Radioactive Chains: Parents and Children

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Calculus

Physics

Students in a second-semester calculus course, or in a calculus-based physics course

We set up and solve the equations for radioactive decay, graph the solution functions, and discuss equilibrium and approximation.

1. Ability to integrate \( \int_0^1 f'(t) \, dt \).
2. Ability to use first and second derivatives as aids in graphing functions.
3. Knowledge of \( \lim_{t \to \infty} e^{-kt} \) (k positive).

1. Know the equations governing radioactive chains.
2. Know the meaning of the terms transient equilibrium and secular equilibrium.
3. Know the approximations relevant to transient and secular equilibrium, and know the circumstances under which they are applicable.

Unit 232: Kinetics of Single Reactant Reactions, by Brindell Horelick and Sinan Koont.
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Parents and Children

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MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS (UMAP) PROJECT

The goal of UMAP was to develop, through a community of users and
developers, a system of instructional modules in undergraduate mathematics
and its applications to be used to supplement existing courses and from
which complete courses may eventually be built.

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1. Introduction

1.1 Radioactive Decay

Radioactive decay is a first-order reaction. This means that if a radioactive substance is not being replenished in any way, then its amount (number of atoms) \( N(t) \) decreases at a rate proportional to that amount:

\[
N'(t) = -\lambda N(t), \tag{1}
\]

where \( \lambda \) is a positive constant known as the disintegration constant or decay constant.

The elementary consequences of (1) are discussed in many elementary calculus textbooks. In Horelick and Koontz 1979] we discuss first-order reactions in greater detail. In that unit you can discover how experimenters determined empirically that radioactive decay is a first-order process, and what this suggests about the mechanism of radioactivity.

1.2 Chains

When a radioactive substance \( A \) decays into a substance \( B \), \( A \) and \( B \) are referred to as the parent and the child, respectively. It may happen that \( B \) itself is radioactive and is the parent of a new child \( C \), and so on. In fact, this is a very common situation. There are three chains like this, beginning respectively with \( U^{238} \) (the uranium series), \( U^{235} \) (the actinium series), and \( Th^{232} \) (the thorium series), whose lengths are 19, 17, and 13. They do not overlap and together account for all naturally-occurring radioactive substances beyond thallium (atomic number 81) on the periodic table. Each of these chains ends with a stable (non-radioactive) form of lead.

2. Setting Up the Equations

2.1 Notation and Assumptions

We shall consider the relationship between one parent \( A \) and its radioactive child \( B \). We shall write \( N_i(t) \) and \( N_\lambda(t) \) for their
amounts at time $t$, and $\lambda_1$ and $\lambda_2$ for their decay constants. Figure 1 may help you remember this notation. Since the rate of decay of $B$ into $C$ depends only on the amount of $B$ present, and not on the amount of $C$, we do not care whether $C$ is stable or radioactive.

Now imagine that at the instant $t = 0$ we have a freshly prepared amount $N_0$ of $A$, and none of $B$. That is, $N_A(0) = N_0$ and $N_B(0) = 0$. Imagine that the chain of reactions in Figure 1 then proceeds without external interference.

2.2 The Equations

Since $A$ is not being replenished, (1) applies directly, and we have

$$N_A'(t) = -\lambda_1 N_A(t).$$

If $B$ were not being replenished, (1) would apply again, and $N_B(t)$ would be changing at the rate $-\lambda_2 N_B(t)$. But $B$ is being replenished. Each atom of $A$ that decays becomes an atom of $B$, and this is happening at the rate $\lambda_1 N_A(t)$. So altogether we have

$$N_B'(t) = \lambda_1 N_A(t) - \lambda_2 N_B(t).$$

We are confronted with the following system of equations:

\begin{align*}
N_A'(t) &= -\lambda_1 N_A(t) \\
N_B'(t) &= \lambda_1 N_A(t) - \lambda_2 N_B(t) \\
N_A(0) &= N_0 \\
N_B(0) &= 0.
\end{align*}
3. Solving the Equations

3.1 Solving for $N_1(t)$

It is fairly straightforward to solve (2) for $N_1(t)$. This was done in [Horelick and Koont 1979] and is probably done in your calculus textbook. We just divide through by $N_1(t)$ and then integrate from 0 to $t$:

\[
\int_0^t \frac{N_1'(t)}{N_1(t)} \, dt = - \int_0^t \lambda_1 \, dt.
\]

This leads to the equation:

\[
\ln(N_1(t)) + \ln(N_1(0)) = -\lambda_1 t,
\]

or

\[
\ln\left(\frac{N_1(t)}{N_0}\right) = -\lambda_1 t,
\]

since $N_1(0) = N_0$. The usual absolute-value signs are not needed, because the quantities involved are positive. Finally,

\[N_1(t) = N_0 e^{-\lambda_1 t}.\] (4)

Exercise 1.

