Module 60-62

The Distribution of Resources

Harry M. Schey

Applications of Calculus to Economics

COMAP, Inc., Suite 210, 57 Bedford Street, Lexington, MA 02173 (617) 862-7878
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# THE DISTRIBUTION OF RESOURCES

by

Harry M. Schey

Department of Biostatistics
University of North Carolina
Chapel Hill, North Carolina 27514

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<td>Answers to Model Exam</td>
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</table>
Intermodular Description Sheet: UMAP Units 60-62

Title: THE DISTRIBUTION OF RESOURCES

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Chapel Hill, NC 27514

Review Stage/Date: IV 4/6/79

Classification: ECON APPL CALC/DIS RESOURCES

Prerequisite Skills:

Unit 60:
1. Be able to read and draw graphs (cartesian coordinates only).

Unit 61:
1. Understand the relationship between a Lorenz curve and the equality of a resource distribution.
2. Understand the concepts of slope and derivative and the relation between them.

Unit 62:
1. Understand the concept of the definite integral.

Output Skills:

Unit 60:
1. Define and describe what is meant by a Lorenz curve.
2. Read and interpret a Lorenz curve.
3. Draw a Lorenz curve from given data.
4. Compare the equality of two or more distributions on the basis of their Lorenz curves.
5. Draw the Lorenz curves corresponding to complete equality and complete inequality.

Unit 61:
1. Define the equal share coefficient.
2. Describe the significance of the slope of a Lorenz curve.
3. Show how the equal share coefficient is a measure of the equality of a resource distribution.
4. Be able to calculate equal share coefficients, graphically, numerically, and by differentiation.

Unit 62:
1. Define the Gini index.
2. Show how the Gini index is a measure of the inequality of a resource distribution.
3. Be able to calculate the Gini index of a given resource distribution.

Other Related Units:
THE DISTRIBUTION OF RESOURCES

INTRODUCTION

The people and the nations of the world own many resources; money, land, energy, food, oil, etc. One of the most important questions about these resources has to do with how they are distributed among people or among groups of people. In this module we investigate several ways in which mathematics can be used to measure the distribution of resources.
1. THE LORENZ CURVE

1.1 Distribution Tables and Their Associated Graphs

Table I shows how total income was distributed among Americans in 1955.* It shows that the lowest .2 of the people (that is, the 20% of the population having the lowest annual income) received only .05 of the total income, the lowest .4 (which includes the lowest .2) received .16 of the total income, etc. This table thus gives us the kind of information we'll work with in this module, since it tells us how a certain resource (income) is distributed among a certain group of people (Americans in 1955). But often a table of numbers does not convey information as well as a picture or a diagram. As we'll see now, a graph of the numbers in Table I will give us a valuable way to visualize the distribution of income.

<table>
<thead>
<tr>
<th>Fraction of People</th>
<th>Fraction of Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>.2</td>
<td>.05</td>
</tr>
<tr>
<td>.4</td>
<td>.16</td>
</tr>
<tr>
<td>.6</td>
<td>.33</td>
</tr>
<tr>
<td>.8</td>
<td>.55</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

The graph shown in Figure 1 is made by drawing a smooth curve through the points that represent the numbers in Table I. Note that we use an ordinary Cartesian

---

coordinate system. The fraction of the population is plotted on the horizontal axis, and the fraction of total income, on the vertical. Since neither of these fractions is ever negative, we need only the first quadrant for the graph. (In fact, since the numbers involved never exceed 1, we only need the unit square.)

![Figure 1.](image)

A graph such as Figure 1 shows how some resource is distributed. Such a monotonic, concave curve is called a **Lorenz curve**. Note that a Lorenz curve always passes through the points (0,0) and (1,1).

It should be emphasized that to draw this Lorenz curve we plotted the points given in Table 1 and then assumed that those points can be connected by a smooth curve. In doing this we "create" values that do not appear in Table 1. Although the fraction of total income corresponding to, say, 0.35 of the population may in reality not be exactly the value given by our curve, our assumption is that it will not be very different.
Exercise 1. Use the curve drawn in Figure 1 to fill in Table II.

<table>
<thead>
<tr>
<th>Fraction of People</th>
<th>Fraction of Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>.35</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td></td>
</tr>
</tbody>
</table>

Exercise 2.
Table III shows how income was distributed in the United States in 1971. * Draw the Lorenz curve for this distribution.

<table>
<thead>
<tr>
<th>Fraction of People</th>
<th>Fraction of Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.06</td>
</tr>
<tr>
<td>0.2</td>
<td>0.18</td>
</tr>
<tr>
<td>0.4</td>
<td>0.36</td>
</tr>
<tr>
<td>0.8</td>
<td>0.60</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

The Lorenz curve in Figure 1 does give a better "picture" of income distribution than the bare numbers in Table I, but it still does not provide any way to decide whether income was divided equally among Americans in 1955 or whether it was divided in a highly unequal way. What we need is some basis for comparison so that we can look at a Lorenz curve and judge how close to (or far from) equality the distribution is. To obtain this basis, we examine the two extreme cases: complete equality and complete inequality.

1.2 Complete equality

What do we mean by complete equality? Table IV shows how income is distributed in the case of complete equality: The lowest .2 of the population (which in this case is any .2) gets .2 of the income, the lowest .4 gets .4 of the income, etc. In Exercise 3, we investigate the Lorenz curve for this completely equal distribution.

<table>
<thead>
<tr>
<th>Fraction of People</th>
<th>Fraction of Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>.2</td>
<td>.2</td>
</tr>
<tr>
<td>.4</td>
<td>.4</td>
</tr>
<tr>
<td>.6</td>
<td>.6</td>
</tr>
<tr>
<td>.8</td>
<td>.8</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Exercise 3. In Figure 2, we have plotted two of the points from Table IV. Plot the remaining points in Figure 2. Connect the points to verify that in this case of complete equality, the Lorenz "curve" is a straigh...
line which makes an angle of 45° with the horizontal axis as shown in Figure 3.

Figure 2.

Figure 3 thus shows a Lorenz curve for the case of perfect equality. Notice that the vertical axis is labeled "Fraction of Resource" instead of "Fraction of Total Income" to emphasize that this is the form of a Lorenz curve in the case of perfect equality in the distribution of any resource, not just total income. A Lorenz curve like the one you drew in Figure 2 and the one shown in Figure 3 is called the "Curve of absolute equality." (See Figure 4).
Exercise 4. Let the fraction of population be denoted \( p \) and the fraction of the resource be denoted \( r \). Use these two symbols to write an equation for the curve of absolute equality (Figure 4).

