Module 468
Calculus of Variations with Applications in Mechanics
Carroll O. Wilde

Applications of Analysis to Mechanical Engineering and Physics

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Title: CALCULUS OF VARIATIONS WITH APPLICATIONS IN MECHANICS

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Prerequisite Skills:
1. Basic physics (kinetic and potential energy)
2. Multivariate calculus (chain rule)
3. Ordinary differential equations

Output Skills:
1. Explain the concept of the definite integral as a functional.
2. Find Euler equations for various definite integral forms.
3. Describe Hamilton's principle.
4. Apply Hamilton's principle to conservative dynamical systems using Euler equations.

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CALCULUS OF VARIATIONS
WITH APPLICATIONS IN MECHANICS

1. INTRODUCTION

Optimization is important in applied mathematics for a variety of reasons that are related to maximizing "payoffs" or minimizing "penalties." However, there is an even more fundamental reason for this importance in science and engineering, namely that forces of nature tend to act in ways that optimize various physical quantities, and therefore the effects of natural forces can be studied by applying mathematics to optimization problems. We cite (in loose terms) some examples to illustrate the general idea.

1. According to Fermat's principle of least time, a light ray tracking from one point to another requires less time along its actual path than it would along any other path between these points.

2. A mechanical system tends to reach equilibrium at a position in which the system has minimum potential energy.

3. According to Hamilton's principle, the motion of a conservative dynamical system will proceed in such a way as to minimize the average difference between the kinetic energy and the potential energy of the system.

In this module we present a brief introduction to a mathematical theory, called the calculus of variations, that will enable us to optimize integrals, i.e., to find functions that will yield the maximum or the minimum value of certain definite integral forms. We
show how these results apply to physical problems, especially in mechanics, and especially via Hamilton's principle.

2. SOME USEFUL NOTATION

Let \( F = F(x_1, x_2, x_3) \) be a function of three variables, such as

\[
F(x_1, x_2, x_3) = x_1 x_2 + x_3.
\]

If \( x \) is a real variable and \( y \) a differentiable function of \( x \), then the symbol \( F(x, y, y') \) has the usual meaning in terms of functional notation: simply replace \( x_1 \) by \( x \), \( x_2 \) by \( y \) and \( x_3 \) by \( y' \) in the formula for \( F \). For the specific example above, we have

\[
F(x, y, y') = xy' + y'.
\]

A similar notation is used in calculus in connection with implicit functions, when an expression of the form

\[
F(x, y) = \text{const}
\]

defines \( y \) implicitly as a function of \( x \). (The point of view is changed, but the notation has the same meaning.) In differential equations we use expressions such as

\[
y'' = F(x, y, y')
\]

for certain second order equations.

You may recall that in studying implicit functions it was useful to treat both \( x \) and \( y \) as independent variables, and to take partial derivatives with respect to them. For, if an equation of the form

\[
F(x, y) = \text{const}
\]

defines \( y \) implicitly as a function of \( x \) at a given point,
then the derivative of the implicit function is given by

\[ \frac{dy}{dx} = -\frac{F_x}{F_y}, \]

provided that \( F_y \) is nonzero at that point. Thus, for the circle

\[ F(x,y) = x^2 + y^2 = c, \]

we have \( F_x = 2x \) and \( F_y = 2y \). For any point \((x,y)\) on the circle

\[ \frac{dy}{dx} = -\frac{2x}{2y} = \frac{x}{y}, \]

except at the two points for which \( y = 0 \).

In the calculus of variations we carry this idea one step further, and find derivatives of \( F(x,y,y') \) with respect to \( y' \), as well as \( x \) and \( y \). In the example above,

\[ F(x,y,y') = xy^2 + y', \]

the three first partial derivatives of \( F \) are:

\[ F_x = y^2, \quad F_y = 2xy, \quad F_{y'} = 1. \]

It should be easy enough for you to become familiar with the formal procedure for finding these partial derivatives. However, the reason for their importance will not become apparent until we have proceeded further with the development.

In connection with notation, we note that often in applied problems \( x \) is replaced by \( t \) and \( y \) by \( x \). We shall then use the familiar "dot" notation for derivatives with respect to time. Thus, if

\[ F(t,x,\dot{x}) = x\sqrt{1 + \dot{x}^2}, \]

then

\[ F_t = 0, \quad F_x = \sqrt{1 + \dot{x}^2}, \quad F_{\dot{x}} = \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}. \]
Exercise 1. For each of the following forms, find the three first partial derivatives:

a. \( F(x,y,y') = xyy' \);
b. \( F(x,y,y') = \sqrt{y^2 + y'^2} \);
c. \( F(t,x,x) = e^x - te^x \);
d. \( F(t,x,x) = 2x \);
e. \( F(t,y,y) = t^2y \sin y \).

3. INTEGRALS AS FUNCTIONALS

In the calculus of variations it is essential to recognize that, for a given interval \([a,b]\), the definite integral over the interval determines a special type of function in the sense that the integral represents the assignment of a unique number to a given function. The domain consists of all real functions \( y = y(x) \) on \([a,b]\) that have a finite Riemann integral. The range is the set of all real numbers, and the value \( I = I(y) \) that is assigned to the function \( y \) is

\[
I(y) = \frac{b}{a} y \, dx = \int_{a}^{b} y(x) \, dx .
\]

For this type of function, \( x \) is a dummy variable of integration, \( y \) is the independent variable, and \( I \) is the dependent variable. Such a function will be called a functional* here to distinguish it from an ordinary function of a real variable.

* Actually the term "functional" has broader meaning in connection with vector spaces. We are using the term here to describe only one special case of the more general notion of a functional.
One way to grasp the concept of $I$ as a functional is to construct a table of values, as we did when learning the ordinary function concept. Table 1 shows values of the functional $I(y) = \int_0^1 y \, dx$, a special case of $\int_a^b y \, dx$, for several choices of $y$ over the interval $[0,1]$.

