin^2 x}{5} \)
38. \( y = \frac{\sec^2 x - \sec^3 x}{2} \)
39. \( y = \ln(\sin^2 x) \)
40. \( y = \frac{1}{2} \ln(\cos^2 x) \)
Implicit Differentiation In Exercises 49 and 50, use implicit differentiation to find $dy/dx$ and evaluate the derivative at the given point.

<table>
<thead>
<tr>
<th>Function</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$49. \sin x + \cos 2y = 1$</td>
<td>$(\pi \quad \pi /4)$</td>
</tr>
<tr>
<td>$50. \tan(x + y) = x$</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

Slope of a Tangent Line In Exercises 51–56, find the slope of the tangent line to the given sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$.

<table>
<thead>
<tr>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>51. $y = \sin \frac{5\pi x}{4}$</td>
</tr>
<tr>
<td>52. $y = \sin \frac{5\pi x}{2}$</td>
</tr>
<tr>
<td>53. $y = \sin 2x$</td>
</tr>
<tr>
<td>54. $y = \sin \frac{3\pi x}{2}$</td>
</tr>
<tr>
<td>55. $y = \sin x$</td>
</tr>
<tr>
<td>56. $y = \sin \frac{\pi x}{2}$</td>
</tr>
</tbody>
</table>

Finding Relative Extrema In Exercises 57–66, find the relative extrema of the trigonometric function in the interval $(0, 2\pi)$. Use a graphing utility to confirm your results. See Examples 7 and 8.

<table>
<thead>
<tr>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>57. $y = \cos \frac{\pi x}{2}$</td>
</tr>
<tr>
<td>58. $y = -3 \sin \frac{x}{3}$</td>
</tr>
<tr>
<td>59. $y = \cos^3 x$</td>
</tr>
<tr>
<td>60. $y = \sec \frac{x}{2}$</td>
</tr>
<tr>
<td>61. $y = 2 \sin x + \sin 2x$</td>
</tr>
<tr>
<td>62. $y = 2 \cos x + \cos 2x$</td>
</tr>
<tr>
<td>63. $y = e^x \cos x$</td>
</tr>
<tr>
<td>64. $y = e^{-x} \sin x$</td>
</tr>
<tr>
<td>65. $y = x - 2 \sin x$</td>
</tr>
<tr>
<td>66. $y = \sin^2 x + \cos x$</td>
</tr>
</tbody>
</table>

Finding Second Derivatives In Exercises 67–70, find the second derivative of the trigonometric function.

<table>
<thead>
<tr>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>67. $y = x \sin x$</td>
</tr>
<tr>
<td>68. $y = \sec x$</td>
</tr>
<tr>
<td>69. $y = \cos x^2$</td>
</tr>
<tr>
<td>70. $y = \csc^2 \pi x$</td>
</tr>
</tbody>
</table>

71. Biology Plants do not grow at constant rates during a normal 24-hour period because their growth is affected by sunlight. Suppose that the growth of a certain plant species in a controlled environment is given by the model

$$h = 0.2t + 0.03 \sin 2\pi t$$

where $h$ is the height of the plant (in inches) and $t$ is the time (in days), with $t = 0$ corresponding to midnight of day 1. During what time of day is the rate of growth of this plant (a) a maximum? (b) a minimum?

72. Meteorology The normal average daily temperature in degrees Fahrenheit for a city is given by

$$T = 55 - \frac{2\pi(\pi - 32)}{365}$$

where $t$ is the time (in days), with $t = 1$ corresponding to January 1. Find the expected date of (a) the warmest day. (b) the coldest day.

73. Construction Workers The numbers $W$ (in thousands) of construction workers employed in the United States during 2010 can be modeled by

$$W = 5488 + 347.6 \sin(0.45t + 4.153)$$

where $t$ is the time (in months), with $t = 1$ corresponding to January. Approximate the month $t$ in which the number of construction workers employed was a maximum. What was the maximum number of construction workers employed? (Source: U.S. Bureau of Labor Statistics)

74. Transportation Workers The numbers $W$ (in thousands) of scenic and sightseeing transportation workers employed in the United States during 2010 can be modeled by

$$W = 27.8 + 8.25 \sin(0.561t - 2.5913)$$

where $t$ is the time (in months), with $t = 1$ corresponding to January. Approximate the month $t$ in which the number of scenic and sightseeing transportation workers employed was a maximum. What was the maximum number of scenic and sightseeing transportation workers employed? (Source: U.S. Bureau of Labor Statistics)

75. Meteorology The number of hours of daylight $D$ in New Orleans can be modeled by

$$D = 12.12 + 1.87 \cos \frac{\pi(t + 5.83)}{6}$$

where $t$ is the time (in months), with $t = 1$ corresponding to January. Approximate the month $t$ in which New Orleans has the maximum number of daylight hours. What is this maximum number of daylight hours? (Source: U.S. Naval Observatory)
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76. Tides Throughout the day, the depth of water $D$ (in meters) at the end of a dock varies with the tides. The depth for one particular day can be modeled by

$$D = 3.5 + 1.5 \cos \frac{\pi t}{6}, \quad 0 \leq t \leq 24$$

where $t$ is the time (in hours), with $t = 0$ corresponding to midnight.

(a) Determine $dD/dt$.
(b) Evaluate $dD/dt$ for $t = 4$ and $t = 20$, and interpret your results.
(c) Find the time(s) when the water depth is the greatest and the time(s) when the water depth is the least.
(d) What is the greatest depth? What is the least depth? Did you have to use calculus to determine these depths? Explain your reasoning.

77. Think About It Rewrite the trigonometric function in terms of sine and/or cosine and then differentiate to prove the following differentiation rules.

(a) \( \frac{d}{dx} [\tan x] = \sec^2 x \)
(b) \( \frac{d}{dx} [\sec x] = \sec x \tan x \)
(c) \( \frac{d}{dx} [\cot x] = -\csc^2 x \)
(d) \( \frac{d}{dx} [\csc x] = -\csc x \cot x \)

78. HOW DO YOU SEE IT? The graph shows the height $h$ (in feet) above ground of a seat on a Ferris wheel at time $t$ (in seconds).

(a) What is the period of the model? What does the period tell you about the ride?
(b) Find the intervals on which the height is increasing and decreasing.
(c) Estimate the relative extrema in the interval $[0, 60]$. 

The graph shows the height $h$ (in feet) above ground of a seat on a Ferris wheel at time $t$ (in seconds).