Go back to the Lorenz curve with which we began, the one shown in Figure 1. It is redrawn in Figure 5, but as you can see, something new has been added: we have also shown the curve of absolute equality, which is how the Lorenz curve would look if income were distributed perfectly equally. So now, at last, we have the beginnings of a basis for comparisons: A realistic Lorenz curve "sags" below the curve of absolute equality, and the more it sags, the greater the inequality of the distribution it represents.

![Graph showing Lorenz curve and curve of absolute equality](image)

**Figure 5.**

Exercise 5. Table V shows how arable land is distributed among farmers in Bolivia, Denmark, and the United States.* Draw the Lorenz curve for each of the three countries. In which country is the land most equally distributed among the farmers? In which country is it the least? Do these curves suggest a way to measure inequality? (See Sections 2 and 3.)

Table V

<table>
<thead>
<tr>
<th>Fraction of Farmers</th>
<th>Fraction of Land (1964)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bolivia (x)</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.010</td>
</tr>
<tr>
<td>0.6</td>
<td>0.016</td>
</tr>
<tr>
<td>0.7</td>
<td>0.022</td>
</tr>
<tr>
<td>0.8</td>
<td>0.03</td>
</tr>
<tr>
<td>0.9</td>
<td>0.04</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

1.3 Complete Inequality

Next we discuss the other extreme case: complete inequality. We already know that a Lorenz curve which represents an unequal distribution of some resource sags below the curve of absolute equality as in Figure 5. The question we’ll now examine is "How much can a Lorenz curve sag?"

The answer is that the Lorenz curve which sags as far as possible is one which represents a completely unequal distribution. What, then, is a completely unequal distribution? One example is the situation in which almost no one has any income; the exception is the one person who gets the entire amount. That distribution is shown in Table VI. Notice that the lowest .2, .4, ..., .9, .99, .999, ... of the population gets nothing. The reason the last entry
in Table VI is a 1 instead of a 0 is that all the people get all the income even though it is divided in such a way that it all goes to one person.

Table VI

<table>
<thead>
<tr>
<th>Fraction of People</th>
<th>Fraction of Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

The Lorenz curve for the distribution in Figure 6a shows the completely unequal distribution recorded in Table VI. Its graph is very discontinuous at 1, since it "jumps" from 0 to 1. When we smoothly connect the data points as in Figure 6b, the curve has a very severe bend or "corner" at the point \( p = 1 \).

Now, even though a well-defined function would not have a vertical line segment in its graph, the curve suggested to us can be idealized as having the right angle shown in Figure 6c: this Lorenz curve lies along the \( p \)-axis almost all the way to \( p = 1 \) and then "shoots" up vertically.
The Lorenz curve in Figure 6 is called the "curve of absolute inequality." It is an abstraction and represents the theoretical curve of "maximum sag." It is not very likely that any real Lorenz curve will look like the one in Figure 6. However, the curve for Bolivia, drawn in Exercise 5 comes close. (See Table V.)

We have now established the limiting cases for Lorenz curves, and these are shown in summary in Figure 7. Any real Lorenz curve (such as the dashed curve in Figure 7) lies in the triangular region between the curve of absolute inequality. The closer the Lorenz curve is to the curve of absolute equality, the closer to equality is the distribution of resources it represents; the closer it is to the curve of absolute inequality, the closer the distribution is to inequality.

![Lorenz Curve Diagram](image)

Figure 7.

The Lorenz curve provides us with a way to visualize and compare the distributions of resources. But a Lorenz curve alone gives only a qualitative measure of how equally a resource is distributed, and we really need some quantitative measures. In the two following sections of this module, we set up some quantitative measures based on the Lorenz curves we have been examining in this section.
1.4 Exercises

Exercise 6.

Table VII shows how population was divided among the 25 largest countries in 1963 and 1968.* Was there much change in the population distribution between 1963 and 1968? How would the Lorenz curves for 1963 and 1968 compare? Graph each using the same set of coordinates.

<table>
<thead>
<tr>
<th>Fraction of Countries</th>
<th>Fraction of Population</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1963 (x)</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>.2</td>
<td>0.046</td>
</tr>
<tr>
<td>.4</td>
<td>0.105</td>
</tr>
<tr>
<td>.6</td>
<td>0.193</td>
</tr>
<tr>
<td>.8</td>
<td>0.345</td>
</tr>
<tr>
<td>1.0</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Question for Discussion:

How do you reconcile the data in Table VII (and in the resulting graph) with the fact that world population is growing rapidly?

Exercise 7.

Table VIII shows how total energy produced and total energy consumed was divided among the 25 largest producers and users of energy in 1963.* Draw the corresponding Lorenz curves on the same set of coordinates. Was production of energy more or less equally divided than was use of energy?

*Data from Comparative Analysis of Political Environments, as quoted by Harf, Ledford, and Thompson, op. cit.

*Data from Comparative Analysis of Political Environments, as quoted by Harf, Ledford, and Thompson, op. cit.
Table VIII

<table>
<thead>
<tr>
<th>Fraction of Countries</th>
<th>Fraction of Energy (x) Used</th>
<th>Produced (o)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
<td>0.00</td>
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<tr>
<td>.2</td>
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<td>0.032</td>
</tr>
<tr>
<td>.4</td>
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<td>.6</td>
<td>0.135</td>
<td>0.170</td>
</tr>
<tr>
<td>.8</td>
<td>0.266</td>
<td>0.3031</td>
</tr>
<tr>
<td>1.0</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Exercise 8.

Tables I and III are for the distribution of income in the United States in 1955 and 1971, respectively. Compare the data in these two tables (also analyse the corresponding Lorenz curves). Can you say whether income was distributed more or less equally in 1971 than it was in 1955? Discuss.

Exercise 9.