**TABLE 1**

Values of $I(y)$ for a Few Choices of $y$ on $[0,1]$

<table>
<thead>
<tr>
<th>$y$</th>
<th>$I(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$x^2$</td>
<td>$1/3$</td>
</tr>
<tr>
<td>$\sqrt{x}$</td>
<td>$2/3$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e-1$</td>
</tr>
<tr>
<td>$\sqrt{1-x^2}$</td>
<td>$\pi/4$</td>
</tr>
</tbody>
</table>

Values are assigned to $y$ by integration; for example, when $y = x^2$ on $[0,1]$, we have

$$I(y) = \int_0^1 x^2 \, dx = 1/3.$$  

Table 1 shows the function concept clearly: the definite integral assigns a unique number to each given function.

For functionals that are used in the calculus of variations we need integrands that also involve derivatives. The simplest use occurs when the functional is of the form

$$I(y) = \int_a^b F(x,y,y') \, dx,$$

where $y$ and $y'$ are continuous on $[a,b]$ and $F$ has continuous first and second partial derivatives.
The notation \( I(y) \) is retained because \( y' \) also depends on \( y \), hence the value of \( I \) is determined by \( y \) alone. We illustrate this type of functional also by constructing a table of values for several choices of \( y \) on the interval \([0,1]\) when the integrand has the form

\[
F(t,y,y') = ty'^2 + \dot{y}.
\]

For example, if \( y = t^2 \), then \( \dot{y} = 2t \), so

\[
I(y) = \int_0^1 (t(t^2)^2 + 2t)dt = \int_0^1 (t^5 + 2t)dt = 7/6.
\]

**TABLE 2**

Values of \( \int_0^1 (ty'^2 + \dot{y})dt \) for a Few Choices of \( y \).
(The Value for \( y = \sqrt{t} \) requires an Improper Integral.)

<table>
<thead>
<tr>
<th>( y )</th>
<th>( I(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>( t )</td>
<td>3/4</td>
</tr>
<tr>
<td>( t^2 )</td>
<td>7/6</td>
</tr>
<tr>
<td>( \sqrt{t} )</td>
<td>4/3</td>
</tr>
<tr>
<td>( e^t )</td>
<td>((e^2 + 4e - 3)/4 )</td>
</tr>
</tbody>
</table>

Table 2 shows values of \( I \) for a few common choices of \( y \). Again, the function concept is apparent: the definite integral of \( F(t,y,y') \) assigns a unique number to a given \( y \).

It may help also to remember that the function concept can be interpreted as an input-output (I/O) device. The function concept shows up clearly in the schematic representation of functionals shown in Figures 1 and 2.
Figure 1. Schematic representation of the functional \( \int_{a}^{b} y \, dx \).

Figure 2. Schematic representation of the functional
\[
I(y) = \int_{a}^{b} F(x, y, y') \, dx
\]

Exercise 2. Construct a table of values for the functional
\[
I(y) = \int_{0}^{1} xy' \, dx
\]
using the following choices of \( y \): 1, \( x \), \( x^2 \), \( \sqrt{x} \), \( e^x \).

4. AN ILLUSTRATIVE EXAMPLE

In this section we present an example to illustrate the way a problem of minimizing a functional of the form
\[
I(y) = \int_{a}^{b} F(x, y, y') \, dx
\]
may arise from physical considerations. This problem is one of the classical problems that provided impetus to the development of the calculus of variations.
Example 1. The Brachistochrone Problem. Let (a,c) and (b,d) be two points in the plane that are situated as shown in Figure 3.

Figure 3. Geometry of the Brachistochrone problem.

Join these points by a curve that is determined by a continuously differentiable function y = y(x). A particle starts from rest at the point (a,c) and moves along the curve to the point (b,d) under the influence of gravity (all other forces are neglected). The problem is to find the curve for which the particle will reach the point (b,d) in minimum time.

To formulate the problem mathematically, we note first that since the speed v at which the particle moves is determined by gravity alone, we must have (assuming that the action takes place at the earth's surface, where the acceleration is g)

\[ g = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}. \]
Thus, we obtain the first order ordinary differential equation

\[ vdv = gdy. \]

with the initial condition \( v(a) = 0 \). Solving this initial value problem, we obtain

\[ v = \sqrt{2g(y-c)}. \tag{1} \]

But the speed \( v \) must also equal the rate of change of arc length along the curve with respect to time, hence

\[ v = \frac{ds}{dt} = \frac{ds}{dx} \frac{dx}{dt} = \frac{\sqrt{1 + y'^2}}{\sqrt{g(y-c)}} \frac{dx}{dt}. \tag{2} \]

Combine (1) and (2) to obtain

\[ \frac{dt}{dx} = \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y-c)}} \]

from which the total time for travel between the two points is

\[ T(y) = \int_a^b \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y-c)}} \, dx. \tag{3} \]

Thus, the basic mathematical problem here is to find the function \( y \) that minimizes the functional

\[ I(y) = \int_a^b \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y-c)}} \, dx \]

and satisfies the boundary conditions \( y(a) = c \), \( y(b) = d \). (Here \( I = \sqrt{2g} \, T \).)

In Section 5 we present a mathematical technique for solving such a minimization problem, and then apply this technique to solve the Brachistochrone problem in Section 6. Before we begin, however, you are invited to formulate a calculus of variations problem yourself.
Exercise 3. In plane geometry we learned that the shortest path between two points is a straight line. Formulate the problem that leads to this result mathematically, as we did for the Brachistochrone problem in Example 3, beginning with two points \((a, c)\) and \((b, d)\) in the plane, with \(a < b\).

