8.8 Linear Programming

In this section, we will learn to

1. Solve linear programming problems.
2. Solve applications of linear programming.

Linear programming is a mathematical technique used to find the optimal allocation of resources in the military, business, telecommunications, and other fields. It got its start during World War II when it became necessary to move huge quantities of people, materials, and supplies as efficiently and economically as possible.

A simple linear programming problem might involve a television program director who wants to schedule comedy skits and musical acts such as LeAnn Rimes for a prime-time variety show. Of course, the director wants to do this in a way that earns the maximum possible income for her network. Such an example is provided in Example 5 in this section.

1. Solve Linear Programming Problems

To solve linear programming problems, we must maximize (or minimize) a function (called the objective function) subject to given restrictions on its variables. These restrictions (called constraints) are usually given as a system of linear inequalities. For example, suppose that the annual profit (in millions of dollars) earned by a business is given by the equation \( P = y + 2x \) and that \( x \) and \( y \) are subject to the following constraints:

\[
\begin{align*}
3x + y &\leq 120 \\
x + y &\leq 60 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]
To find the maximum profit $P$ that can be earned by the business, we solve the system of inequalities as shown in Figure 8-16(a) and find the coordinates of each corner point of the region $R$. This region is often called a **feasibility region**. We can then write the profit equation

$$P = y + 2x$$

in the form

$$y = -2x + P$$

The equation $y = -2x + P$ is the equation of a set of parallel lines, each with a slope of $-2$ and a $y$-intercept of $P$. The graph of $y = -2x + P$ for three values of $P$ is shown as red lines in Figure 8-16(b). To find the red line that passes through region $R$ and provides the maximum value of $P$, we locate the red line with the greatest $y$-intercept. Since line $l$ has the greatest $y$-intercept and intersects region $R$ at the corner point $(30, 30)$, the maximum value of $P$ (subject to the given constraints) is

$$P = y + 2x = 30 + 2(30) = 90$$

Thus, the maximum profit $P$ that can be earned is $90$ million. This profit occurs when $x = 30$ and $y = 30$.

![Graph of linear inequalities and profit equation](image)

**FIGURE 8-16**

The preceding discussion illustrates the following important fact.

<table>
<thead>
<tr>
<th>Maximum or Minimum of an Objective Function</th>
</tr>
</thead>
</table>

If a linear function, subject to the constraints of a system of linear inequalities in two variables, attains a maximum or a minimum value, that value will occur at a corner point or along an entire edge of the region $R$ that represents the solution of the system.

**EXAMPLE 1** Finding the Maximum Value of an Objective Function Given Certain Constraints

If $P = 2x + 3y$, find the maximum value of $P$ subject to the following constraints:

$$
\begin{align*}
\begin{cases}
x + y & \leq 4 \\
2x + y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{cases}
\end{align*}
$$
SOLUTION We solve the system of inequalities to find the feasibility region $R$ shown in Figure 8-17. The coordinates of its corner points are $(0, 0), (3, 0), (0, 4),$ and $(2, 2).

$$
\begin{align*}
\text{Point} & & P = 2x + 3y \\
(0, 0) & & P = 2(0) + 3(0) = 0 \\
(3, 0) & & P = 2(3) + 3(0) = 6 \\
(2, 2) & & P = 2(2) + 3(2) = 10 \\
(0, 4) & & P = 2(0) + 3(4) = 12 \\
\end{align*}
$$

The maximum value $P = 12$ occurs when $x = 0$ and $y = 4$.

Self Check 1 Find the maximum value of $P = 4x + 3y$ subject to the constraints of Example 1. Now Try Exercise 7.

EXAMPLE 2 Finding the Minimum Value of an Objective Function Given Certain Constraints

If $P = 3x + 2y$, find the minimum value of $P$ subject to the following constraints:

$$
\begin{align*}
& x + y \geq 1 \\
& x - y \leq 1 \\
& x - y \geq 0 \\
& x \leq 2
\end{align*}
$$

SOLUTION We refer to the feasibility region shown in Figure 8-18 with corner points at $\left(\frac{1}{2}, \frac{1}{2}\right)$, $(2, 2), (2, 1),$ and $(1, 0)$.
Since the minimum value of $P$ occurs at a corner point of region $R$, we substitute the coordinates of each corner point into the objective function $P = 3x + 2y$ and find the one that gives the minimum value of $P$.