Letting $p$ stand for the fraction of the population and $r$ for the fraction of the resource, consider the family of Lorenz curves given by the equation

$$r = p^n, \; n = 1, 2, 3, 4, \ldots$$

a) To what situation does the case $n = 1$ correspond? (See Exercise 4.)

b) In what way is the number $n$ a measure of the equality of distribution? To answer this question, plot $r = p^n \; (0 \leq p \leq 1)$ for $n = 2, 4, 8, 10$. (Although $n$ is a quantitative measure of equality, it is not generally useful because Lorenz curves are certainly not often of the form $r = p^n$.)

c) To what situation does the Lorenz curve $r = p^n$ correspond in the limiting case $n \rightarrow \infty$? To answer the question try drawing the curve $r = p^{100}$ for $0 \leq p \leq 1$. 12
Exercise 10 (Advanced).

Can a Lorenz curve have the shape shown in Figure 8? Discuss.
1.5 Model Exam

The accompanying table shows how gross national product (GNP) is distributed among the countries of the world:

Table IX

<table>
<thead>
<tr>
<th>Fraction of Countries</th>
<th>Fraction of GNP</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>.1</td>
<td>.001</td>
</tr>
<tr>
<td>.2</td>
<td>.002</td>
</tr>
<tr>
<td>.3</td>
<td>.005</td>
</tr>
<tr>
<td>.4</td>
<td>.010</td>
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<td>.5</td>
<td>.018</td>
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<td>.6</td>
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</tr>
<tr>
<td>.7</td>
<td>.058</td>
</tr>
<tr>
<td>.8</td>
<td>.11</td>
</tr>
<tr>
<td>.9</td>
<td>.21</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

1. In the space provided sketch a graph of the Lorenz curve for the data in this table. Label the axes carefully.

2. On the same graph used in question 1, draw the Lorenz curve that corresponds to a completely unequal distribution of GNP, and the Lorenz curve that corresponds to a completely equal distribution of GNP. Label each.

3. On the basis of the curves drawn in question 1 and 2 would you say that GNP is divided equally or unequally among the countries of the world? Write a brief paragraph in support of your answer.
1.6 Answers to Exercises

1. | Fraction of People | Fraction of Income |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
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</tr>
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<td>.35</td>
<td>.13</td>
</tr>
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<td>.24</td>
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<tr>
<td>.75</td>
<td>.48</td>
</tr>
<tr>
<td>.9</td>
<td>.75</td>
</tr>
</tbody>
</table>

2. 

3. 

1.0

0.8

0.6

0.4

0.2

0

Fraction of Population

0

0.2

0.4

0.6

0.8

1.0

Fraction of Total Income

Fraction of Population

0

0.2

0.4

0.6

0.8

1.0
4. \( r = p \)

5.

![Graph showing the distribution of farmers and land ownership with lines for Bolivia, Denmark, and the United States.](image)

- (Bolivia)
- (Denmark)
- (United States)
6.

![Graph showing the fraction of population against the fraction of countries.](image)

- x (1963)
- o (1968)

9. a) Complete equality \( r = p^1 = p \).

b) The larger the \( n \), the less equal the distribution. Note the larger the \( n \), the more of a "corner" near 1.

c) Complete inequality.

10. No, the smallest values are at the start of the curve. We must put the smallest shares at the beginning so the slope is always increasing as we go from left to right.
1.7 Answers to Model Exam

1 and 2.

3. Very unequal. The Lorenz curve is very close to the curve of complete inequality.
2. THE EQUAL SHARE COEFFICIENT

2.1 Introduction

Section 1 of this module discussed Lorenz curves and how they can help us to visualize the way in which resources are distributed. But a Lorenz curve by itself can give us only a qualitative measure of equality. Several different methods have been invented which provide quantitative measures of equality. In this Section, we study one of these; it is called the equal share coefficient (ESC) and involves the mathematical concepts of slope and derivative.

2.2 The Slope of a Lorenz Curve

In order to define the ESC, we need to learn the significance of the slope of a Lorenz curve. To do this, we go back to the 1955 distribution of income studied in Section 1. The Lorenz curve for this distribution is repeated as Figure 1 of this Section.

The point marked A in Figure 1 corresponds to the lowest .3 of the population and .10 of the total income. The point B corresponds to the lowest .4 of the population and .16 of the income. Thus, the .1 of the population between the lowest .3 and the lowest .4 receives .16 - .10 = .06 of the total income. That is, this .1 of the population, since it gets only .06 of the income, receives less than an equal share. (An equal share for .1 of the population would, of course, be .1 of the income.) Notice that the slope of the line through the points A and B is .6; that is

\[
\frac{\text{slope}}{\text{of AB}} = \frac{.16 - .10}{.4 - .3} = \frac{.06}{.1} = .6.
\]
This slope is less than 1, and as we see, that is a geometric way of saying that the part of the population between .3 and .4 gets less than an equal share of the income.

Now look at two other points on the curve, C and D. The point C corresponds to the lowest .7 of the population and .42 of the income, and D corresponds to the whole population (.1) and the whole income (.1). Thus, this \( 1 - .7 = .3 \) of the population earns \( 1 - .42 = .58 \) of the income which is more than an equal share. (An equal share for .3 of the population would be .3 of the income.) Notice that the slope of
the line CD is approximately 1.93:

\[
\frac{\text{slope of CD}}{1 - 0.7} = \frac{0.58}{\frac{0.5}{1}} = 1.93.
\]

This slope is greater than one, which is a geometric way of saying that the .3 of the population between .7 and 1.0 gets more than an equal share of the income.

In general, the slope of a line joining two points on a Lorenz curve is less than one if the fraction of the population between these points gets less than an equal share of the income (Figure 2a). The slope is greater than one if the fraction of the population gets more than an equal share (Figure 2b). The slope is equal to one if the fraction of the population receives exactly an equal share (Figure 2c).

Figure 2.
Exercise 1.

Figure 3 is the Lorenz curve for the distribution of land among farmers in the United States (see Section 1, Table V, Problem 5). By measuring slopes from this curve, determine whether each of the following groups of farmers own more or less than an equal share of the land:

a. The .1 between .1 and .2
b. The .2 between .3 and .5
c. The .05 between .6 and .65
d. The .1 between .9 and 1.0

Exercise 2.

Use Figure 3 to find the .2 of the farmers who own an exactly equal share of the land. (Your answer, since it will be determined from a graph, can only be approximate.)
2.3 The Definition

So far our discussion has dealt only with the *average* slope of Lorenz curves (that is, a slope over some interval between two points). This is not really precise enough a quantity for our purposes for the following reason. Figure 4a shows a hypothetical Lorenz curve for some resource. You should be able to convince yourself that the .2 of the population between .4 and .6 has an exactly equal share (.2) of the resource (draw the line whose slope shows this). The trouble with this statement is that it is only true in an *average* sense: A group on the lower end of this .2 of the population (as for example, the .1 between .4 and .5) gets *less* than an equal share, and a group near the upper end (between .5 and .6 for example) gets *more* than an equal share. The .2 of the population between .4 and .6 gets an equal share only on the average.