5. EULER EQUATIONS: THE SIMPLEST CASE

In elementary and advanced calculus we solve maximum and minimum problems for real functions \(f(x)\), where \(x\) is a real variable or a vector variable, using differentiation, a procedure that has roots in the concept of an increment. In the calculus of variations we maximize or minimize real valued functionals \(I(y)\) using differentiation procedures that are also based on the concept of an increment, but here we must increment the function \(y\), since it is the independent variable of the functional.

The type of increment used in calculus of variations problems is described in the Appendix as part of the derivation of Euler's equation. In this section, we simply describe the result and indicate the way it is applied.

If \(f\) is a differentiable function of a single variable \(x\), then \(f'(x)\) must vanish at any interior point \(x_0\) of its domain that yields a relative extremum. Thus, in maximizing or minimizing a function that has a derivative everywhere, we solve the equation \(f'(x) = 0\) to find critical points, and then find the appropriate maxima and minima. Similarly, if \(f\) is a real function of a vector variable \(x = (x_1, \ldots, x_n)\) that has all its partial derivatives existing everywhere, then each derivative \(\partial f/\partial x_i\) must vanish at any vector \(x_0\) that yields a relative extremum for \(f\). Again, we find
critical values by solving the vector equation \( df = 0 \)
(which is equivalent to solving the system \( \partial f / \partial x_i = 0 \)
for \( i = 1, \ldots, n \)), and then finding the appropriate
maxima and minima.

For a functional of the form

\[
I(y) = \int_a^b F(x, y, y') \, dx ,
\]

which is the simplest case, the condition that corresponds
to "setting the derivative equal to zero" is called
Euler's equation. It has the basic form

\[
F_y - \frac{d}{dx} (F_{y'}) = 0 .
\]

This equation is a differential equation that we must
solve for the "critical points," which are actually
functions. We must then find the appropriate maxima
and minima, making sure that all prescribed conditions
on \( y \) are satisfied. Thus, the overall procedure for
finding extrema of functionals is quite analogous to
the procedures in elementary and advanced calculus.

Equation (4) is derived in the Appendix. We have
chosen to show how this result is applied before
presenting its derivation, as a simple matter of
presentation style. Some of you may well prefer to
study the derivation before proceeding with the text.

There are some simplifications that can sometimes
be made in using Euler's equation. First, if the
integrand \( F(x, y, y') \) does not involve \( y \), then \( F_y = 0 \), so
in this case Equation (4) reduces to the form

\[
\frac{d}{dx} (F_{y'}) = 0 .
\]

Thus, when \( F_y = 0 \) we can carry out the first integration
to obtain the simplified differential equation
where $C$ is a constant.

In addition, an expanded form of Euler's equation can be obtained by applying the chain rule to the term $(d/dx)(F_y')$ in Equation (4). We obtain

$$\frac{d}{dx} (F_y') = F_y' x \frac{dx}{dx} + F_y' y \frac{dy}{dx} + F_y' y' \frac{dy'}{dx}$$

which leads to the equation

$$F_y' y'' + F_y' y' + F_y' y - F_y = 0.$$  

Finally, we obtain one more useful simplification that emerges from Equation (6) when we consider some further results of applying the chain rule. First note that

$$\frac{d}{dx} (F) = F_x + F_y' y' + F_y' y''$$

and

$$\frac{d}{dx} (F_y, y') = F_y' y' + y(F_y' x + F_y' x + F_y' y + F_y' y''').$$

Subtract the first of these results from the second to find

$$\frac{d}{dx} (F_y, y' - F) = y (F_y' y' + F_y' y' + F_y'' x - F_y' - F_x).$$

The expression inside the parentheses on the right side of this last equation equals the left side of Equation (6), so we may also write Euler's equation in the form

$$F_x - \frac{d}{dx} (F - F_y', y') = 0.$$  

Therefore, when $F_x = 0$, we may carry out the first integration to obtain the simplified differential equation

$$F - F_y' y' = C,$$
where C is a constant. This result is particularly useful because in many practical problems, $F_x$ is indeed zero.

In practice we should be prepared to use whichever of Equations (4)-(8) may be appropriate, depending on the particular situation and the intended use. We illustrate the various forms by an example.

**Example 2.** For the functional

$$I(y) = \int_a^b y\sqrt{1 + y'^2} \, dx,$$

we have

$$F(x,y,y') = y\sqrt{1 + y'^2}. $$

Hence

$$F_x = 0, \quad F_y = \sqrt{1 + y'^2}, \quad F_{y'} = \frac{yy'}{\sqrt{1 + y'^2}}. $$

In turn, differentiating $F_{y'}$, we obtain

$$F_{y'y} = 0, \quad F_{y'y} = \frac{y'}{\sqrt{1 + y'^2}}, \quad F_{y'y'} = \frac{y}{(1 + y'^2)^{3/2}}. $$

Thus: Equation (4) becomes

$$\frac{d}{dx} \left( \frac{yy'}{\sqrt{1 + y'^2}} \right) = 0.$$  

Equation (5) is inappropriate since $F_y \neq 0$. Equation (6) becomes

$$\frac{YY''}{(1 + y'^2)^{3/2}} + \frac{y'^2}{\sqrt{1 + y'^2}} - \sqrt{1 + y'^2} = 0,$$

which reduces to

$$yy'' - y'^2 - 1 = 0.$$
when simplified algebraically. Equation (7) becomes
\[ \frac{d}{dx} \left\{ y^{1/2} + y'^2 - \frac{yy'^2}{\sqrt{1 + y'^2}} \right\} = 0. \]

Equation (8) takes the form
\[ y^{1/2} + y'^2 - \frac{yy'^2}{\sqrt{1 + y'^2}} = C, \]
which reduces to
\[ y' = \pm \frac{1}{C} \sqrt{y^2 - C^2} \]
when simplified algebraically.