(a) What is the period of the model? What does the period tell you about the ride?
(b) Find the intervals on which the height is increasing and decreasing.
(c) Estimate the relative extrema in the interval $[0, 60]$. 

79. Project: Meteorology For a project analyzing the mean monthly temperature and precipitation in Sioux City, Iowa, visit this text’s website at www.cengagebrain.com. (Source: National Oceanic and Atmospheric Administration)
14.5 Integrals of Trigonometric Functions

- Learn the trigonometric integration rules that correspond directly to differentiation rules.
- Integrate the six basic trigonometric functions.
- Use trigonometric integrals to solve real-life problems.

Trigonometric Integrals
For each trigonometric differentiation rule, there is a corresponding integration rule. For instance, corresponding to the differentiation rule

$$\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$$

is the integration rule

$$\int \cos u \, du = \sin u + C$$

and corresponding to the differentiation rule

$$\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$$

is the integration rule

$$\int \sin u \, du = -\cos u + C.$$
Example 1  Integrating a Trigonometric Function

Find \( \int 2 \cos x \, dx \).

**SOLUTION**  Let \( u = x \). Then \( du = dx \).

\[
\int 2 \cos x \, dx = 2 \int \cos x \, dx
\]

Apply Constant Multiple Rule.

\[
= 2 \int \cos u \, du
\]

Substitute for \( x \) and \( dx \).

\[
= 2 \sin u + C
\]

Integrate.

\[
= 2 \sin x + C
\]

Substitute for \( u \).

Example 2  Integrating a Trigonometric Function

Find \( \int 3x^2 \sin x^3 \, dx \).

**SOLUTION**  Let \( u = x^3 \). Then \( du = 3x^2 \, dx \).

\[
\int 3x^2 \sin x^3 \, dx = \int (\sin x^3)3x^2 \, dx
\]

Rewrite integrand.

\[
= \int \sin u \, du
\]

Substitute for \( x^3 \) and \( 3x^2 \, dx \).

\[
= -\cos u + C
\]

Integrate.

\[
= -\cos x^3 + C
\]

Substitute for \( u \).

**Checkpoint 1 and 2**

Find (a) \( \int 5 \sin x \, dx \) and (b) \( \int 4x \cos x^4 \, dx \).

Example 3  Integrating a Trigonometric Function

Find \( \int \sec 3x \tan 3x \, dx \).

**SOLUTION**  Let \( u = 3x \). Then \( du = 3 \, dx \).

\[
\int \sec 3x \tan 3x \, dx = \frac{1}{3} \int (\sec 3x \tan 3x)3 \, dx
\]

Multiply and divide by 3.

\[
= \frac{1}{3} \int \sec u \tan u \, du
\]

Substitute for \( 3x \) and \( 3 \, dx \).

\[
= \frac{1}{3} \sec u + C
\]

Integrate.

\[
= \frac{1}{3} \sec 3x + C
\]

Substitute for \( u \).

**Checkpoint 3**

Find \( \int \sec^2 5x \, dx \).
Section 14.5  Integrals of Trigonometric Functions

Example 4  Integrating a Trigonometric Function

Find \( \int e^x \sec^2 e^x \, dx \).

**SOLUTION**  Let \( u = e^x \). Then \( du = e^x \, dx \).

\[
\int e^x \sec^2 e^x \, dx = \int \sec^2 u \, du
\]

Rewrite integrand.

\[
= \tan u + C
\]

Substitute for \( u \).

\[
= \tan e^x + C
\]

Integrate.

Checkpoint 4

Find \( \int 2 \csc 2x \cot 2x \, dx \).

The next two examples use the General Power Rule for integration and the General Log Rule for integration. Recall from Chapter 11 that these rules are

<table>
<thead>
<tr>
<th>General Power Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int u^n , du = \frac{u^{n+1}}{n+1} + C ),  ( n \neq -1 )</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>General Log Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int \frac{du}{u} , dx = \ln</td>
</tr>
</tbody>
</table>

The key to using these two rules is identifying the proper substitution for \( u \).

Example 5  Using the General Power Rule

Find \( \int \sin^2 4x \cos 4x \, dx \).

**SOLUTION**  Let \( u = \sin 4x \). Then \( du = 4 \cos 4x \, dx \).

\[
\int \sin^2 4x \cos 4x \, dx = \frac{1}{4} \int (\sin 4x)^2 (4 \cos 4x) \, dx
\]

Rewrite integrand.

\[
= \frac{1}{4} \int u^2 \, du
\]

Substitute for \( \sin 4x \) and \( 4 \cos 4x \, dx \).

\[
= \frac{1}{4} \frac{u^3}{3} + C
\]

Integrate.

\[
= \frac{1}{4} \frac{(\sin 4x)^3}{3} + C
\]

Substitute for \( u \).

\[
= \frac{1}{12} \sin^3 4x + C
\]

Simplify.

Checkpoint 5

Find \( \int \cos^3 2x \sin 2x \, dx \).
Example 6 Using the Log Rule

Find \( \int \frac{\sin x}{\cos x} \, dx \).

**SOLUTION** Let \( u = \cos x \). Then \( du/dx = -\sin x \).

\[
\int \frac{\sin x}{\cos x} \, dx = - \int \frac{\sin x}{\cos x} \, dx = - \int \frac{du}{u} \, dx
\]

Rewrite integrand.

Substitute for \( \cos x \) and \( -\sin x \).

\[
= - \ln |u| + C
\]

Apply Log Rule.

Substitute for \( u \).

\[
= - \ln |\cos x| + C
\]

**Checkpoint 6**

Find \( \int \frac{\cos x}{\sin x} \, dx \).

Example 7 Evaluating a Definite Integral

Evaluate \( \int_{0}^{\pi/4} \cos 2x \, dx \).

**SOLUTION**

\[
\int_{0}^{\pi/4} \cos 2x \, dx = \left[ \frac{1}{2} \sin 2x \right]_{0}^{\pi/4} = \frac{1}{2} - 0 = \frac{1}{2}
\]

**Checkpoint 7**

Find \( \int_{0}^{\pi/2} \sin 2x \, dx \).

Example 8 Finding Area by Integration

Find the area of the region bounded by the \( x \)-axis and the graph of \( y = \sin x \) for \( 0 \leq x \leq \pi \).