![Figure 4a](image)

![Figure 4b](image)

A more precise way to measure equality of distribution is provided by the *single point* on the Lorenz curve at which the slope becomes equal to 1. That point Q is indicated in Figure 4b. The slope of the curve everywhere to the left of that point is less than one; everywhere to the
right, it is greater than 1. (This is because a
Lorenz curve is concave upwards and monotonically in-
creasing.) The value of p at which the slope of the
Lorenz curve becomes 1 is called the equal share
coefficient (ESC) which we denote by ε. The ESC is
the fraction of the population that receives less
than an equal share of the resource. Note that
0 ≤ ε ≤ 1.

If, for example, ε = .4, that means that .4 of
the population owns less than an equal share of the
resource, but if ε = .75, then .75 of the population
owns less than an equal share of the resource.

If we denote a Lorenz curve by r = f(p), then ε
is the (smallest) value of p for which

\[ \frac{dr}{dp} = 1, \]

since the derivative is a measure of the slope. Below
we show a few idealized cases in which r is a given
function of p. In those cases, we calculate ESC's by
differentiation. As a rule, however, we have tables or
graphs but no formulas and cannot calculate formulas for
derivatives. In those cases, we have to work geo-
metrically or arithmetically as we now show.

2.4 A Geometric Approach

First we discuss the geometric way to determine ε.
This method is most useful when we have a carefully
drawn Lorenz curve. The procedure is illustrated in
Figures 5a-e. Place a plastic drafting triangle so
that one leg passes through the points (0,0) and (1,1)
of the Lorenz curve (Figure 5a). Note that the slope of
this leg is 1. Now place a ruler against the base of
the triangle (Figure 5b). Hold the triangle firmly as
you do this so it cannot move. Then, holding the ruler
firmly, begin to slide the triangle to the right (Figure
5c). Note that the slope of the triangle leg remains
equal to one, since, by holding the ruler firmly, you ensure that the triangle does not rotate as it slides. Continue sliding the triangle until the leg just touches or grazes the Lorenz curve at a single point (Figure 5d). Mark that point on the curve. Finally, remove the triangle and ruler (Figure 5e) and read the horizontal coordinate of the point. That coordinate is $\epsilon$, the ESC.

![Diagram of Lorenz curve](image)

Figure 5.

**Exercise 3.**

Use the graphical method described above to determine the ESC for each of the following Lorenz curves from Section 1:

- a. Figure 1.
- b. Answer to Exercise 2.
- c. Answer to Exercise 5.
- d. Answer to Exercise 6.
- e. Answer to Exercise 7.
Exercise 4.

Using part (c) of Exercise 3 above as a guide, state whether, in general, a value of ε greater than .5 indicates an equal or an unequal distribution.

Exercise 5.

Figure 6 is the Lorenz curve for a perfectly equal distribution. Note that the slope of this curve is one for all values of p. What value would you assign to the ESC here? (Remember ε is the fraction of the population that receives less than an equal share of the resource.)

Figure 6.

Exercise 4 touches upon an important issue. In what way is ε a measure of equality? Put in different terms, of two distributions does the one with the larger ESC represent a more or less equal distribution? To do this, we consider two extreme cases. In the most equal of all possible distributions, represented by the curve of absolute equality (Figure 6), everyone gets an equal share, so no one has less than an equal share. The fraction of the population with less than an equal share is therefore zero and so, for perfect equality, ε = 0. On the other hand, if a resource is distributed in a very unequal way (as in Figure 7) then almost everyone gets less than an equal share and ε comes very close to 1.
Exercise 6.

Check your results in Exercise 3 to determine if they bear out the assertion made above that a value of $c$ near zero implies a high degree of equality while one close to zero implies a high degree of inequality.

Exercise 7.

Using your results from Exercise 6, correlate in a qualitative way the value of $c$ for a Lorenz curve, with the amount that the curve sags below the curve of absolute equality.

### 2.5 A Numerical Approach

Next, we turn to the numerical method for calculating an ESC. We know that if $r$ is given as a function of $p$, $r = f(p)$, the ESC is the (smallest) value of $p$ for which

$$\frac{dr}{dp} = 1.$$ 

An approximate (though often quite accurate) way to calculate such a derivative is illustrated in Figure 8. The derivative at some point $p = p_0$ is approximated by the average slope between $p_0 - h$ and $p_0 + h$. If $h$ is small enough, this should be a good approximation.
because by the definition of the derivative

\[ r'(p_0) = \lim_{h \to 0} \frac{r(p_0 + h) - r(p_0 - h)}{2h} \]

Thus

\[ r'(p_0) = \frac{r(p_0 + h) - r(p_0 - h)}{2h} \]

as long as \( h \) is small.

We'll show how this works in the case of the 1955 income distribution quoted in Section 1, Table I. Note that, for example,

\[ r(.4) = .16 \]

and

\[ r(.6) = .33. \]

Thus, according to our approximation (with \( h = .1 \))

\[ r'(5) = \frac{r(.6) - r(.4)}{2(.1)} = \frac{.33 - .16}{.2} = .85. \]

The numbers from Table I, Section 1 are repeated in Table I of this section and values for \( r' \) are also included, calculated as above.

Exercise 8.

Use the formula \( r'(p_0) = \frac{r(p_0 + h) - r(p_0 - h)}{2h} \) to verify the numbers in Column 3 of Table 1.

We are interested in the value of \( p \) for which \( r' = 1 \). The value 1 does not, however, appear in Table 1, so let's make a graph of \( r' \) vs. \( p \) as in Figure 9, where we have plotted the five points given in Table 1 and then drawn a smooth curve through them. The graph shows that \( r'(p) = 1 \) at \( p = .61 \) which is thus
our estimate of the ESC. How does this compare with the value you obtained by the graphical method in Problem 37

Table 1

<table>
<thead>
<tr>
<th>p</th>
<th>r</th>
<th>r'</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.25</td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
<td>0.55</td>
</tr>
<tr>
<td>0.2</td>
<td>0.16</td>
<td>0.85</td>
</tr>
<tr>
<td>0.3</td>
<td>0.33</td>
<td>1.1</td>
</tr>
<tr>
<td>0.4</td>
<td>0.55</td>
<td>2.25</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Figure 9.
Exercise 9.