The final equation above is a separable first order ordinary differential equation, so it seems that Equation (8) (hence also Equation (7)) produced an especially desirable result. But the equation produced by (6), \( yy'' - y' - 1 = 0 \), can also be solved conveniently via the standard substitution \( y' = p \), with \( y'' = p \frac{dp}{dy} \). We note only that the solution has the form of a catenary curve \( y = c_1 \cosh \left( \frac{x}{c_1} + c_2 \right) \).

**Exercise 4.** Apply all the appropriate forms of Euler's equation to the functional
\[ I(y) = \int_{0}^{\pi/2} (y^2 - y'^2) \, dx. \]

---

**6. SOLUTION OF THE BRACHISTOCHRONO PROBLEM**

In Section 4 we formulated the Brachistochrone problem. We now apply Euler's equation to obtain the solution.
Example 3. Solution of the Brachistochrone Problem. The basic problem is to minimize the functional

$$I(y) = \int_a^b \frac{\sqrt{1 + y'^2}}{\sqrt{y - c}} \, dx,$$

with boundary conditions $y(a) = b, y(c) = d$. We have

$$F(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{y - c}},$$

hence

$$F_x = 0 \quad \text{and} \quad F_y' = \frac{y'}{(y - c) (1 + y'^2)}.$$

Thus, we may use Equation (8) to obtain

$$\frac{\sqrt{1 + y'^2}}{\sqrt{y - c}} \cdot \frac{y'^2}{\sqrt{(y - c) (1 + y'^2)}} = C.$$

Simplifying algebraically, we reach the form

$$C^2 (y - c) (1 + y'^2) = 1,$$

and then the form

$$y' = \pm \sqrt{\frac{1 - C^2 (y - c)}{C^2 (y - c)}}.$$

This differential equation can be solved by using the substitution

$$y - c = \frac{1}{C^2} \sin^2(\theta/2),$$

where $\theta$ is a parameter. We find

$$y' = \pm \sqrt{\frac{1 - \sin^2(\theta/2)}{\sin^2(\theta/2)}} \cdot \frac{\cos(\theta/2)}{\sin(\theta/2)}.$$
where we have taken the positive square root to match the geometry of the problem. Since
\[
\frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx} = \left( \frac{1}{C^2} \sin(\theta/2)\cos(\theta/2) \right) \frac{d\theta}{dx}
\]
and
\[
\frac{dy}{dx} = \frac{\cos(\theta/2)}{\sin(\theta/2)},
\]
we have
\[
dx = \frac{1}{C^2} \sin^2(\theta/2),
\]
from which
\[
x = \frac{1}{2C^2} (\theta - \sin\theta) + k.
\]
We evaluate k using the condition \(x = a\) when \(\theta = 0\). We find that \(k = a\), and if we let \(\frac{1}{2C^2} = A\), we obtain the following parametric representation of the solution:
\[
x = A(\theta - \sin\theta) + a \\
y = A(1 - \cos\theta) + c
\]
Thus, the minimizing curve is a cycloid. The constant \(A\) is determined by the condition \(y = d\) when \(x = b\), which leads to the pair of transcendental equations.
\[
A(\theta - \sin\theta) = b - a \\
A(1 - \cos\theta) = d - c.
\]
To interpret the solution geometrically, we note from the expression
\[
y' = \frac{\cos(\theta/2)}{\sin(\theta/2)}
\]
that \(y' = 0\) when \(\theta = \pi\). Hence the cycloid has its minimum at the point
\[
(a + \pi A, c + 2A),
\]
which lies on the line $L$ that passes through the point $(a,c)$ with slope $2/\pi$. If the endpoint $(b,d)$ lies below the line $L$, the particle will reach the endpoint while $y'$ is still positive. If $(b,d)$ lies on $L$, the particle will reach the endpoint at the bottom of the cycloid arc. If $(b,d)$ lies above $L$, the particle will reach the end point while traveling upward after reaching the bottom of the arc. (See Figure 4.)

![Diagram showing minimum time paths, depending on the location of the point $(b,d)$ relative to the line $L$.](image)

Figure 4. Minimum time paths, depending on the location of the point $(b,d)$ relative to the line $L$.

We note that the analysis is not really complete, because Euler's condition is only a necessary condition for a minimizing function. On physical grounds it does seem reasonable that a minimizing curve should exist, but even then we would need to know that the differential equation that produced the cycloid has a unique solution. While we are giving you an idea of what the calculus of variations is all about, we are leaving unanswered many questions that require a more comprehensive treatment of the subject.
7. EULER EQUATIONS FOR OTHER FUNCTIONALS

In the applications we present in Section 8, the functionals involved have a more complicated form than the simplest case, in which \( I(y) = \int_a^b F(x,y,y') \, dx \). We therefore consider functionals in which:

(i) the dummy variable \( x \) (or \( t \)) is replaced by a vector, with the integration over a region \( R \) in a space of appropriate dimension;

(ii) the independent variable of the functional, \( y \) (or perhaps \( x \) when \( t \) is the dummy variable), is replaced by a vector of functions, i.e., we have parametric equations;

(iii) \( y' \) (or \( \dot{x} \)) is replaced by a vector of first partial derivatives.

Examples of such functionals are

\[
\int_a^b F(x,y,z,y',z') \, dx, \text{ or } \int_a^b F(t,x,y,\dot{x}, \dot{y}) \, dt ;
\]

\[
\iint_R F(x,y,z,z_x,z_y) \, dy \, dx, \quad \text{or } \iiint_R F(s,t,x,x_s,x_t) \, dt \, ds ;
\]
\[ \int_{\mathbb{R}} F(x,y,u,v,u_x, u_y, v_x, v_y) \, dy \, dx, \]

or \[ \int_{\mathbb{R}} \int_{\mathbb{R}} F(s,t,x,y,x_s, x_t, y_s, y_t) \, ds \, dt. \]

In each case we can identify three sets of variables, the first representing the dummy variable of integration, the second the independent variable of the functional, and the third the derivative. The analysis required to obtain the appropriate Euler equation(s) becomes more complicated, but the underlying principles are much the same. In particular, the concept of an increment is still fundamental, and the chain rule and Leibniz's rule are useful in the development (see Appendix). The results are quite similar to the results for the simplest case, and we present a sample of them in Table 3.