**SOLUTION** As indicated in Figure 14.36, this area is given by

\[
\text{Area} = \int_{0}^{\pi} \sin x \, dx = \left[ -\cos x \right]_{0}^{\pi} = -(-1) - (-1) = 2.
\]

So, the region has an area of 2 square units.

**Checkpoint 8**

Find the area of the region bounded by the graphs of \( y = \cos x \) and \( y = 0 \) for \( 0 \leq x \leq \frac{\pi}{2} \).
Integrals of the Six Basic Trigonometric Functions

At the beginning of this section, the integration rules for the sine and cosine functions were listed. Now, using the result of Example 6, you have an integration rule for the tangent function. That rule is

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| + C. \]

Integration formulas for the other three trigonometric functions can be developed in a similar way. For instance, to obtain the integration formula for the secant function, you can integrate as shown.

\[
\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{du}{u} \, dx \quad \text{Substitute: } u = \sec x + \tan x.
\]

\[ = \ln |u| + C \quad \text{Apply Log Rule.} \]

\[ = \ln |\sec x + \tan x| + C \quad \text{Substitute for } u. \]

The integrals of the six basic trigonometric functions are summarized below.

### Integrals of the Six Basic Trigonometric Functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin u , du )</td>
<td>( -\cos u + C )</td>
</tr>
<tr>
<td>( \cos u , du )</td>
<td>( \sin u + C )</td>
</tr>
<tr>
<td>( \tan u , du )</td>
<td>( -\ln</td>
</tr>
<tr>
<td>( \sec u , du )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( \cot u , du )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( \csc u , du )</td>
<td>( \ln</td>
</tr>
</tbody>
</table>

**Example 9** Integrating a Trigonometric Function

Find \( \int \tan 4x \, dx \).

**SOLUTION** Let \( u = 4x \). Then \( du = 4 \, dx \).

\[
\int \tan 4x \, dx = \frac{1}{4} \int (\tan 4x) \, 4 \, dx = \frac{1}{4} \int \tan u \, du \quad \text{Rewrite integrand.}
\]

\[
= \frac{1}{4} \ln|\cos u| + C \quad \text{Substitute for } 4x \text{ and } 4 \, dx.
\]

\[
= \frac{1}{4} \ln|\cos 4x| + C \quad \text{Integrate.}
\]

\[
\int \sec 2x \, dx. \quad \text{Substitute for } u.
\]

**Checkpoint 9**

Find \( \int \sec 2x \, dx \).
Application

In the next example, recall from Section 11.4 that the average value of a function \( f \) over an interval \([a, b]\) is given by

\[
\frac{1}{b - a} \int_a^b f(x) \, dx.
\]

**Example 10** Finding an Average Temperature

The temperature \( T \) (in degrees Fahrenheit) during a 24-hour period can be modeled by

\[
T = 72 + 18 \sin \frac{\pi(t - 8)}{12}
\]

where \( t \) is the time (in hours), with \( t = 0 \) corresponding to midnight. Will the average temperature during the four-hour period from noon to 4 P.M. be greater than 85°F?

**SOLUTION** To find the average temperature \( \bar{A} \), use the formula for the average value of a function on an interval.

\[
\bar{A} = \frac{1}{16} \int_0^8 \left( 72 + 18 \sin \frac{\pi(t - 8)}{12} \right) \, dt
\]

\[
= \frac{1}{4} \left[ 72t + 18 \left( \frac{12}{\pi} \right) \left( -\cos \frac{\pi(t - 8)}{12} \right) \right]_0^8
\]

\[
= \frac{1}{4} \left[ 72(16) + 18 \left( \frac{12}{\pi} \right) \left( \frac{1}{2} \right) - 72(0) + 18 \left( \frac{12}{\pi} \right) \left( \frac{1}{2} \right) \right]
\]

\[
= \frac{1}{4} \left( 288 + \frac{216}{\pi} \right)
\]

\[
= 72 + \frac{54}{\pi}
\]

\[
= 89.2^\circ
\]

So, the average temperature is about 89.2°F, as indicated in Figure 14.37. You can conclude that the average temperature from noon to 4 P.M. will be greater than 85°F.

**Checkpoint 10**

Use the function in Example 10 to find the average temperature from 9 A.M. to noon.

**SUMMARIZE** (Section 14.5)

1. For each trigonometric integration rule on page 1037, state the corresponding differentiation rule (page 1037). For examples of using these integration rules, see Examples 1–8.

2. State the integration rules for the six basic trigonometric functions (page 1041). For examples of using these integration rules, see Examples 1, 2, 5, 7, 8, and 9.

3. Describe a real-life example of how the integral of a trigonometric function can be used to find an average temperature (page 1042, Example 10).
Section 14.5 ■ Integrals of Trigonometric Functions

**SKILLS WARM UP 14.5**

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 11.4 and 14.2.

In Exercises 1–8, evaluate the trigonometric function.

1. \[ \cos \frac{5\pi}{4} \]
2. \[ \sin \frac{7\pi}{6} \]
3. \[ \sin \left( -\frac{\pi}{3} \right) \]
4. \[ \cos \left( -\frac{\pi}{6} \right) \]
5. \[ \tan \frac{5\pi}{6} \]
6. \[ \cot \frac{5\pi}{3} \]
7. \[ \sec \pi \]
8. \[ \cos \frac{\pi}{2} \]

In Exercises 9–16, simplify the expression using the trigonometric identities.

9. \[ \sin x \sec x \]
10. \[ \csc x \cos x \]
11. \[ \cos^2 x (\sec^2 x - 1) \]
12. \[ \sin^2 x (\csc^2 x - 1) \]
13. \[ \sec x \sin \left( \frac{\pi}{2} - x \right) \]
14. \[ \cot x \cos \left( \frac{\pi}{2} - x \right) \]
15. \[ \cot x \sec x \]
16. \[ \cot x (\sin^2 x) \]

In Exercises 17–20, evaluate the definite integral.

17. \[ \int_0^1 (x^2 + 3x - 4) \, dx \]
18. \[ \int_{-1}^1 (1 - x^2) \, dx \]
19. \[ \int_0^2 (4 - x^2) \, dx \]
20. \[ \int_0^1 x(9 - x^2) \, dx \]

---

**Exercises 8.5**


**Integrating Trigonometric Functions** In Exercises 1–32, find the indefinite integral. See Examples 1, 2, 3, 4, 5, 6, and 9.