Using the method just described, calculate and plot $r'$ vs. $p$ for the distributions given in Tables III, V, and VI of Section 1, and determine a value for the ESC in each case. Compare your results with those obtained by the sliding triangle method. Which method, the present one, or the sliding triangle, do you believe gives the more accurate results? Discuss.

2.6 Linear Interpolation

There is another procedure for finding $c$ that does not involve reading numbers from a graph. It is called the method of linear interpolation. In Figure 10, we plot the two points from Table 1 which bracket the value $r' = 1$ (that is, $p = .5, r' = .85$, and $p = .7, r' = 1.1$) and then connect these two points with a straight line.

![Figure 10](image)

This straight line is an approximation to the curve $r'$ vs. $p$ in the interval between $p = .5$ and $p = .7$. The underlying idea (or pious hope) is that any well-mannered curve will be approximately straight over a short interval. In any case, $c$ is the value of $p$ at which $r' = 1$. To find $c$ we use the fact that triangles ABC and ADE in Figure 10 are similar:

$$\frac{AB}{AD} = \frac{BC}{DE}.$$
However,

\[
\begin{align*}
AB &= c - .5 \\
AD &= .7 - .5 = .2 \\
BC &= 1.0 - .85 = .15 \\
DE &= 1.1 - .85 = .25 \\
\end{align*}
\]

so

\[
\frac{c - .5}{.2} = \frac{.15}{.25}
\]

or

\[
0.5 + 0.2 \left( \frac{0.15}{0.25} \right) = 0.62
\]

which is close to the value we read from Figure 9.

2.7 Exercises

**Exercise 10.**

Recalculate the ESC's found in Problem 9 by the method of linear interpolation. How do these values compare with those calculated in Problem 8? Which set do you think is the more accurate? Discuss.

**Exercise 11.**

(a) Let \( r = p^n \) (\( n > 1 \)) be the equation of a Lorenz curve. Find an expression for the ESC in terms of \( n \).

(b) Use the result of (a) to calculate \( c \) for \( n = 2, 4, 8, 16 \). Are these results reasonable (see Section 1, Problem 9)?

(c) Calculate \( \lim_{n \to \infty} c \) using for \( c \) the formula you derived in (a).

**Exercise 12.** (Advanced).

If a Lorenz curve is given by \( r = f(p) \), the ESC is the value of \( p \) for which

\[
\frac{dr}{dp} = f'(p) = 1.
\]

But how can we be sure that \( f'(p) \) will equal 1 at some \( p \) between 0 and 1? Making the necessary assumptions about the continuity and differentiability of \( f(p) \), use the mean value theorem of differential calculus to prove that \( f'(p) = 1 \) for at least one value of \( p \) between 0 and 1.
Exercise 13.

Let a Lorenz curve be given by

\[ r = \frac{1}{2} p^\frac{3}{2} (1 + p). \]

Show that the ESC \( \epsilon \) is a solution of

\[ \frac{3}{4} \epsilon^x + \frac{5}{4} \epsilon^y = 1. \]
2.8 Model Exam

1. Define the equal share coefficient, and discuss how it measures the equality of a resource distribution.

2. In the Lorenz curve shown below, does the fraction of the population \( p \) between \( p = .3 \) and \( p = .4 \) own more than an equal share, less than an equal share, or an equal share of the resource \( r \)? Discuss how you arrive at your answer.

![Lorenz Curve Diagram](image)

3. Estimate the equal share coefficient for the Lorenz curve shown above by graphical means. Show your work.

4. A Lorenz curve is given by \( r = \frac{1}{4} (p + p^2) \). Find the equal share coefficient. On the basis of this calculation and your answer to Question 3, state which distribution is more nearly equal, the one shown in the graph above, or the one given by this equation. Explain.
2.9 Answers to Exercises

1. a) less than  
   b) less than  
   c) less than  
   d) more than  

2. The .2 between .7 and .9.  

3. a) .64  
   b) .6  
   c) .8, .82, .9  
   d) .8, .8  
   e) .8, .8  

4. Values larger than .5 indicate an unequal distribution in favor of the few near the top.  

5. $\epsilon = 0$  

6. All answers in Exercise 3 are greater than .5. This, as was asserted, indicates a high degree of inequality in all cases.  

7. $0 \leq \epsilon \leq 1$.  
   $\epsilon = 0$ = absolute equality.  
   $\epsilon = 1$ = absolute inequality.  
   $\epsilon = 0$ means no sag  
   $\epsilon = .5$ means a sag half way  
   $\epsilon = 1$ means complete sag  

   The larger the $\epsilon$, the more the curve will sag.  

8. $r'(1) = \frac{r(.2) - r(0)}{2(.1)} = \frac{.05 - 0}{.2} = .25$  
   $r'(1) = \frac{r(.4) - r(.2)}{2(.1)} = \frac{.16 - .05}{.2} = .55$  
   $r'(1) = \frac{r(.8) - r(.6)}{2(.1)} = \frac{.55 - .33}{.2} = 1.1$  
   $r'(1) = \frac{r(1.0) - r(.8)}{2(.1)} = \frac{1.0 - 5.5}{.2} = -2.25$
Table III

<table>
<thead>
<tr>
<th>p</th>
<th>r</th>
<th>r'</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>.1</td>
<td>.06</td>
<td>.6</td>
</tr>
<tr>
<td>.2</td>
<td>.06</td>
<td>.6</td>
</tr>
<tr>
<td>.3</td>
<td>.18</td>
<td>.9</td>
</tr>
<tr>
<td>.4</td>
<td>.36</td>
<td>1.2</td>
</tr>
<tr>
<td>.5</td>
<td>.60</td>
<td>2.0</td>
</tr>
<tr>
<td>.6</td>
<td>.60</td>
<td>2.0</td>
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<tr>
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<tr>
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<tr>
<td>.9</td>
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<td>2.0</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
<td>2.0</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
r'(.1) &= \frac{r(.2) - r(0)}{2(.1)} = \frac{.06}{.2} = .3 \\
r(.3) &= \frac{r(.4) - r(.2)}{2(.1)} = \frac{.18 - .06}{.2} = .6 \\
r(.5) &= \frac{r(.6) - r(.4)}{2(.1)} = \frac{.36 - .18}{.2} = .9 \\
r(.7) &= \frac{r(.8) - r(.6)}{2(.1)} = \frac{.60 - .36}{.2} = 1.2 \\
r(.9) &= \frac{r(1.0) - r(.8)}{2(.1)} = \frac{1.0 - .6}{.2} = 2.0 \\
\end{align*}
\]

ESC = .6
Triangle method also gives .6.
9. (cont.)