<table>
<thead>
<tr>
<th>Point</th>
<th>$P = 3x + 2y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2)$</td>
<td>$P = 3(1) + 2(2) = 7$</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$P = 3(2) + 2(2) = 10$</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$P = 3(2) + 2(1) = 8$</td>
</tr>
<tr>
<td>$(1, 0)$</td>
<td>$P = 3(1) + 2(0) = 3$</td>
</tr>
</tbody>
</table>

The minimum value $P = \frac{7}{2}$ occurs when $x = \frac{1}{2}$ and $y = \frac{1}{2}$.

**Self Check 2**  
Find the minimum value of $P = 2x + y$ subject to the constraints of Example 2.

**Now Try Exercise 15.**

### 2. Solve Applications of Linear Programming

Linear programming problems can be very complex and involve hundreds of variables. In this section, we will consider only a few simple problems. Since they involve only two variables, we can solve them using graphical methods.

To solve a linear programming problem, we will follow these steps.

<table>
<thead>
<tr>
<th>Strategy for Solving Linear Programming Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Find the objective function and constraints.</td>
</tr>
<tr>
<td>2. Find the feasibility region by graphing the system of inequalities and identifying the coordinates of its corner points.</td>
</tr>
<tr>
<td>3. Find the maximum (or minimum) value by substituting the coordinates of the corner points into the objective function.</td>
</tr>
</tbody>
</table>

**EXAMPLE 3**  
Solving an Application Problem

An accountant prepares tax returns for individuals and for small businesses. On average, each individual return requires 3 hours of her time and 1 hour of computer time. Each business return requires 4 hours of her time and 2 hours of computer time. Because of other business considerations, her time is limited to 240 hours, and the computer time is limited to 100 hours. If she earns a profit of $80 on each individual return and a profit of $150 on each business return, how many returns of each type should she prepare to maximize her profit?

**SOLUTION**  
First, we organize the given information into a table.

<table>
<thead>
<tr>
<th>Individual tax return</th>
<th>Business tax return</th>
<th>Time available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accountant’s time</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Computer time</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Profit</td>
<td>$80</td>
<td>$150</td>
</tr>
</tbody>
</table>

Then we solve the problem using the following steps.

Find the objective function and constraints  
Suppose that $x$ represents the number of individual returns to be completed and $y$ represents the number of business
returns to be completed. Since each of the \( x \) individual returns will earn an $80 profit and each of the \( y \) business returns will earn a $150 profit, the total profit is given by the equation

\[
P = 80x + 150y
\]

Since the number of individual returns and business returns cannot be negative, we know that \( x \geq 0 \) and \( y \geq 0 \).

Since each of the \( x \) individual returns will take 3 hours of her time and each of the \( y \) business returns will take 4 hours of her time, the total number of hours she will work will be \((3x + 4y)\) hours. This amount must be less than or equal to her available time, which is 240 hours. Thus, the inequality \( 3x + 4y \leq 240 \) is a constraint on the accountant’s time.

Since each of the \( x \) individual returns will take 1 hour of computer time and each of the \( y \) business returns will take 2 hours of computer time, the total number of hours of computer time will be \((x + 2y)\) hours. This amount must be less than or equal to the available computer time, which is 100 hours. Thus, the inequality \( x + 2y \leq 100 \) is a constraint on the computer time.

We have the following constraints on the values of \( x \) and \( y \):

\[
\begin{align*}
x &\geq 0 & \text{The number of individual returns is nonnegative.} \\
y &\geq 0 & \text{The number of business returns is nonnegative.} \\
3x + 4y &\leq 240 & \text{The accountant’s time must be less than or equal to 240 hours.} \\
x + 2y &\leq 100 & \text{The computer time must be less than or equal to 100 hours.}
\end{align*}
\]

**Find the feasibility region** To find the feasibility region, we graph each of the constraints to find region \( R \), as in Figure 8-19. The four corner points of this region have coordinates of \((0, 0)\), \((80, 0)\), \((40, 30)\), and \((0, 50)\).

![FIGURE 8-19](image)

**Find the maximum profit** To find the maximum profit, we substitute the coordinates of each corner point into the objective function \( P = 80x + 150y \).

<table>
<thead>
<tr>
<th>Point</th>
<th>( P = 80x + 150y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>( P = 80(0) + 150(0) = 0 )</td>
</tr>
<tr>
<td>((80, 0))</td>
<td>( P = 80(80) + 150(0) = 6,400 )</td>
</tr>
<tr>
<td>((40, 30))</td>
<td>( P = 80(40) + 150(30) = 7,700 )</td>
</tr>
<tr>
<td>((0, 50))</td>
<td>( P = 80(0) + 150(50) = 7,500 )</td>
</tr>
</tbody>
</table>

From the table, we can see that the accountant will earn a maximum profit of $7,700 if she prepares 40 individual returns and 30 business returns.