1. \[ \int 4 \sin x \, dx \]
2. \[ \int 8 \cos x \, dx \]
3. \[ \int \sin 2x \, dx \]
4. \[ \int \cos 6x \, dx \]
5. \[ \int 4x^3 \cos x^4 \, dx \]
6. \[ \int 2x \sin x^2 \, dx \]
7. \[ \int \sec^2 \frac{x^5}{5} \, dx \]
8. \[ \int \csc^4 4x \, dx \]
9. \[ \int \sec 2x \tan 2x \, dx \]
10. \[ \int \csc \frac{x}{3} \cot \frac{x}{3} \, dx \]
11. \[ \int \tan^3 x \sec^2 x \, dx \]
12. \[ \int \sqrt{\cot x} \csc^2 x \, dx \]
13. \[ \int \sec^2 x \, dx \]
14. \[ \int \csc^2 x \cot x \, dx \]
15. \[ \int \frac{\sec x \tan x}{\sec x - 1} \, dx \]
16. \[ \int \frac{\cos t}{1 + \sin t} \, dt \]
17. \[ \int \frac{\sin x}{1 + \cos x} \, dx \]
18. \[ \int \frac{1 - \cos \theta}{\theta - \sin \theta} \, d\theta \]
19. \[ \int \cot \pi x \, dx \]
20. \[ \int \tan 5x \, dx \]
21. \[ \int \csc 2x \, dx \]
22. \[ \int \sec \frac{x}{2} \, dx \]
23. \[ \int \sec^4 \frac{x}{4} \, dx \]
24. \[ \int \csc^3 5x \cot 5x \, dx \]
25. \[ \int \csc^2 x \cot^3 x \, dx \]
26. \[ \int \frac{\sin x}{\cos^2 x} \, dx \]
27. \[ \int e^t \sin e^t \, dt \]
28. \[ \int e^{-t} \tan e^{-t} \, dx \]
29. \[ \int e^{\sin x} \cos \pi x \, dx \]
30. \[ \int e^{e^{4x}} \sec 4x \, dx \]
31. \[ \int (\sin x + \cos x)^2 \, dx \]
32. \[ \int (1 + \tan x)^2 \, dx \]
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Integration by Parts In Exercises 33–38, use integration by parts to find the indefinite integral.

33. \( \int x \cos 2x \, dx \)  
34. \( \int x \sin 5x \, dx \)

35. \( \int 6x \sec^2 x \, dx \)  
36. \( \int \frac{\theta}{3} \sec \theta \tan \theta \, d\theta \)

37. \( \int x \csc 3t \cot 3t \, dt \)  
38. \( \int 4x \csc^2 x \, dx \)

Evaluating Definite Integrals In Exercises 39–46, evaluate the definite integral. See Example 7.

39. \( \int_{0}^{\pi/4} \cos \frac{4x}{3} \, dx \)  
40. \( \int_{0}^{\pi/6} \sin 6x \, dx \)

41. \( \int_{\pi/12}^{\pi/3} \sec \frac{x}{2} \, dx \)  
42. \( \int_{0}^{\pi/2} (x + \cos x) \, dx \)

43. \( \int_{\pi/12}^{\pi/4} \csc 2x \cot 2x \, dx \)  
44. \( \int_{0}^{\pi/8} \sin 2x \cos 2x \, dx \)

45. \( \int_{0}^{1} \tan(1 - x) \, dx \)  
46. \( \int_{0}^{\pi/4} \sec x \tan x \, dx \)

Finding Area by Integration In Exercises 47–52, find the area of the region. See Example 8.

47. \( y = \cos \frac{x}{4} \)  
48. \( y = \tan x \)

49. \( y = x + \sin x \)  
50. \( y = \frac{x}{2} + \cos x \)

51. \( y = \sin x + \cos 2x \)  
52. \( y = 2 \sin x + \sin 2x \)

Finding the Area Bounded by Two Graphs In Exercises 53–56, sketch the region bounded by the graphs of the functions and find the area of the region.

53. \( y = \cos x, y = 2 - \cos x, x = 0, x = 2\pi \)

54. \( y = \sin x, y = \cos 2x, x = -\frac{\pi}{2}, x = \frac{\pi}{6} \)

55. \( y = 2 \sin x, y = \tan x, x = 0, x = \frac{\pi}{3} \)

56. \( y = \sec^2 \frac{x}{4}, y = 4 - x^2, x = -1, x = 1 \)

57. Meteorology The average monthly precipitation \( P \) (in inches), including rain, snow, and ice, for Sacramento, California can be modeled by

\[ P = 2.47 \sin(0.40t + 1.80) + 2.08 \]  
where \( t \) is the time (in months), with \( t = 1 \) corresponding to January. Find the total annual precipitation for Sacramento.  
(Source: National Climatic Data Center)

58. Meteorology The average monthly precipitation \( P \) (in inches), including rain, snow, and ice, for Bismarck, North Dakota can be modeled by

\[ P = 1.07 \sin(0.59t + 3.94) + 1.52 \]  
where \( t \) is the time (in months), with \( t = 1 \) corresponding to January. Find the total annual precipitation for Bismarck.  
(Source: National Climatic Data Center)

59. Consumer Trends Energy consumption in the United States is seasonal. The primary residential energy consumption \( Q \) (in trillion Btu) in the United States during 2009 can be modeled by

\[ Q = 936 + 737.3 \cos(0.31t + 0.928) \]  
where \( t \) is the time (in months), with \( t = 1 \) corresponding to January. Find the average primary residential energy consumption during

(a) the first quarter \( (0 \leq t \leq 3) \).

(b) the fourth quarter \( (9 \leq t \leq 12) \).

(c) the entire year \( (0 \leq t \leq 12) \).  
(Source: U.S. Energy Information Administration)

60. Construction Workers The number \( W \) (in thousands) of construction workers employed in the United States during 2010 can be modeled by

\[ W = 5488 + 347.6 \sin(0.45t - 4.153) \]  
where \( t \) is the time (in months), with \( t = 1 \) corresponding to January. Find the average number of construction workers during

(a) the first quarter \( (0 \leq t \leq 3) \).

(b) the second quarter \( (3 \leq t \leq 6) \).

(c) the entire year \( (0 \leq t \leq 12) \).  
(Source: U.S. Bureau of Labor Statistics)
61. **Cost** The temperature \( T \) (in degrees Fahrenheit) in a house is given by
\[
T = 72 + 12 \sin \left( \frac{\pi(t - 8)}{12} \right)
\]
where \( t \) is the time (in hours), with \( t = 0 \) corresponding to midnight. The hourly cost of cooling a house is $0.30 per degree.