Find a relationship between $\lambda_1$ and the half-life of $A$ (the half-life is the time $t^*$ at which $N_1(t^*) = \frac{1}{2}N_0$). (For more about half-life, see [Horelick and Koont 1979].)

3.2 Solving for $N_2(t)$

Finding $N_2(t)$ is a bit more tricky. Applying (4) to (3) we get

\[N_2'(t) = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N_2(t).\] (5)

Equation (5) probably looks quite different from any you have seen before. Let's try to make a shrewd guess what kind of solution it has. It says that the derivative of $N_2(t)$ is the sum of two terms, $\lambda_1 N_0 e^{-\lambda_1 t}$ and $-\lambda_2 N_2(t)$. With luck, this might remind us of the
product rule:

\[ N_2(t) = u(t) \cdot v(t) \]

then \[ N_2'(t) = u(t) \cdot v'(t) + v(t) \cdot u'(t). \]  

Can we pick \( u(t) \) and \( v(t) \), so the terms in (6) match up with the terms in (5)? In other words, can we pick \( u(t) \) and \( v(t) \) so that

\[ u(t) \cdot v'(t) = \lambda_1 N_0 e^{-\lambda_1 t} \]  

(7)

and

\[ v(t) \cdot u'(t) = -\lambda_2 N_2(t). \]  

(8)

Since \( N_2(t) = u(t) \cdot v(t) \), (8) can be rewritten

\[ v(t) \cdot u'(t) = -\lambda_2 u(t) v(t) \]

and we are in business! The \( v(t) \) factors cancel out, leaving us with

\[ u'(t) = -\lambda_2 u(t) \]

which looks very much like (2) and can be solved in the same way. First,

\[ \int_0^t \frac{u'(t)}{u(t)} dt = -\int_0^t \lambda_2 dt. \]

Then, writing \( R = u(0) \),

\[ \ln \left( \frac{u(t)}{R} \right) = -\lambda_2 t \]

\[ u(t) = Re^{-\lambda_2 t}. \]

Putting this into (7) gives

\[ Re^{-\lambda_2 t} v'(t) = \lambda_1 N_0 e^{-\lambda_1 t} \]

\[ v'(t) = \frac{\lambda_1 N_0 e^{\lambda_2 - \lambda_1} v(t)}{R}. \]

If \( \lambda_1 = \lambda_2 \) we feel confident you can complete this solution yourself (see Exercise 2).
If \( \lambda_1 \neq \lambda_2 \), then \( \lambda_2 - \lambda_1 \neq 0 \) and we can write

\[
v(t) = \frac{\lambda_1 N_0}{R(\lambda_2 - \lambda_1)} e^{(\lambda_2 - \lambda_1)t} + K,
\]

where \( K \) is the constant of integration. Then

\[
N_2(t) = u(t) \cdot v(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + KR e^{-\lambda_2 t}.
\]

Using the fact that \( N_2(0) = 0 \), we get

\[
0 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} + KR
\]

\[
KR = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}
\]

\[
N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}).
\]

(9)

**Exercises**

2. Find \( N_2(t) \) if \( \lambda_1 = \lambda_2 \).

3. Assuming \( C \) is stable, find the time at which the total radioactivity (i.e., the total number of disintegrations of \( A \)-atoms and \( B \)-atoms per unit time) is greatest.