There are two patterns that emerge from Table 3. First of all, there are as many equations as there are independent variables of the functional. Second, the number of terms that are differentiated equals the dimension of the dummy variable vector. Use the next exercise to check that you have the pattern.

**Exercise 6.** Find the Euler equations for the functionals:

a. \[ \int_{\mathbb{R}} \int_{\mathbb{R}} F(x,y,u,v,u_x, u_y, v_x, v_y) \, dy \, dx; \]

b. \[ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} F(x,y,z,u,v,u_x, u_y, u_z, v_x, v_y, v_z) \, dz \, dy \, dx. \]
### Table 3

#### Euler Equations for Selected Functional Forms

<table>
<thead>
<tr>
<th>Form of the Functional</th>
<th>Euler Equation(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_{a}^{b} F(x,y,y') , dx$</td>
<td>$F_y - \frac{d}{dx} (F_y) = 0$</td>
</tr>
<tr>
<td>$\int_{a}^{b} F(t,x,y,\dot{x},\dot{y}) , dt$</td>
<td>$\begin{cases} F_x - \frac{d}{dt} (F_x) = 0 \ F_y - \frac{d}{dt} (F_y) = 0 \ F_z - \frac{d}{dt} (F_z) = 0 \end{cases}$</td>
</tr>
<tr>
<td>$\int_{a}^{b} F(t,x,y,z,\dot{x},\dot{y},\dot{z}) , dt$</td>
<td>$\begin{cases} F_x - \frac{d}{dt} (F_x) = 0 \ F_y - \frac{d}{dt} (F_y) = 0 \ F_z - \frac{d}{dt} (F_z) = 0 \end{cases}$</td>
</tr>
<tr>
<td>$\int_{R} F(x,y,u,u_x,u_y) , dydx$</td>
<td>$F_u - \left[ \frac{2}{\partial x} (F_{u_x}) + \frac{2}{\partial y} (F_{u_y}) \right] = 0$</td>
</tr>
<tr>
<td>$\int_{R} F(x,y,u,v,u_x,u_y,v_x,v_y) , dydx$</td>
<td>$\begin{cases} F_u - \left[ \frac{2}{\partial x} (F_{u_x}) + \frac{3}{\partial y} (F_{u_y}) \right] = 0 \ F_v - \left[ \frac{2}{\partial x} (F_{v_x}) + \frac{3}{\partial y} (F_{v_y}) \right] = 0 \end{cases}$</td>
</tr>
</tbody>
</table>

In the last two formulas the symbol $\partial / \partial x$ indicates a partial differentiation in which $y$ is held fixed but the remaining variables are regarded as functions of $x$ and $y$. Thus, in the first of these two formulas we have

$$\frac{\partial}{\partial x} (F_{u_x}) = F_{u_x} + F_{u_x} u_x + F_{u_x} u_{xx} + F_{u_x} u_{xy} + F_{u_x} u_{y},$$

and in the second

$$\frac{\partial}{\partial x} (F_{u_x}) = F_{u_x} + F_{u_x} u_x + F_{u_x} v_x + F_{u_x} u_{xx} + F_{u_x} u_{yy} + F_{u_x} u_{xy} + F_{u_x} v_{xy}.$$  

A corresponding interpretation holds for the symbol $\partial / \partial y$.  

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8. APPLICATIONS IN MECHANICS

Hamilton's principle* roughly asserts that the motion of a conservative system will proceed in a way that minimizes the average difference between the kinetic energy and the potential energy. If we denote the kinetic energy by $T$ and the potential by $V$, then between the time points $t = a$ and $t = b$ the system will proceed so that the functional

$$\int_{a}^{b} (T - V) \, dt$$

is minimized over the set of all curves between two given points, provided that $b - a$ is not "too large." The integrand $T - V$ is called the Lagrangian of the system, and is often denoted by $L$. The variables involved in $L$ will depend on the particular physical situation. We illustrate by examples.

Example 4. Consider the (undamped, undriven) simple, harmonic oscillator illustrated in Figure 5. At time $t = 0$, the (frictionless) mass is set in

![Diagram of a simple harmonic oscillator]

Figure 5. A simple harmonic oscillator.

* See any of the books listed as references for more insight into Hamilton's principle.
motion by an initial displacement from rest position and/or an initial velocity, which are assumed to be known. Let \( x = x(t) \) be the displacement of the mass from rest position (say in centimeters). We wish to find the equation of motion using Hamilton’s principle.

If the masses of the spring and the connecting rods are negligible by comparison with \( m \), then the kinetic energy of the system at any instant of time is given by

\[
T = \frac{1}{2}m\dot{x}^2.
\]

The potential energy at any time is the energy stored in the spring, and its value must equal the work done in stretching or compressing the spring from its natural length to its length at that time. If we assume that the displacements are small so that Hooke’s Law applies and we have linear stress-strain relations, then the work is given by

\[
W = \int_0^x kds = \frac{1}{2}kx^2.
\]

Thus, we have

\[
L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.
\]

By Hamilton’s principle, the motion will proceed between any time points \( a \) and \( b \) in such a way as to minimize the functional

\[
\int_a^b L dt = \left[ \int_a^b \left( \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right) dt \right].
\]

The integrand has the form

\[
L(t,x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,
\]

and the Euler equation for this form,

\[
l_x - \frac{d}{dt} (l_{\dot{x}}) = 0,
\]
becomes

\[-kx - \frac{d}{dt}(mx) = 0,\]

or

\[m\ddot{x} + kx = 0.\]

Thus, Euler's equation is equivalent to the familiar equation that governs the motion of a simple harmonic oscillator, which may be solved by elementary techniques of ordinary differential equations.