(a) Find the cost \( C \) of cooling this house between 8 A.M. and 8 P.M., when the thermostat is set at 72°F (see figure) by evaluating the integral
\[
C = 0.3 \int_{8}^{20} \left[ 72 + 12 \sin \left( \frac{\pi(t - 8)}{12} \right) - 72 \right] dt.
\]

(b) Find the savings realized by resetting the thermostat to 78°F (see figure) by evaluating the integral
\[
C = 0.3 \int_{10}^{18} \left[ 72 + 12 \sin \left( \frac{\pi(t - 8)}{12} \right) - 78 \right] dt.
\]

62. **Water Supply** The flow rate \( R \) (in thousands of gallons per hour) of water at a pumping station during a day can be modeled by
\[
R = 53 + 7 \sin \left( \frac{\pi t}{6} + 3.6 \right) + 9 \cos \left( \frac{\pi t}{12} + 8.9 \right),
\]
where \( t \) is the time in hours, with \( t = 0 \) corresponding to midnight.

(a) Find the average hourly flow rate from midnight to noon (0 \( \leq t \leq 12 \)).

(b) Find the average hourly flow rate from noon to midnight (12 \( \leq t \leq 24 \)).

(c) Find the total volume of water pumped in one day.

63. **Sales** In Example 9 in Section 14.4, the sales of a seasonal product can be modeled by
\[
F = 100,000 \left[ 1 + \sin \left( \frac{\pi(t - 60)}{365} \right) \right], \quad t \geq 0
\]
where \( F \) is the amount sold (in pounds) and \( t \) is the time (in days), with \( t = 1 \) corresponding to January 1. The manufacturer of this product wants to set up a manufacturing schedule to produce a uniform amount each day. What should this amount be? (Assume that there are 200 production days during the year.)

64. **HOW DO YOU SEE IT?** The graph shows the sales \( S \) (in thousands of units) of a seasonal product, where \( t \) is the time (in months), with \( t = 1 \) corresponding to January.

(a) Which is greater, the average monthly sales from January through March, or the average monthly sales from October through December? Explain your reasoning.

(b) Estimate the average monthly sales for the entire year. Explain your reasoning.

**Using Simpson's Rule** In Exercises 65 and 66, use the Simpson's Rule program in Appendix I to approximate the integral.

\[
\text{Integral} \quad \int_{a}^{b} f(x) \, dx
\]

65. \( \int_{0}^{\pi} \sqrt{x} \sin x \, dx \) 8

66. \( \int_{0}^{\pi} \sqrt{1 + \cos^2 x} \, dx \) 20

**True or False?** In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

67. \( \int_{a}^{b} \sin x \, dx = \int_{a}^{b+2\pi} \sin x \, dx \)

68. \( \int \sin x \cos x \, dx = 0 \)
Solving Trigonometric Equations

Solving a trigonometric equation requires the use of trigonometry, but it also requires the use of algebra. Some examples of solving trigonometric equations were presented on pages 1012 and 1013. Here are several others.

**Example 1** Solving Trigonometric Equations

Solve each trigonometric equation. Assume \(0 \leq x \leq 2\pi\).

a. \(\sin x + \sqrt{2} = -\sin x\)

b. \(3 \tan^2 x = 1\)

c. \(\cot x \cos^2 x = 2 \cot x\)

**SOLUTION**

**a.**

Write original equation.
\[
\sin x + \sqrt{2} = -\sin x
\]
Subtract \(\sqrt{2}\) from each side.
\[
\sin x = -\sin x - \sqrt{2}
\]
Add \(\sin x\) to each side.
\[
2 \sin x = -\sqrt{2}
\]
Combine like terms.
\[
\sin x = -\frac{\sqrt{2}}{2}
\]
Divide each side by 2.

\[
x = \frac{5\pi}{4}, \frac{7\pi}{4}, \quad 0 \leq x \leq 2\pi
\]

**b.**

Write original equation.
\[
3 \tan^2 x = 1
\]
Divide each side by 3.
\[
\tan^2 x = \frac{1}{3}
\]
Extract square roots.
\[
\tan x = \pm \frac{\sqrt{3}}{3}, \quad 0 \leq x \leq 2\pi
\]

**c.**

Write original equation.
\[
\cot x \cos^2 x = 2 \cot x
\]
Subtract \(2 \cot x\) from each side.
\[
\cot x (\cos^2 x - 2) = 0
\]
Factor.

Setting each factor equal to zero, you obtain the solutions in the interval \(0 \leq x \leq 2\pi\) as shown.

\[
\cot x = 0 \quad \text{and} \quad \cos^2 x - 2 = 0
\]
\[
x = \frac{\pi}{2}, \frac{3\pi}{2}, \quad \cos^2 x = 2
\]
\[
\cos x = \pm \sqrt{2}
\]

No solution is obtained from \(\cos x = \pm \sqrt{2}\) because \(\pm \sqrt{2}\) are outside the range of the cosine function. So, the equation has two solutions

\[
x = \frac{\pi}{2} \quad \text{and} \quad x = \frac{3\pi}{2}
\]
in the interval \(0 \leq x \leq 2\pi\).
Example 2  Solving Trigonometric Equations

Solve each trigonometric equation in the interval $[0, 2\pi]$.

a. $2\sin^2 x - \sin x - 1 = 0$

b. $2\sin^2 x + 3\cos x - 3 = 0$

c. $\sin t - \cos 2t = 0$

SOLUTION

a. $2\sin^2 x - \sin x - 1 = 0$

Write original equation.

$(2\sin x + 1)(\sin x - 1) = 0$

Factor.

Set each factor equal to zero. The solutions in the interval $[0, 2\pi]$ are

$2\sin x + 1 = 0$ and $\sin x - 1 = 0$

$2\sin x = -1$ and $\sin x = 1$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$ and $x = \frac{\pi}{2}$.

b. $2\sin^2 x + 3\cos x - 3 = 0$

Write original equation.

$2(1 - \cos^2 x) + 3\cos x - 3 = 0$

Pythagorean Identity

$2 - 2\cos^2 x + 3\cos x - 3 = 0$

Multiply.

$-2\cos^2 x + 3\cos x - 1 = 0$

Combine like terms.