### 3.3 Graphs of \( N_1(t) \) and \( N_2(t) \)

It is easy to confirm from (4) that \( N_1'(t) < 0 \) for all \( t \), that \( N_1''(t) > 0 \) for all \( t \), and that \( \lim_{t \to \infty} N_1(t) = 0 \). With a little more work (see Exercise 4) it can be confirmed that \( \lim_{t \to \infty} N_2(t) = 0 \) and that

\[
N_2'(t) = \begin{cases} 
> 0 & \text{if } t < t_0 \\
= 0 & \text{if } t = t_0 \\
< 0 & \text{if } t > t_0
\end{cases}
\]

(10)
and

\[
N_2''(t) = \begin{cases} 
< 0 & \text{if } t < 2t_0 \\
0 & \text{if } t = 2t_0 \\
> 0 & \text{if } t > 2t_0,
\end{cases}
\]  

(11)

where

\[
t_0 = \frac{\ln \lambda_1 - \ln \lambda_2}{\lambda_1 - \lambda_2}.
\]

Therefore, the graphs of \(N_1(t)\) and \(N_2(t)\) have the shapes shown in Figures 2 and 3.

![Figure 2](image1)

**Figure 2.** Typical graph of \(N_1(t)\) (amount of \(A\) as a function of time).

![Figure 3](image2)

**Figure 3.** Typical graph of \(N_2(t)\) (amount of \(B\) as a function of time).
Exercises

4a. Show that \( \lim_{t \to \infty} N_A(t) = 0 \).

b. Confirm Equation (10).

c. Confirm Equation (11).

5. Find the time at which the greatest amount of \( B \) will be present.

6. For the chain \( \text{Bi}^{210} \to \text{Po}^{210} \to \text{Pb}^{206} \), \( \lambda_1 = 1.37 \times 10^{-1} \text{ day}^{-1} \) and \( \lambda_2 = 5.1 \times 10^{-3} \text{ day}^{-1} \).

a. Use Exercise 5 to determine when the amount of \( \text{Po}^{210} \) will be greatest.

b. If initially there are \( 10^{-8} \) grams of \( \text{Bi}^{210} \), how many grams of \( \text{Po}^{210} \) will there be when it is at its maximum amount?

4. Equilibrium

4.1 What is Equilibrium?

In a continuing process such as the one we are discussing, it is natural to ask about "equilibrium" of the process. Webster's Seventh New Collegiate Dictionary (1965) defines equilibrium as "a static or dynamic state of balance between opposing forces or actions." To a scientist, "state of balance" means that certain measurable quantities remain constant. But in practice scientists frequently use the word "equilibrium" when the measurements under consideration are nearly constant rather than actually constant. There is good reason for this. The process we are discussing illustrates that reason nicely. We have been devoting our attention to \( N_1(t) \) and \( N_2(t) \). For either of these actually to be constant over any time interval, its derivative would have to be zero throughout that interval. But \( N_1'(t) < 0 \) for all \( t \), and \( N_2'(t) = 0 \) for only one \( t \). So, strictly speaking, it is impossible for either \( N_1(t) \) or \( N_2(t) \) to be "in equilibrium."

But \( N_1(t) \) and \( N_2(t) \) involve negative exponential functions. In fact, functions involving negative exponentials occur fairly commonly in the description of physical and chemical processes. The most basic negative exponential function is \( e^{-t} \), and if you know anything at all about it you know that it approaches zero very fast, so that, although it is never constant, it is before long practically equal to zero and therefore practically constant. This characteristic
"never constant but practically constant") carries through to many of the more complicated functions involving negative exponentials. The quantities they describe never actually reach their limiting values, but usually come (and remain) extremely close to them within a reasonable length of time—perhaps even so close that the difference is not measurable. Scientists often apply the words "equilibrium" or "steady state" to this situation.

There is one more thing we should say about the word "equilibrium" before we move on to discussing specific cases of it. The functions $N_1(t)$ and $N_2(t)$ are examples of functions that come and remain extremely close to a constant value (zero). But it would be wrong to say that they are therefore in equilibrium, even allowing for the stretching of the definition which we just discussed. The reason lies in the other part of the definition: "opposing forces or actions." There are no "opposing forces or actions." Rather than having two things happen which cancel each other out, we have nothing happening at all (in the limit). In plain English, virtually all of $A$ will have decayed into $B$ and then into $C$, so that there will be virtually none of $A$ or $B$ left. It takes no fancy mathematics to see this. If $C$ is radioactive it will eventually decay, and so on, so that the limiting situation is that only the stable substance at the end of the chain will remain.

There are, however, two situations involving radioactive chains to which the word "equilibrium" is usually applied. The first of these is known as transient equilibrium. Another, known as secular equilibrium, can be regarded as a special case of the first.