<table>
<thead>
<tr>
<th>Fraction of Farmers</th>
<th>Fraction of Land (1964)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B(x)</td>
</tr>
<tr>
<td>0.0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.01</td>
</tr>
<tr>
<td>0.6</td>
<td>0.016</td>
</tr>
<tr>
<td>0.7</td>
<td>0.022</td>
</tr>
<tr>
<td>0.8</td>
<td>0.03</td>
</tr>
<tr>
<td>0.9</td>
<td>0.04</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

ESC(x) = .812
ESC(o) = .715
ESC(a) = .8
9. (cont.)

<table>
<thead>
<tr>
<th>Fraction of People</th>
<th>Fraction of Income</th>
<th>$r'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.7</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>.8</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1.0</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

ESC = .84

Both the graphical and the sliding triangle methods give only approximate values.
10. For Table III: ESC = .5667
   For Table V: ESC(x) = .8191
   ESC(o) = .7286
   ESC(,) = .8
   For Table VI: ESC = .74

11. a) np^n = 1.
    b) For n = 2, p = .5; for n = 4, p = \sqrt{\frac{1}{4}} = .63.
    For n = 8, p = \sqrt[8]{\frac{1}{8}} = .77; for n = 16, p = \sqrt[16]{\frac{1}{16}} = .83.
    c) \lim_{n \to \infty} p^n = 1

12. The mean value theorem says: If f(x) is continuous for 
a \leq x \leq b and f'(x) exists for a < x < b, then there exists 
at least one value c, such that a < c < b, for which 
\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]

For this problem f(p) is continuous on 1 \leq p \leq 0 and f'(p) 
exists on 0 < p < 1. Therefore, by the mean value theorem, 
we can write 
\[ f'(p) = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1 \]
for at least one value of p on 0 < p < 1.

13. a) \[ r' = \frac{1}{2} \rho^2 + (1 + p)^\frac{3}{4} \rho^\frac{1}{4} \]
\[ \therefore \frac{1}{2} \epsilon^2 + (1 + p)^\frac{3}{4} \epsilon^\frac{1}{4} = 1. \]
and \[ \frac{5}{4} \epsilon^2 + \frac{3}{4} \epsilon^\frac{1}{4} = 1. \]
b) \[ \epsilon = .5149 \]
2.10 Answers to Model Exam

1. Given that the Lorenz curve is defined by \( r = f(p) \), the value of \( p \) at which the Lorenz curve has slope 1 is called the equal share coefficient \( \epsilon \) or ESC. The value of \( \epsilon \) lies in the interval from 0 to 1, inclusive. A value of \( \epsilon \) close to 0 indicates a very nearly equal sharing of resources. A value close to 1 indicates a very unequal sharing.

2. Less than an equal share. The slope is less than \( \frac{1}{4} \).

3. Approximately .78.

4. To find \( \epsilon \), find \( r' \) and set it equal to 1.

\[
r' = \frac{1}{2} + p \\
\therefore \quad \epsilon = \frac{1}{2} = .5.
\]

This distribution is more nearly equal than the one shown on the graph.
3. THE GINI INDEX

3.1 Introduction

Section 2 of this module developed a quantitative measure of the equality of distributions, the equal share coefficient. This quantity is found by locating the point on the Lorenz curve at which the slope is 1. Calculating the ESC is therefore a problem in differential calculus. In this section, we introduce another measure of equality, one which is called the Gini index. To determine the Gini index of a distribution, it is necessary to calculate the area under a Lorenz curve, and so we will be dealing here with a problem in integral calculus.

3.2 Area Under the Lorenz Curve

In Section 1 we learned that in the distribution of an resource, there are two idealized cases, that in which the resource is divided equally among the population (absolute equality), and that in which it is divided completely unequally (absolute inequality). The Lorenz curve for absolute equality is shown in Figure 1a and the Lorenz curve for absolute inequality is shown in Figure 1b.
Lorenz curves for distributions generally lie somewhere between the two extremes pictured in Figures 1a and b. Figures 2a-d show four Lorenz curves illustrating distributions which move further and further from complete

![Lorenz curves](image)

Figure 2.

equality. The area between the curve of absolute equality (the dashed line in Figures 2a-d) and the Lorenz curve grows larger as we move further from absolute equality. This area (cross hatched in Figure 3) is called the area of inequality, and we observe that the larger the area of inequality, the less equal the distribution of the resource.

![Area of inequality](image)

Figure 3.

How big is the area of inequality for the case of absolute equality (Figure 1a)? How big is the area of inequality for the case of absolute inequality (Figure 1b)?

When you determined the area of inequality for the two extreme cases, you should have found that that area is zero for absolute equality and .5 for absolute inequality, as indicated in Figures 4a and 4b. Thus the area
of inequality for any real Lorenz curve must lie between 0 and .5.

3.3 The Definition

Although the area of inequality is a measure of the extent to which a resource is divided equally, another quantity closely related to the area of inequality is more often used. This quantity, called the Gini index, is the ratio of the area of inequality to the largest possible area of inequality. In other words,

\[
\text{Gini Index} = \left( \frac{\text{Area of Inequality}}{1/2} \right) = 2 \times \left( \frac{\text{Area of Inequality}}{\text{Area of Inequality}} \right).
\]

If we let \( g \) stand for the Gini index, and \( A_I \) for the area of inequality, then \( g = 2A_I \).

Exercise 1.

a) What is the value of \( g \) for the case of absolute equality (Figure 4a)?

b) What is the value of \( g \) for the case of absolute inequality (Figure 4b)?

Exercise 2.

The Gini indexes for five different distributions of a resource are as follows:

a) .73  c) .41  e) .33
b) .01  d) .86

Arrange these distributions in order from the most to the least equal.
As Exercise 1 shows, a Gini index can be any number from 0 to 1. A value of $g$ close to 0 indicates a distribution close to absolute equality and a value close to 1 indicates a distribution close to absolute inequality.