$2\cos^2 x - 3\cos x + 1 = 0$

Multiply each side by $-1$.

$(2\cos x - 1)(\cos x - 1) = 0$

Factor.

Set each factor equal to zero. The solutions in the interval $[0, 2\pi]$ are

$2\cos x - 1 = 0$ and $\cos x - 1 = 0$

$2\cos x = 1$ and $\cos x = 1$

$\cos x = \frac{1}{2}$ and $\cos x = 1$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$c. \sin t - \cos 2t = 0$

Write original equation.

$\sin t - (1 - 2\sin^2 t) = 0$

Double-Angle Identity

$\sin t - 1 + 2\sin^2 t = 0$

Remove parentheses.

$2\sin^2 t + \sin t - 1 = 0$

Rewrite.

$(2\sin t - 1)(\sin t + 1) = 0$

Factor.

Set each factor equal to zero. The solutions in the interval $[0, 2\pi]$ are

$2\sin t - 1 = 0$ and $\sin t + 1 = 0$

$2\sin t = 1$ and $\sin t = -1$

$\sin t = \frac{1}{2}$ and $\sin t = -1$

$t = \frac{\pi}{6}, \frac{5\pi}{6}$
After studying this chapter, you should have acquired the following skills. The exercise numbers are keyed to the Review Exercises that begin on page 1050. Answers to odd-numbered Review Exercises are given in the back of the text.*

### Section 14.1
- Find coterminal angles.
- Convert from degree to radian measure and from radian to degree measure. \( \pi \text{ radians} = 180^\circ \)
- Use formulas relating to triangles.
- Use formulas relating to triangles to solve real-life problems.

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### Section 14.2
- Evaluate trigonometric functions.

**Right Triangle Definition:** \( 0 < \theta < \frac{\pi}{2} \)

\[
\begin{align*}
\sin \theta &= \frac{\text{opp}}{\text{hyp}} \\
\cos \theta &= \frac{\text{adj}}{\text{hyp}} \\
\tan \theta &= \frac{\text{opp}}{\text{adj}} \\
\csc \theta &= \frac{\text{hyp}}{\text{opp}} \\
\sec \theta &= \frac{\text{hyp}}{\text{adj}} \\
\end{align*}
\]

**Circular Function Definition:** Let \( \theta \) be an angle in standard position with \((x, y)\) a point on the terminal ray of \( \theta \) and \( r = \sqrt{x^2 + y^2} \neq 0 \).

\[
\begin{align*}
\sin \theta &= \frac{y}{r} \\
\csc \theta &= \frac{r}{y} \\
\cos \theta &= \frac{x}{r} \\
\sec \theta &= \frac{r}{x} \\
\tan \theta &= \frac{y}{x} \\
\cot \theta &= \frac{x}{y} \\
\end{align*}
\]

- Use a calculator to approximate values of trigonometric functions.
- Solve right triangles.
- Solve trigonometric equations.
- Use right triangles to solve real-life problems.

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### Section 14.3
- Find the period and amplitude of trigonometric functions.
- Sketch graphs of trigonometric functions.
- Use trigonometric functions to model real-life situations.

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*A wide range of valuable study aids are available to help you master the material in this chapter. The Student Solutions Manual includes step-by-step solutions to all odd-numbered exercises to help you review and prepare. The student website at www.cengagebrain.com offers algebra help and a Graphing Technology Guide, which contains step-by-step commands and instructions for a wide variety of graphing calculators.*
Section 14.4

Review Exercises

67–84

Find derivatives of trigonometric functions.

\[
\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx} \quad \frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}
\]

\[
\frac{d}{dx}[\tan u] = \sec^2 u \frac{du}{dx} \quad \frac{d}{dx}[\cot u] = -\csc^2 u \frac{du}{dx}
\]

\[
\frac{d}{dx}[\sec u] = \sec u \tan u \frac{du}{dx} \quad \frac{d}{dx}[\csc u] = -\csc u \cot u \frac{du}{dx}
\]

Find the equations of tangent lines to graphs of trigonometric functions.

85–90

Find the relative extrema of trigonometric functions.

91–96

Use derivatives of trigonometric functions to answer questions about real-life situations.

97, 98

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Section 14.5

Find indefinite integrals of trigonometric functions.

99–110

\[
\int \cos u \, du = \sin u + C \quad \int \sin u \, du = -\cos u + C
\]

\[
\int \sec^2 u \, du = \tan u + C \quad \int \sec u \tan u \, du = \sec u + C
\]

\[
\int \csc^2 u \, du = -\cot u + C \quad \int \csc u \cot u \, du = -\csc u + C
\]

\[
\int \tan u \, du = -\ln|\cos u| + C \quad \int \sec u \, du = \ln|\sec u + \tan u| + C
\]

\[
\int \cot u \, du = \ln|\sin u| + C \quad \int \csc u \, du = \ln|\csc u - \cot u| + C
\]

Evaluate definite integrals of trigonometric functions.

111–118

Find the areas of regions in the plane.

119–122

Use trigonometric integrals to solve real-life problems.

123–126

---

Study Strategies

Degree and Radian Modes  When using a computer or calculator to evaluate or graph a trigonometric function, be sure that you use the proper mode—radian mode or degree mode.

Checking the Form of an Answer  Because of the abundance of trigonometric identities, solutions of problems in this chapter can take a variety of forms. For instance, the expressions \(-\ln|\cot x| + C\) and \(\ln|\tan x| + C\) are equivalent. So, when you are checking your solutions with those given in the back of the text, remember that your solution might be correct, even if its form doesn’t agree precisely with that given in the text.

Using Technology  Throughout this chapter, remember that technology can help you graph trigonometric functions, evaluate trigonometric functions, differentiate trigonometric functions, and integrate trigonometric functions. Consider, for instance, the difficulty of sketching the graph of the function below without using a graphing utility.
Review Exercises

Finding Coterminal Angles  In Exercises 1–4, determine two coterminal angles (one positive and one negative) for each angle. Give the answers in degrees.

1. $\theta = 390^\circ$
2. $\theta = 230^\circ$
3. $\theta = -405^\circ$
4. $\theta = -210^\circ$

Converting from Degrees to Radians  In Exercises 5–12, express the angle in radian measure as a multiple of $\pi$. Use a calculator to verify your result.