We note that the situation in Example 4 represents a slight variation from the preceding discussion in that we obtained an initial value problem with \(x(0)\) and \(\dot{x}(0)\) prescribed, rather than a boundary value problem with \(x(a)\) and \(x(b)\) prescribed. Hamilton's principle still yields the desired results.

Example 5. Suppose now that the harmonic oscillator is arranged vertically, and that at the equilibrium position the mass stretches the spring a small distance \(d\) by its weight alone, as shown in Figure 6. Let \(u = u(t)\) be

![Diagram](image)

Figure 6. Simple harmonic oscillator of Example 5.
the displacement from the zero stress position, and
\( x = x(t) \) the displacement from equilibrium. We apply
Hamilton's principle to both \( x \) and \( u \) to find the equa-
tions of motion, and we also reconcile these equations.

If we use \( x \), then expressions for the energy are
found relative to the equilibrium position, and we have
\[
L = T - V = \frac{1}{2}m\dddot{x}^2 - \frac{1}{2}kx^2.
\]
from which Euler's equation becomes
\[
-kx - \frac{d}{dt}(m\dddot{x}) = 0,
\]
or
\[
m\dddot{x} + kx = 0,
\]
which is the equation found in Example 4.

If we use \( u \), the kinetic energy will be
\[
T = \frac{1}{2}m\dot{u}^2,
\]
but the potential energy becomes,
\[
V = \frac{1}{2}ku^2 - mg,
\]
to account for the original shift. Thus, in this case
\[
L = \frac{1}{2}m\dot{u}^2 - \frac{1}{2}ku^2 + mgu,
\]
from which Euler's equation becomes
\[
-ku + mg - \frac{d}{dt}(m\dot{u}) = 0,
\]
or
\[
m\ddot{u} + ku = mg.
\]

This equation yields the same harmonic motion with
the value \( u(t) \) shifted (translated) from the value of
\( x(t) \) by an amount \( mg/k \). We can reconcile this shift
by observing that if \( W \) is the weight \( W = mg \), then by 
Hooke's law the weight (or force) must satisfy \( W = kd \), 
so that \( kd = mg \), and \( d = mg/k \), which is the difference 
we found between \( u \) and \( x \) at any time \( t \).

**Example 6.** Suppose now that we have a coupled 
harmonic oscillator system of springs and masses, 
as illustrated in Figure 7. The Lagrangian of this 
system is given by

\[
L = T - V = \left( \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \right) - \left( \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_3x_2^2 \right).
\]

In this case we wish to minimize a functional of the 
form

\[
\int_a^b L(t,x_1,x_2,\dot{x}_1,\dot{x}_2) \, dt
\]

between any two time points \( a \) and \( b \). We therefore have 
two Euler equations

\[
(-k_1x_1 + k_2(x_2 - x_1)) - \frac{d}{dt}(m_1\dot{x}_1) = 0
\]

and

\[
(-k_2(x_2 - x_1) - k_3x_2) - \frac{d}{dt}(m_2\dot{x}_2) = 0.
\]

Simplifying these expressions, we obtain a system of 
equations that governs the motion:

\[
m_1\ddot{x}_1 = -(k_1 + k_2)x_1 + k_2x_2
\]

\[
m_2\ddot{x}_2 = k_2x_1 - (k_2 + k_3)x_2.
\]

This system of equations can be solved for \( x_1 \) and 
\( x_2 \) by standard matrix methods for systems of differential 
equations.
Example 7. Consider the spring-mass-pulley system shown in Figure 8. The pulley has radius R, mass M and moment of inertia I. We assume that the total weight involved stretches the spring only a small amount from its natural length (as was the case in Example 5). The pulley is displaced from zero-stress.
(where $y = y_0$) and/or given an initial velocity at
time $t = 0$. In addition, we take $m_1 > m_2$, raise the
mass $m_1$ to full height and release it at time $t = 0$
to fall under the force of gravity. The problem is
to find equations that govern the motion of the system.

The kinetic energy of the system is given by
\[ T = \frac{1}{2}m_1(\dot{y} + \dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y} - \dot{y}_1)^2 + \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}I(\frac{\dot{y}_1}{R})^2, \]
and the potential energy by
\[ V = \frac{1}{2}k(y - y_0)^2 + (m_1 - m_2)g(d - y_1)
- (m_1 + m_2 + M)g(y - y_0). \]

Then
\[ T_y = m_1(\dot{y} + \dot{y}_1) + m_2(\dot{y} - \dot{y}_1) + \frac{1}{2}m_1\dot{y}_1 \]
\[ = (m_1 + m_2 + M)\dot{y} + (m_1 - m_2)\dot{y}_1 \]
and
\[ T_{y_1} = m_1(\dot{y} + \dot{y}_1) - m_2(\dot{y} - \dot{y}_1) + (I/R^2)\dot{y}_1 \]
\[ = (m_1 - m_2)\dot{y} + (m_1 + m_2 + I/R^2)\dot{y}_1. \]

If we introduce the notation
\[ A = m_1 + m_2 + M, \ B = m_1 + m_2 + I/R^2, \ C = m_1 - m_2, \]
we obtain the simplified expressions
\[ T_y = Ay + C\dot{y}_1, \]
\[ T_{y_1} = C\dot{y} + B\dot{y}_1, \]
and the expression for $V$ becomes
\[ V = \frac{1}{2}k(y - y_0)^2 + Cg(d - y_1) - Ag(y - y_0). \]
Since
\[-V_y = -k(y - y_0) + Ag\]
and
\[-V_{y_1} = Cg,\]
we obtain the Euler equations
\[-k(y - y_0) + Ag - \frac{d}{dt}(Ay' + Cy') = 0\]
\[Cg - \frac{d}{dt}(Cy' + By') = 0.\]

These equations simplify to the form
\[Ay' + Cy' = -k(y - y_0) + Ag,\]
\[Cy' + By' = Cg.\]

By an easy application of Cramer's rule, we obtain the system
\[
\ddot{y} = -\frac{k\theta}{AC - BT}(y - y_0) + g, \\
Cy' + By' = Cg,
\]
which we can readily solve for \(y\) and \(y_1\).