![Figure 5.](image)

**Exercise 3.**

Let $A_L$ stand for the area under the Lorenz curve (Figure 5). Write a formula for the Gini index $g$ in terms of $A_L$.

![Figure 6.](image)

Your result in Problem 3 should be $g = 1 - 2A_L$. Since this expression will be used in what follows to calculate $g$, it is worth showing how it is derived. Figure 6 is a "diagram equation" which shows that the area under the curve of absolute equality (which is 1/2) equals the area of inequality ($A_I$) plus the area under the Lorenz curve ($A_L$). That is,

\[
\frac{1}{2} = A_I + A_L.
\]

But, as we know, $g = 2A_I$ so $A_I = g/2$, and

\[
\frac{1}{2} = \frac{1}{2} g + A_L
\]
or

\[ \frac{1}{2}g = \frac{1}{2} - A_L. \]

From this we get our formula

\[ g = 1 - 2A_L. \]

3.4 Using the Definite Integral

This expression is useful because it reduces the calculation of the Gini index, g, to the calculation of the area under a curve (to be precise, the area under the Lorenz curve), and finding areas under curves is the most famous application of integral calculus. If, as usual, we let \( r \) represent the fraction of the resource and \( p \) the fraction of the population, then as we see in Figure 7,

\[ A_L = \int_0^1 r \, dp \]

So that

\[ g = 1 - 2\int_0^1 r \, dp \]

If \( r \) is given as a function of \( p \), we can use this expression to find \( g \).

![Figure 7.](image)

Exercise 4.

Calculate the Gini index \( g \) in each of the following cases:

a. \( r = p^2 \)

b. \( r = p^3 \)

c. \( r = p^4 \)

Which of these represents the most nearly equal distribution? The least?
3.5 The Rectangle Method

Now we calculate \( g \) for some actual Lorenz curves. Table 1 shows the distribution of income in the United States in 1955 (this is also Table 1 of Section 1). What is the Gini index for this distribution? We know that we can calculate \( g \) once we know \( A_L \), the area under the Lorenz curve. In this situation (and in most situations) we have no formula which gives \( r \) in terms of \( p \) so we can't calculate \( \int_0^1 r dp \) exactly by any of the standard "textbook" methods. We can, however, use numerical and graphical methods to approximate the value of the integral.

Table 1

<table>
<thead>
<tr>
<th>Fraction of People</th>
<th>Fraction of Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>.2</td>
<td>.05</td>
</tr>
<tr>
<td>.4</td>
<td>.16</td>
</tr>
<tr>
<td>.6</td>
<td>.33</td>
</tr>
<tr>
<td>.8</td>
<td>.55</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

To show how this can be done, in Figure 8 we draw the Lorenz curve for the distribution in Table 1. Notice that a horizontal line has been drawn on the graph at \( r = .36 \). This line creates two triangle-like regions, A and B, which appear to have the same area. If they do have the same area, then the area under the Lorenz curve is equal to the area of the rectangle under the horizontal line. The dimensions of this rectangle are 1 and .36 so its area is \( 1 \times .36 = .36 \). This is also (approximately) the area \( A_L \) under the Lorenz curve in Figure 8. Thus,
In the sense that this number is closer to 0 than to 1, the income distribution in Table 1 is closer to equality than inequality.

It is easy to improve upon this method for calculating $g$ by doing a better job of finding $A_L$, the area under the Lorenz curve. Figure 9 is a graph of the Lorenz curve also pictured in Figure 8. In Figure 9, however, we have used two rectangles, one extending from $p = 0$ to .5, and the second from $p = .5$ to $p = 1$, to estimate the area under the curve. The idea here is that it is easier to
"eyeball" the height of each of the two narrow rectangles in Figure 9 than it is for the one wide rectangle in Figure 8. The two rectangles in Figure 9 are chosen so that, to the best of our ability to judge by eye, the area of $A_1$ equals the area of $B_1$, and the area of $A_2$ equals the area of $B_2$.

The height of the left rectangle in Figure 9 is .1 and its width is .5. Thus its area is .05. The area of the right-hand rectangle is .26 = (.52 x .5). Thus, the area of the two rectangles is .05 + .26 = .31. Using our formula for $g$ we find

$$g = 1 - 2(.31) = 1 - .62 = .38,$$

which is substantially larger than the value from the one rectangle approximation.
Exercise 5.

Figure 10 is the Lorenz curve of Figures 8 and 9. We have divided the graph into five strips each of width .2. We also have drawn a horizontal line in the fourth strip at a height of .44. The area of the rectangle from $p = .6$ to $p = .8$ whose height is .44 is .2 x .44 = .088. Estimate the heights of the other four rectangles; compute the areas of each. Then add the areas of the five rectangles to get an approximation to the area $A_L$ under the curve. Use your value $A_L$ to compute $g$. (Note how this approach is leading us towards the definition of the integral in terms of Riemann sums.)
Exercise 6.
Using five rectangles of equal width, estimate the Gini index for each of the three distributions of land given in Table V, Section 1. Order these three distributions from least to most equal on the basis of their Gini indexes. Does this order agree with the order you made in Exercise 5, Section 1?

Exercise 7.
Use five rectangles of equal width to calculate the Gini index for the distribution of energy production and for the distribution of energy use (data given in Exercise 7, Section 1).

Exercise 8.

a. Draw graphs for each of the three distributions given in Exercise 4 of this Section and make a one-rectangle estimate of the area under each of the curves. Compute the three Gini indexes. How do these values compare with the exact values you found in Exercise 4?

b. Repeat the calculations of (a) using a five-rectangle approximation in each of the three cases. How do these approximate Gini indexes compare with the exact values?

3.6 A Numerical Method

The method of calculating the Gini index we have been discussing involves estimating areas by eye. We now describe another closely related method which is purely numerical and involves no "eyeball" estimation. There is no guarantee that this method provides more accurate answers, but it will permit us to work directly with tabulated values so that we do not need to draw a graph of the Lorenz curve.