5. $340^\circ$
6. $300^\circ$
7. $-60^\circ$
8. $-30^\circ$
9. $-480^\circ$
10. $-540^\circ$
11. $110^\circ$
12. $320^\circ$

Converting from Radians to Degrees  In Exercises 13–16, express the angle in degree measure. Use a calculator to verify your result.

13. $\frac{4\pi}{3}$
14. $\frac{5\pi}{6}$
15. $-\frac{2\pi}{3}$
16. $-\frac{11\pi}{6}$

Using Triangles  In Exercises 17–20, solve the triangle for the indicated side and/or angle.

17. $\theta = 30^\circ$
18. $\theta = 60^\circ$
19. $\theta = 540^\circ$
20. $\theta = 300^\circ$

21. Height  A ladder of length 16 feet leans against the side of a house. The bottom of the ladder is 4.4 feet from the house (see figure). Find the height $h$ of the top of the ladder.

22. Length  To stabilize a 75-foot tower for a radio antenna, a guy wire must be attached from the top of the tower to an anchor 50 feet from the base (see figure). How long is the wire?

Evaluating Trigonometric Functions  In Exercises 23 and 24, determine all six trigonometric functions of the angle $\theta$.

23. $\theta = \theta$
24. $\theta = \theta$
Evaluating Trigonometric Functions In Exercises 25–32, evaluate the six trigonometric functions of the angle without using a calculator.

25. \(-45°\)     26. \(240°\)
27. \(\frac{5\pi}{3}\)     28. \(\frac{4\pi}{3}\)
29. \(-225°\)     30. \(180°\)
31. \(-\frac{11\pi}{6}\)     32. \(\frac{5\pi}{2}\)

Evaluating Trigonometric Functions In Exercises 33–40, use a calculator to evaluate the trigonometric function to four decimal places.

33. \(\tan 33°\)     34. \(\cot 216°\)
35. \(\sec \frac{12\pi}{5}\)     36. \(\csc \frac{2\pi}{9}\)
37. \(\sin \left(-\frac{\pi}{9}\right)\)     38. \(\cos \left(-\frac{3\pi}{7}\right)\)
39. \(\cos 105°\)     40. \(\sin 224°\)

Solving a Right Triangle In Exercises 41–44, solve for \(x\), \(y\), or \(r\) as indicated.

41. Solve for \(r\).

42. Solve for \(y\).

43. Solve for \(x\).

44. Solve for \(r\).

Solving Trigonometric Equations In Exercises 45–50, solve the equation for \(\theta\). Assume \(0 \leq \theta \leq 2\pi\). For some of the equations, you should use the trigonometric identities listed in Section 14.2. Use the trace feature of a graphing utility to verify your results.

45. \(2\cos \theta + 1 = 0\)     46. \(2\cos^2 \theta = 1\)
47. \(2\sin^2 \theta + 3\sin \theta + 1 = 0\)
48. \(\cos^3 \theta = \cos \theta\)
49. \(\sec^2 \theta - \sec \theta - 2 = 0\)
50. \(2\sec^2 \theta + \tan^2 \theta - 3 = 0\)

51. Height The length of the shadow of a tree is 125 feet when the angle of elevation of the sun is \(33°\) (see figure). Approximate the height \(h\) of the tree.

52. Length A guy wire runs from the ground to a cell tower. The wire is attached to the cell tower 150 feet above the ground. The angle formed between the wire and the ground is \(43°\) (see figure).

(a) How long is the guy wire?
(b) How far from the base of the tower is the guy wire anchored to the ground?

Finding the Period and Amplitude In Exercises 53–56, find the period and amplitude of the trigonometric function.

53. \(y = -2\sin 4x\)     54. \(y = \cos 2x\)
55. \(y = -2\cos \frac{3x}{2}\)     56. \(y = \sin \frac{x}{2}\)
Graphing Trigonometric Functions  In Exercises 57–64, sketch the graph of the trigonometric function by hand.

57. \( y = 2 \cos 6x \)  
58. \( y = \sin 2\pi x \)  
59. \( y = \frac{1}{3} \tan x \)  
60. \( y = \cot \frac{x}{2} \)  
61. \( y = 3 \sin \frac{2x}{5} \)  
62. \( y = 8 \cos \frac{x}{4} \)  
63. \( y = \sec 2\pi x \)  
64. \( y = 3 \csc 2x \)

Seasonal Sales  The jet ski sales \( S \) (in units) of a company are modeled by

\[ S = 74 - 40 \cos \frac{\pi t}{6} \]

where \( t \) is the time (in months), with \( t = 1 \) corresponding to January.

(a) Use a graphing utility to graph \( S \).
(b) Will the sales exceed 110 units during any month? If so, during which month(s)?

Seasonal Sales  The bathing suit sales \( S \) (in thousands of units) of a company are modeled by

\[ S = 25 + 20 \sin \frac{\pi t}{6} \]

where \( t \) is the time (in months), with \( t = 1 \) corresponding to January.

(a) Use a graphing utility to graph \( S \).
(b) Will the sales exceed 42,000 units during any month? If so, during which month(s)?

Differentiating Trigonometric Functions  In Exercises 67–84, find the derivative of the trigonometric function.

67. \( y = \sin 5\pi x \)  
68. \( y = \cos \frac{x}{4} \)  
69. \( y = \tan 3x^3 \)  
70. \( y = \sec 3x \)  
71. \( y = \cot \frac{3x^2}{5} \)  
72. \( y = \csc(3x + 4) \)  
73. \( y = \sqrt{\cos 2\pi x} \)  
74. \( y = \sqrt[3]{\csc 5x} \)  
75. \( y = -x \tan x \)  
76. \( y = \csc 3x + \cot 3x \)  
77. \( y = \frac{\cos x}{x^2} \)  
78. \( y = \frac{\sin(2x - 1)}{x + 3} \)  
79. \( y = \sin^2 x + x \)

Finding an Equation of a Tangent Line  In Exercises 85–90, find an equation of the tangent line to the graph of the function at the given point.

85. \( y = \cos 2x \)  
86. \( y = \tan 2x \)  
87. \( y = \csc x \)  
88. \( y = \sin 2x \)  
89. \( y = \frac{1}{2} \sin^2 x \)  
90. \( y = -x \cos x \)

Finding Relative Extrema  In Exercises 91–96, find the relative extrema of the trigonometric function in the interval \((0, 2\pi)\). Use a graphing utility to confirm your results.