**Self Check 3** In Example 3, if the accountant earns a profit of $100 on each individual return and a profit of $175 on each business return, find the maximum profit.

**Now Try Exercise 21.**
Vigortab and Robust are two diet supplements. Each Vigortab tablet costs 50¢ and contains 3 units of calcium, 20 units of vitamin C, and 40 units of iron. Each Robust tablet costs 60¢ and contains 4 units of calcium, 40 units of vitamin C, and 30 units of iron. At least 24 units of calcium, 200 units of vitamin C, and 120 units of iron are required for the daily needs of one patient. How many tablets of each supplement should be taken daily for a minimum cost? Find the daily minimum cost.

**SOLUTION**

First, we organize the given information into a table.

<table>
<thead>
<tr>
<th>Supplement</th>
<th>Calcium</th>
<th>Vitamin C</th>
<th>Iron</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vigortab</td>
<td>3</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>Robust</td>
<td>4</td>
<td>40</td>
<td>30</td>
</tr>
<tr>
<td>Amount</td>
<td>24</td>
<td>200</td>
<td>120</td>
</tr>
</tbody>
</table>

Cost: 50¢ for Vigortab, 60¢ for Robust.

Find the objective function and constraints

We can let $x$ represent the number of Vigortab tablets to be taken daily and $y$ the corresponding number of Robust tablets. Because each of the $x$ Vigortab tablets will cost 50¢ and each of the $y$ Robust tablets will cost 60¢, the total cost will be given by the equation

$$C = 0.50x + 0.60y$$

Since there are requirements for calcium, vitamin C, and iron, there is a constraint for each. Note that neither $x$ nor $y$ can be negative.

$$\begin{align*}
3x + 4y & \geq 24 & \text{The amount of calcium must be greater than or equal to 24 units.} \\
20x + 40y & \geq 200 & \text{The amount of vitamin C must be greater than or equal to 200 units.} \\
40x + 30y & \geq 120 & \text{The amount of iron must be greater than or equal to 120 units.} \\
x & \geq 0, y & \geq 0 & \text{The number of tablets taken must be greater than or equal to 0.}
\end{align*}$$

Find the feasibility region

We graph the inequalities to find the feasibility region and the coordinates of its corner points, as in Figure 8-20.

![Figure 8-20](image)

Find the minimum cost

In this case, the feasibility region is not bounded on all sides. The coordinates of the corner points are (0, 6), (4, 3), and (10, 0). To find the minimum cost, we substitute each pair of coordinates into the objective function.

<table>
<thead>
<tr>
<th>Point</th>
<th>$C = 0.50x + 0.60y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 6)</td>
<td>$C = 0.50(0) + 0.60(6) = 3.60$</td>
</tr>
<tr>
<td>(4, 3)</td>
<td>$C = 0.50(4) + 0.60(3) = 3.80$</td>
</tr>
<tr>
<td>(10, 0)</td>
<td>$C = 0.50(10) + 0.60(0) = 5.00$</td>
</tr>
</tbody>
</table>

A minimum cost will occur if no Vigortab and 6 Robust tablets are taken daily. The minimum daily cost is $3.60.
Self Check 4  If the cost of each Robust tablet increases to 75¢ and the cost of each Vigortab increases to 80¢, find the minimum cost.

Now Try Exercise 24.

EXAMPLE 5  Solving an Application Problem

A television program director must schedule comedy skits and musical numbers for prime-time variety shows. Each comedy skit requires 2 hours of rehearsal time, costs $3,000, and brings in $20,000 from the show’s sponsors. Each musical number requires 1 hour of rehearsal time, costs $6,000, and generates $12,000. If 250 hours are available for rehearsal and $600,000 is budgeted for comedy and music, how many segments of each type should be produced to maximize income? Find the maximum income.

SOLUTION  First, we organize the given information into a table.