Exercise 7. Apply Hamilton's principle and the calculus of variations to find the differential equations that govern the spring-mass system shown in Figure 9, where \(x_1\) and \(x_2\) represent displacement from equilibrium.

Figure 9. Spring-mass system for Exercise 7.
9. WHAT WE DIDN'T SAY

This module represents only a modest introduction to the calculus of variations and its applications, and we have barely scratched the surface. For example, we have not even considered constrained extremes for functionals, a topic that encompasses the so-called Sturm-Liouville problems, among others. Nor have we considered applications for functionals whose Euler equations are partial differential equations. In addition, we have not considered the concept of the variation of a functional \( \delta (I) \), and the properties that show why we call the theory a "calculus," such as rules for differentiating sums and products: \( \delta (I + J) = \delta (I) + \delta (J) \) and \( \delta (IJ) = I \delta (J) + J \delta (I) \). For further study of this fascinating subject, the books listed as references for this module represent an excellent way to go from here.
10. MODEL EXAMINATION

1. Find all appropriate forms of Euler's equation for the functional

\[ I(y) = \int_{0}^{1} (xy' + y''^2) \, dx. \]

(Refer to the forms (4)-(8).)

2. Find Euler's equation for the functional

\[ I(\phi) = \int_{0}^{1} \int_{0}^{1} (\phi_x^2 + \phi_y^2) \, dy \, dx. \]

(You should first identify the form of the integral.)

3. Carry out the complete calculus of variations problem to find the only function that could yield an extremum for the functional

\[ I(y) = \int_{0}^{1} (y''^2 + 12xy) \, dx \]

and satisfy the conditions \( y(0) = 0, \ y(1) = 1. \)

4. Find the system of differential equations that governs the motion of the mechanical system of coupled oscillators illustrated in the diagram.

![Diagram of coupled oscillators](image-url)
11. ANSWERS TO EXERCISES

1. a. \( y y', xy', xy; \)
   b. 0, \( \sqrt{y^2 + y'^2}, y'/y^2 + y'^2; \)
   c. \( e^x - e^t, e^t, -te^t; \)
   d. 0, 2, 0;
   e. \( 2ty \sin \gamma, t^2 \sin \gamma, t^2y \cos \gamma. \)

2. \[
\begin{array}{|c|c|c|c|c|}
\hline
y & 1 & x & x^2 & \sqrt{x} & e^x \\
\hline
I(y) & 0 & 1/3 & 2/5 & 1/4 & (e^2 + 1)/4 \\
\hline
\end{array}
\]

3. **Minimize the Functional**

\[
I(y) = \int_a^b \sqrt{1 + y'^2} \, dx,
\]

subject to the conditions \( y(a) = c, y(b) = d. \)

4. (4) \( 2y - \frac{d}{dx}(-2y') = 0, \) or \( y'' + y = 0 \)

   (5) inappropriate

   (6) \( (-2y')y' - 2y = 0, \) or \( y'' + y = 0 \)

   (8) \( (y^2 - y'^2) - (-2y'y') = 0, \) or \( y'' + y = 0 \)

5. \( y = \frac{d}{b - a}(x - a) + c. \)

6. a. \( F_u - \left[ \frac{\partial}{\partial x}(F_{ux}) + \frac{\partial}{\partial y}(F_{uy}) \right] = 0 \)
   
\( F_v - \left[ \frac{\partial}{\partial x}(F_{vx}) + \frac{\partial}{\partial y}(F_{vy}) \right] = 0 \)

\( F_w - \left[ \frac{\partial}{\partial x}(F_{wx}) + \frac{\partial}{\partial y}(F_{wy}) \right] = 0 \)

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6. \( F_u = \mu \frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_y + \frac{\partial}{\partial z} u_z \) = 0

\( F_v = \mu \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z \) = 0

7. The energy expressions are

\( T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2} I \dot{y}_1^2 \)

\( V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 \).

The Euler equations are

\[ (m_1 + m_2 + \frac{1}{2} k_2) \ddot{x}_1 + m_2 \ddot{x}_2 + k_1 x_1 = 0 \]

\[ m_2 \ddot{x}_1 + m_2 \ddot{x}_2 + k_2 \ddot{x}_2 = 0. \]

12. ANSWERS TO MODEL EXAM

1. \(\frac{d}{dx}(x + 2y') = 0\), or \(1 + 2y'' = 0\)

2. \(\dot{y} - \frac{d}{dx} (xy' + y'^2) - (xy'' + 2y'y') = 0\), or \(y'(1 + 2y'') = 0\).

3. \(\text{Inappropriate.}\)

4. \(\begin{align*}
\dot{x}_1 &= -(k_1 + k_2) x_1 + k_2 x_2 \\
\dot{x}_2 &= k_2 x_1 - (k_2 + k_3) x_2 + k_3 x_3 \\
\dot{x}_3 &= k_3 x_2 - (k_3 + k_4) x_3.
\end{align*}\)
13. APPENDIX: DERIVATION OF EULER EQUATIONS

Here we derive Euler's equation for the simplest case, that in which the functional \( I(y) \) has the form

\[
I(y) = \int_a^b F(x,y,y') \, dx ,
\]

and in which \( y \) must satisfy the conditions \( y(a) = c \)
and \( y(b) = d \). We assume that \( F \) has continuous first
partial derivatives with respect to each of its variables.