91. \( y = \sin \frac{\pi x}{4} \)  
92. \( y = \cos \frac{3x}{2} \)  
93. \( f(x) = \frac{x}{2} + \cos x \)  
94. \( f(x) = \sin x \cos x \)  
95. \( f(x) = \sin^2 x + \sin x \)  
96. \( f(x) = \frac{1}{2 + \sin x} \)

Seasonal Sales  Refer to the model given in Exercise 65. Approximate the month \( t \) in which the sales of jet skis were a maximum. What was the maximum number of jet skis sold?

Seasonal Sales  Refer to the model given in Exercise 66.

(a) Approximate the month \( t \) in which the sales of bathing suits were a maximum. What was the maximum number of bathing suits sold?
(b) Approximate the month \( t \) in which the sales of bathing suits were a minimum. What was the minimum number of bathing suits sold?
**Integrating Trigonometric Functions** In Exercises 99–110, find the indefinite integral.

99. $\int \sin 3x \, dx$

100. $\int \cos \frac{x}{4} \, dx$

101. $\int \csc 5x \cot 5x \, dx$

102. $\int 2x \sec^2 x^2 \, dx$

103. $\int x \csc^2 x^2 \, dx$

104. $\int \sec 8x \tan 8x \, dx$

105. $\int \sec^2 2x \tan 2x \, dx$

106. $\int \cos x \sin^3 x \, dx$

107. $\int \tan 3x \, dx$

108. $\int \csc^2 \frac{3x}{4} \, dx$

109. $\int \sin^3 x \cos x \, dx$

110. $\int e^{\cos 3x} \sin 3x \, dx$

**Evaluating Definite Integrals** In Exercises 111–118, evaluate the definite integral.

111. $\int_0^\pi (1 + \sin x) \, dx$

112. $\int_{-\pi/2}^{\pi/2} (1 + \cos 2x) \, dx$

113. $\int_{-\pi/6}^{\pi/6} \sec^2 x \, dx$

114. $\int_{-\pi/6}^{\pi/6} \csc^2 x \, dx$

115. $\int_{-\pi/3}^{\pi/3} 4 \sec x \tan x \, dx$

116. $\int_{-\pi/6}^{\pi/6} \csc x \cot x \, dx$

117. $\int_{-\pi/2}^{\pi/2} (2x + \cos x) \, dx$

118. $\int_0^\pi 2x \sin x^2 \, dx$

**Finding Area by Integration** In Exercises 119–122, find the area of the region.

119. $y = \sin 3x$

120. $y = \cot x$

121. $y = 2 \sin x + \cos 3x$

122. $y = 2 \cos x + \cos 2x$

**Meteorology** The average monthly precipitation $P$ (in inches), including rain, snow, and ice, for Dodge City, Kansas can be modeled by

$$P = 1.27 \sin(0.58t - 2.05) + 1.98, \ 0 \leq t \leq 12$$

where $t$ is the time (in months), with $t = 1$ corresponding to January. Find the total annual precipitation for Dodge City. (Source: National Climatic Data Center)

**Health** For a person at rest, the velocity $v$ (in liters per second) of air flow into and out of the lungs during a respiratory cycle is given by

$$v = 0.9 \sin \frac{\pi t}{3}$$

where $t$ is the time (in seconds). Find the volume in liters of air inhaled during one cycle by integrating this function over the interval $[0, 3]$.

**Sales** The sales $S$ (in billions of dollars) for Lowe’s for the years 2000 through 2009 can be modeled by

$$S = 15.31 \sin(0.37t - 1.27) + 33.66, \ 0 \leq t \leq 9$$

where $t$ is the year, with $t = 0$ corresponding to 2000. Find the average sales for Lowe’s from 2000 through 2009. (Source: Lowe’s Companies, Inc.)

**Electricity** The oscillating current in an electrical circuit can be modeled by

$$I = 2 \sin(60\pi t) + \cos(120\pi t)$$

where $I$ is measured in amperes and $t$ is measured in seconds. Find the average current for the time intervals (a) $0 \leq t \leq \frac{\pi}{30}$, (b) $0 \leq t \leq \frac{1}{60}$, and (c) $0 \leq t \leq \frac{1}{30}$.
Take this test as you would take a test in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–6, copy and complete the table. Use a calculator if necessary.

<table>
<thead>
<tr>
<th>Function</th>
<th>θ (deg)</th>
<th>θ (rad)</th>
<th>Function Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. sin</td>
<td>67.5°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. cos</td>
<td></td>
<td>π/5</td>
<td></td>
</tr>
<tr>
<td>3. tan</td>
<td>15°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. cot</td>
<td></td>
<td>−π/6</td>
<td></td>
</tr>
<tr>
<td>5. sec</td>
<td>−40°</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. csc</td>
<td></td>
<td>−5π/4</td>
<td></td>
</tr>
</tbody>
</table>

7. A digital camera tripod has a height of 25 inches, and an angle of 24° is formed between the vertical and the leg of length ℓ (see figure). What is ℓ?

In Exercises 8–10, solve the equation for θ. Assume 0 ≤ θ ≤ 2π.

8. 2 sin θ − √2 = 0
9. cos² θ − sin² θ = 0
10. csc θ = √3 sec θ

In Exercises 11–13, sketch the graph of the trigonometric function by hand.

11. y = 3 sin 2x
12. y = 4 cos 3πx
13. y = cot πx/5

In Exercises 14–16, (a) find the derivative of the trigonometric function and (b) find the relative extrema of the trigonometric function in the interval (0, 2π).

14. y = cos x − cos³ x
15. y = sec (x − π/4)
16. y = 1/³ [3 − sin(x + π)]

In Exercises 17–19, find the indefinite integral.

17. ∫ sin 5x dx
18. ∫ sec³ x/4 dx
19. ∫ x csc x² dx

In Exercises 20–23, evaluate the definite integral.

20. ∫₁/² cos πx dx
21. ∫₀ π sec² x/3 tan x/3 dx
22. ∫₅π/₆ csc 2x cot 2x dx
23. ∫₅π/₄ sin 4x cos 4x dx

24. The monthly sales $S$ (in thousands of dollars) of a company that produces insect repellent can be modeled by

$S = 20.3 − 17.2 \cos \frac{\pi t}{6}$

where $t$ is the time (in months), with $t = 1$ corresponding to January.

(a) Find the total sales during the year (0 ≤ t ≤ 12).
(b) Find the average monthly sales from April through October (3 ≤ t ≤ 10).