<table>
<thead>
<tr>
<th></th>
<th>Comedy</th>
<th>Musical</th>
<th>Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rehearsal time (hours)</td>
<td>2</td>
<td>1</td>
<td>250</td>
</tr>
<tr>
<td>Cost (in $1,000s)</td>
<td>3</td>
<td>6</td>
<td>600</td>
</tr>
<tr>
<td>Generated income (in $1,000s)</td>
<td>20</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Find the objective function and constraints  We can let $x$ represent the number of comedy skits and $y$ the number of musical numbers to be scheduled. Since each of the $x$ comedy skits generates $20\,000, the income generated by the comedy skits is $20x$ thousand. The musical numbers produce $12y$ thousand. The objective function to be maximized is

$$I = 20x + 12y$$

Since there are limits on rehearsal time and budget, there is a constraint for each. Note that neither $x$ nor $y$ can be negative.

$$
\begin{align*}
2x + y & \leq 250 & \text{The total rehearsal time must be less than or equal to 250 hours.} \\
3x + 6y & \leq 600 & \text{The total cost must be less than or equal to 600 thousand.} \\
x \geq 0, \ y \geq 0 & & \text{The numbers of skits and musical numbers must be greater than or equal to 0.}
\end{align*}
$$

Find the feasibility region  We graph the inequalities to find the feasibility region shown in Figure 8-21 and find the coordinates of each corner point.

Find the maximum income  The coordinates of the corner points of the feasible region are $(0, \ 0), \ (0, \ 100), \ (100, \ 50),$ and $(125, \ 0).$ To find the maximum income, we substitute each pair of coordinates into the objective function.
Maximum income will occur if 100 comedy skits and 50 musical numbers are scheduled. The maximum income will be 2,600 thousand dollars, or $2,600,000.

Self Check 5

If during the following year it is predicted that each comedy skit will generate $30 thousand and each musical number $20 thousand, find the maximum income for the year.

Now Try Exercise 25.

Self Check Answer
1. 14
2. \( \frac{3}{2} \)
3. $9,250
4. $4.80
5. $4,000,000

Exercises 8.8

Getting Ready
You should be able to complete these vocabulary and concept statements before you proceed to the practice exercises.

Fill in the blanks.
1. In a linear program, the inequalities are called ________.
2. Ordered pairs that satisfy the constraints of a linear program are called ________ solutions.
3. The function to be maximized (or minimized) in a linear program is called the ________ function.
4. The objective function of a linear program attains a maximum (or minimum), subject to the constraints, at a _____ or along an _____ of the feasibility region.

Practice
Maximize \( P \) subject to the following constraints.
5. \( P = 2x + 3y \)
   \[
   \begin{align*}
   x &\geq 0 \\
y &\geq 0 \\
x + y &\leq 4
   \end{align*}
   \]

9. \( P = 2x + y \)
   \[
   \begin{align*}
y &\geq 0 \\
y - x &\leq 2 \\
2x + 3y &\leq 6 \\
3x + y &\leq 3
   \end{align*}
   \]

13. \( P = 5x + 12y \)
   \[
   \begin{align*}
x &\geq 0 \\
y &\geq 0 \\
x + y &\leq 4
   \end{align*}
   \]

Minimize \( P \) subject to the following constraints.
10. \( P = x - 2y \)
   \[
   \begin{align*}
x &+ y \leq 5 \\
y &\leq 3 \\
x &\leq 2 \\
y &\geq 0
   \end{align*}
   \]

14. \( P = 3x + 6y \)
   \[
   \begin{align*}
x &\geq 0 \\
y &\geq 0 \\
x + y &\leq 4
   \end{align*}
   \]

15. \( P = 3y + x \)
   \[
   \begin{align*}
x &\geq 0 \\
y &\geq 0 \\
2y - x &\leq 1 \\
y - 2x &\geq -2
   \end{align*}
   \]
17. \( P = 6x + 2y \)  
\[
\begin{align*}
  y &\geq 0 \\
x - y &\leq 2 \\
2x + 3y &\leq 6 \\
3x + y &\leq 3
\end{align*}
\]
18. \( P = 2y - x \)  
\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
x + y &\leq 5 \\
x + 2y &\geq 2
\end{align*}
\]
19. \( P = 2x - 2y \)  
\[
\begin{align*}
x &\leq 1 \\
x &\geq -1 \\
y - x &\leq 1 \\
y - y &\leq 1
\end{align*}
\]
20. \( P = y - 2x \)  
\[
\begin{align*}
x + 2y &\leq 4 \\
2x + y &\leq 4 \\
x + 2y &\geq 2 \\
2x + y &\geq 2
\end{align*}
\]

Applications
Write the objective function and the inequalities that describe the constraints in each problem. Graph the feasibility region, showing the corner points. Then find the maximum or minimum value of the objective function.

21. Making furniture Two woodworkers, Tom and Carlos, get $100 for making a table and $80 for making a chair. On average, Tom must work 3 hours and Carlos 2 hours to make a chair. Tom must work 2 hours and Carlos 6 hours to make a table. If neither wishes to work more than 42 hours per week, how many tables and how many chairs should they make each week to maximize their income? Find the maximum income.