Suppose that a minimizing (or a maximizing)
function does indeed exist, and denote it by \( \bar{y} \) (so that
\( \bar{y} \) is analogous to a critical point in calculus). The
increment \( \delta \bar{y} \) is formed by choosing any continuously
differentiable function \( \eta = \eta(x) \) that vanishes at both
\( a \) and \( b \), and multiplying \( \eta \) by a (small) real number \( \varepsilon \),
as shown in Figure 10. (For example, we could choose
\( \eta(x) = (x - a)(x - b) \).)

The functions \( \eta \) and \( \bar{y} \) remain fixed, and \( \varepsilon \) is
allowed to vary. The new function \( \bar{y} + \delta \bar{y} \), or \( \bar{y} + \varepsilon \eta \),
also yields a value of \( I \), which is formed by substituting
\( \bar{y} + \delta \bar{y} = \bar{y} + \varepsilon \eta \) for \( \bar{y} \), and the derivative
\( (\bar{y} + \delta \bar{y})' = \bar{y}' + \varepsilon \eta' \) for \( \bar{y} \). This procedure determines a real function
of \( \varepsilon \), i.e.,

\[
f(\varepsilon) = \int_a^b F(x,\bar{y} + \varepsilon \eta, \bar{y}' + \varepsilon \eta') \, dx ,
\]

since both \( \bar{y} = \bar{y}(x) \) and \( \eta = \eta(x) \) are fixed functions of
\( x \), and \( x \) is a dummy variable that is replaced by the
constants \( a \) and \( b \) in the integration.
The function $f(\varepsilon)$ is a real function that attains its minimum (or its maximum) at $\varepsilon = 0$, so if the derivative $f'(\varepsilon)$ exists, it must vanish at $\varepsilon = 0$. But since $F$, $\bar{y}$ and $\eta$ all have continuous first derivatives, we may apply Leibnitz's rule for differentiating under the integral to obtain

$$f'(\varepsilon) = \frac{d}{d\varepsilon} \int_{\alpha}^{b} F(x, \bar{y} + \varepsilon \eta(x), \bar{y}'(c) + \varepsilon \eta'(c)) \, dx$$

$$= \int_{\alpha}^{b} \frac{d}{d\varepsilon} [F(x, \bar{y} + \varepsilon \eta(x), \bar{y}'(x) + \varepsilon \eta'(x))] \, dx .$$

The function to be differentiated is a function of $\varepsilon$ and $x$ that is in fact a composite function

$$F(x,y,y'),$$

where

$$\begin{align*}
  x &= x \\
  y &= \bar{y}(x) + \varepsilon \eta(x) \\
  y' &= \bar{y}'(x) + \varepsilon \eta'(x)
\end{align*}$$

To differentiate this function with respect to $\varepsilon$, hold $x$ fixed and apply the chain rule:

$$\frac{\partial F}{\partial \varepsilon} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \varepsilon} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial \varepsilon} .$$
The first term is zero since x does not involve \( \epsilon \). Thus we find \( \partial F \partial \epsilon \) as a function of x and \( \epsilon \) to be

\[
\frac{\partial F}{\partial \epsilon} = F_x(x, \bar{y}(x)) + \epsilon \eta(x), \bar{y}'(x) + \epsilon \eta'(x))n(x) \\
+ F_y(x, \bar{y}(x)) + \epsilon \eta(x), \bar{y}'(x) + \epsilon \eta'(x))n'(x),
\]

Now substitute this result into the expression for \( f'(\epsilon) \) above to find

\[
f'(\epsilon) = \int_a^b [F_x(x, \bar{y}(x)) + \epsilon \eta(x), \bar{y}'(x) + \epsilon \eta'(x))n(x) \\
+ F_y(x, \bar{y}(x)) + \epsilon \eta(x), \bar{y}'(x) + \epsilon \eta'(x))n'(x)] \, dx.
\]

Since \( f \) has a minimum at \( \epsilon = 0 \), the derivative must vanish there, i.e., we must have \( f'(0) = 0 \):

\[
f'(0) = \int_a^b [F_x(x, \bar{y}(x), \bar{y}'(x))n(x) \\
+ F_y(x, \bar{y}(x), \bar{y}'(x))n'(x)] \, dx = 0.
\]

Thus, any minimizing function \( \bar{y} \) must satisfy the condition

\[
\int_a^b (F_x(x, \bar{y}, \bar{y}') + F_y(x, \bar{y}, \bar{y}') \eta') \, dx = 0
\]

which we abbreviate further to the form

\[
\int_a^b (F_x + F_y \eta') \, dx = 0.
\]

The second term in the last integrand can be integrated by parts. As a result we obtain the condition.

\[
\int_a^b (F_y - \frac{d}{dx}(F_y')) \eta \, dx + [F_y \eta]_a^b = 0.
\]

The integrated term vanishes since \( \eta(a) = \eta(b) = 0 \), so we obtain

\[
\int_a^b (F_y - \frac{d}{dx}(F_y')) \eta \, dx = 0.
\]
This condition must hold for all functions n that are continuously differentiable and satisfy \( n(a) = n(b) = 0 \), hence the factor inside the parentheses in the integrand must equal zero. Thus, any minimizing (or maximizing) function \( \bar{y} \) must be a solution of the differential equation

\[
F_y - \frac{d}{dx}F_{y'} = 0.
\]

This is Euler's equation for the functional \( I(y) = \int_a^b F(x,y,y') \, dx \), which we listed as Equation (4).