To see how this numerical method works, we show in Figure 11a, a Lorenz curve with the region beneath it divided into strips. Figure 11b shows one of these strips in detail. Our problem is to draw the (dotted) horizontal line which is the top of the rectangle whose
area is the same as the area of the strip under the curve. In the graphical method discussed above, this horizontal line was chosen by eye. This method we now propose is to chose this horizontal line so that its height, \( h \), above the horizontal axis is the average of \( h_L \), the height of the curve at the left edge of the strip, and \( h_R \), the height of the right edge. That is,

\[
h = \frac{h_L + h_R}{2}.
\]

If the width of this strip is \( w \), the area of this rectangle is

\[
A = wh = w \left( \frac{h_L + h_R}{2} \right).
\]

This area is then an approximation to the area of the strip beneath the curve. If the strip is narrow, the curve's height will be pretty much the same everywhere in the strip (that is \( h_L \) and \( h_R \) won't differ by much). If that is so, we can choose a rectangle whose height \( h \) is any value between \( h_L \) and \( h_R \) and use it to calculate a reasonably good approximation to the area of the strip. The method we are using here is to let \( h \) be the average of \( h_L \) and \( h_R \). This choice is both sensible and easy to calculate.

To see how this method works, let's use the numbers given in Table I. They are repeated in Table II with some new features added. The column marked "Height (Fraction of Income)" gives the height of the Lorenz curve at each value of \( p \). The column marked "Average Height" is the average of the left and right edge heights. For
example, at .6 the height (fraction of income) is .33 and at .8 it is .55 (see Figure 12). The average of these two numbers is

\[
\frac{.33 + .55}{2} = \frac{.88}{2} = .44.
\]

This number is entered in the table between the .6 row and the .8 row because it is the average height of the rectangle which extends from .6 to .8.

The area of this rectangle is its height \( \times \) width. The area is therefore \(.2 \times .44 = .088\), and this number appears in the corresponding row of the last column of Table II.
3.7 Exercises

Exercise 9.

Fill in the missing number in the third column of Table II and the missing number in the fourth column.

The area under the Lorenz curve is approximately the sum of the area of the rectangles in the fourth column of Table II. That is,

\[ A_L = .005 + (.021) + .049 + .088 + .155 = .318. \]

From this we find

\[ g = 1 - 2(.318) = 1 - .636 = .364 \]

as the value of the Gini index. Notice that this agrees very well with the five strip geometric method we used above.

Exercise 10.

Using the numerical method we have just discussed, calculate the Gini index for each of the three distributions of land given in Table V, Section 1. How do your results compare with those you obtained in Exercise 7? Which set do you think is the more accurate? Discuss.

Exercise 11 (Advanced).

a) Calculate the Gini index \( g \) for the Lorenz curve \( r = p^n \) where \( n \) is a positive number.

b) Evaluate your result from (a) for the case \( n = 1 \). Is this the result you would expect? Discuss.

c) Find the limit of \( g \) as \( n \to \infty \). How can you explain your result?
3.8 Model Exam

1. Define the Gini index and discuss how it measures the equality of a resource distribution.

2. Which of the three distributions whose Lorenz curves are pictured below has the largest Gini index?

   ![Lorenz curves](image)

   The smallest? Which is the most nearly equal distribution? The least?

3. The accompanying table gives the distribution of a resource. Use these numbers to find a good approximate value of the Gini index.

<table>
<thead>
<tr>
<th>p</th>
<th>r</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.2</td>
<td>.04</td>
</tr>
<tr>
<td>.4</td>
<td>.15</td>
</tr>
<tr>
<td>.6</td>
<td>.30</td>
</tr>
<tr>
<td>.8</td>
<td>.50</td>
</tr>
<tr>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>

4. Find the Gini Index of a distribution whose Lorenz curve is given by \( r = \frac{1}{2} (p \cdot p^2) \). On the basis of this calculation and your answer to question 3, state which distribution is more nearly equal, the one shown in the table above, or the one given by this equation. Explain.
3.9 Answers to Exercises

1. a) 0
   b) 1

2. b) .01, a) .33, c) .41, a) .73, d) .86.

3. \( g = 1 - 2A_L \).

4. a) \( 1/3 = .3333 \)
   b) \( 1/2 = .5 \)
   c) \( 3/5 = .6 \)

5. 1st rectangle: \( A_1 = .02(.2) = .004 \)
    2nd rectangle: \( A_2 = .1(.2) = .02 \)
    3rd rectangle: \( A_3 = .24(.2) = .048 \)
    5th rectangle: \( A_5 = .74(.2) = .148 \)

    \[ A_L = .004 + .02 + .048 + .088 + .148 = .308. \]
    \[ g = 1 - 2(.308) = .384. \]

6. Gini Index: Denmark U.S. Bolivia
   \[ .324 \quad .652 \quad .782 \]
   The distribution of land is most equal in Denmark, next most equal in the U.S., and least equal in Bolivia.

7. \( g_{used} = .62 \)
   \( g_{produced} = .56 \)
a) Gini Index for one rectangle = .32.

b) Gini Index for five rectangles = .32.
8. (cont.)

\[ r = p^3 \]

a) Gini index for one rectangle = .48.
b) Gini index for five rectangles = .48.
8. (cont.)

\[ r = p^b \]

a) Gini index for one rectangle = .6.
b) Gini index for five rectangles = .5712.
9. 0.245, 0.021

10. Answers should be virtually the same as those obtained in Exercise 7.

11. a) \( \int_0^1 p^n \, dp = \frac{p^{n+1}}{n+1} \bigg|_0^1 = \frac{1}{n+1} \).

\[
g = 1 - 2 \left( \frac{1}{n+1} \right) = \frac{n - 1}{n + 1}.
\]

b) For \( n = 1 \), \( g = \frac{1 - 1}{1 + 1} = \frac{0}{2} = 0 \).

c) \( \lim_{n \to \infty} \frac{n - 1}{n + 1} = \lim_{n \to \infty} \frac{1 - \frac{1}{n}}{1 + \frac{1}{n}} = \frac{1 - 0}{1 + 0} = 1 \).
3.10 Answers to Model Exam

1. The Gini Index is the ratio of the area of inequality to the largest possible area of inequality, i.e., \( g = 2A_I \).

   The smaller the index the more equal the distribution.

   The larger the index the less equal the distribution.

2. b, c, c, b.

3. .404

4. \( \frac{1}{6} = .1667 \). The distribution given by \( r = \frac{1}{2} (p + p^2) \) is more nearly equal than the distribution of question 4, because it has a smaller Gini index.

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