22. Making crafts Two artists, Nina and Rob, make yard ornaments. They get $80 for each wooden snowman they make and $64 for each wooden Santa Claus. On average, Nina must work 4 hours and Rob 2 hours to make a snowman. Nina must work 3 hours and Rob 4 hours to make a Santa Claus. If neither wishes to work more than 20 hours per week, how many of each ornament should they make each week to maximize their income? Find the maximum income.

23. Inventories An electronics store manager stocks from 20 to 30 IBM-compatible computers and from 30 to 50 Apple computers. There is room in the store to stock up to 60 computers. The manager receives a commission of $50 on the sale of each IBM-compatible computer and $40 on the sale of each Apple computer. If the manager can sell all of the computers, how many should she stock to maximize her commissions? Find the maximum commission.

24. Diet problems A diet requires at least 16 units of vitamin C and at least 34 units of vitamin B complex. Two food supplements are available that provide these nutrients in the amounts and costs shown in the table. How much of each should be used to minimize the cost?

25. Production Manufacturing DVRs and TVs requires the use of the electronics, assembly, and finishing departments of a factory, according to the following schedule:

26. Production problems A company manufactures one type of computer chip that runs at 2.0 GHz and another that runs at 2.8 GHz. The company can make a maximum of 50 fast chips per day and a maximum of 100 slow chips per day. It takes 6 hours
to make a fast chip and 3 hours to make a slow chip, and the company’s employees can provide up to 360 hours of labor per day. If the company makes a profit of $20 on each 2.8-GHz chip and $27 on each 2.0-GHz chip, how many of each type should be manufactured to earn the maximum profit?

27. **Financial planning** A stockbroker has $200,000 to invest in stocks and bonds. She wants to invest at least $100,000 in stocks and at least $50,000 in bonds. If stocks have an annual yield of 9% and bonds have an annual yield of 7%, how much should she invest in each to maximize her income? Find the maximum return.

28. **Production** A small country exports soybeans and flowers. Soybeans require 8 workers per acre, flowers require 12 workers per acre, and 100,000 workers are available. Government contracts require that there be at least 3 times as many acres of soybeans as flowers planted. It costs $250 per acre to plant soybeans and $300 per acre to plant flowers, and there is a budget of $3 million. If the profit from soybeans is $1,600 per acre and the profit from flowers is $2,000 per acre, how many acres of each crop should be planted to maximize profit? Find the maximum profit.

29. **Band trips** A high school band trip will require renting buses and trucks to transport no fewer than 100 students and 18 or more large instruments. Each bus can accommodate 40 students plus three large instruments; it costs $350 to rent. Each truck can accommodate 10 students plus 6 large instruments and costs $200 to rent. How many of each type of vehicle should be rented for the cost to be minimum? Find the minimum cost.

30. **Making ice cream** An ice cream store sells two new flavors: Fantasy and Excess. Each barrel of Fantasy requires 4 pounds of nuts and 3 pounds of chocolate and has a profit of $500. Each barrel of Excess requires 4 pounds of nuts and 2 pounds of chocolate and has a profit of $400. There are 16 pounds of nuts and 18 pounds of chocolate in stock, and the owner does not want to buy more for this batch. How many barrels of each should be made for a maximum profit? Find the maximum profit.

**Discovery and Writing**

31. Does the objective function attain a maximum at the corners of a region defined by following nonlinear inequalities? Attempt to maximize $P(x) = x + y$ on the region and write a paragraph on your findings. 

\[
\begin{align*}
x &\geq 0 \\
y &\geq 0 \\
y &\leq 4 - x^2
\end{align*}
\]

32. Attempt to minimize the objective function of Exercise 31.

**Review**

Write each matrix in reduced row echelon form. Problem 33 cannot be done with a calculator.

33. 
\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & -2 & 3 \\
0 & 2 & -3 \\
2 & 0 & 6
\end{bmatrix}
\]

34. 
\[
\begin{bmatrix}
1 & 3 & -2 & 1 \\
2 & 0 & 6 & 3 \\
3 & 9 & -3 & 2
\end{bmatrix}
\]

35. The matrix in Exercise 34 is the system matrix of a system of equations. Find the general solution of the system.

36. Find the inverse of 
\[
\begin{bmatrix}
7 & 2 & 5 \\
3 & 1 & 2 \\
3 & 1 & 3
\end{bmatrix}
\].