4.2 Some Comments on Approximation

Before we get into the mathematics of transient and secular equilibrium, it will be wise to take a moment to discuss just what we mean by a "good approximation." When scientists say two numbers $r$ and $s$ are approximately equal, they almost always mean that the difference between $r$ and $s$ is small compared to either of the numbers. For example they might say $1002 = 1000$ (depending on the context), but would almost never say $2 = 1$.

Saying $r = s$ is small compared to (for example) $s$ means $\frac{r - s}{s} = \frac{r}{s} - 1$ is small, or $\frac{r}{s}$ is near 1. In the numerical examples we just gave, $\frac{1002}{1000} = 1.002$, which is very near 1, but $\frac{1}{2} = 0.5$, which is much further from 1.

This interpretation of approximation can be applied to functions too. Let's look specifically at negative exponential functions. If $P$
and $Q$ are any non-zero constants, and if $a$ and $b$ are constants such that $0 < a < b$, then
\[
\frac{Pe^{-at} + Qe^{-bt}}{Pe^{-at}} = 1 + \frac{Q}{P} e^{(a-b)t} \to 1
\] (12)
as $t \to \infty$, since $a - b$ is negative. Therefore, for $t$ large enough, $Pe^{-at} + Qe^{-bt}$ can be approximated by $Pe^{-at}$. We shall use this fact in the next section.

4.3 Transient Equilibrium

It has often been observed that in many chains involving a parent $A$ and a radioactive child $B$, after a while both parent and child appear to be decaying at the same rate, in the sense that in any given time interval (say from $t_1$ to $t_2$, with $t_1$ large enough) parent and child each lose the same fraction of their initial amount—
\[
\frac{N_A(t_2)/N_A(t_1)}{N_B(t_2)/N_B(t_1)} = \frac{N_A(t_1)/N_A(t_1)}{N_B(t_1)/N_B(t_1)}.
\]
This phenomenon is called transient equilibrium. Let us try to explain it mathematically.

We can rewrite the equation of the preceding paragraph
\[
\frac{N_A(t_2)/N_A(t_1)}{N_B(t_2)/N_B(t_1)} = \frac{N_A(t_1)/N_A(t_1)}{N_B(t_1)/N_B(t_1)}.
\]
in other words, the observed result is that $N_A(t)/N_A(t)$ is a constant. (This is why the word “equilibrium” is used in describing this phenomenon.) Why should this be so? Let’s investigate this quotient, starting with the formulas for $N_A(t)$ and $N_B(t)$ given in (4) and (9) respectively.

We know from (9) that
\[
N_B(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}).
\]
If $\lambda_1 < \lambda_2$, Section 4.2 tells us that for $t$ large enough
\[
N_B(t) \approx \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t}.
\] (13)

We also know, from (4), that
\[
N_A(t) = N_0 e^{-\lambda_1 t}
\]
(which is exact). Dividing (13) by (4), we get
\[
\frac{N_B(t)}{N_A(t)} = \frac{\lambda_1}{\lambda_2 - \lambda_1}.
\] (14)
On the other hand, if $\lambda_2 < \lambda_1$ then this does not go through as neatly. Equation (13) has to be replaced by

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_1 - \lambda_2} e^{-\lambda_1 t}$$

and then (14) becomes

$$\frac{N_2(t)}{N_1(t)} = \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{(\lambda_1 - \lambda_2)t}.$$ 

This is a positive exponential. It does not have a finite limit.

So the mathematics tell us that transient equilibrium should be observed when $\lambda_1 < \lambda_2$, but not otherwise. Sure enough, this is exactly what happens.

Another way of looking at transient equilibrium is to compare the approximation given in (13) directly with (4). The exponent $-\lambda_1 t$ is the same in both cases. So for $t$ large enough $B$ behaves as if it had the same decay constant (and therefore the same half-life) as $A$. Since $\lambda_1 < \lambda_2$ this apparent half-life is longer than $B$'s natural half-life, an observation which should appeal to your common sense even with no mathematics at all. After all, two things are happening to $B$. It is decaying at its natural rate, and it is being replenished at a certain rate. Therefore you would expect its actual rate of disappearance to be somewhat slower than if it were not being replenished. This common sense observation may help you remember which way the inequality $\lambda_1 < \lambda_2$ goes for transient equilibrium.

4.4 Secular Equilibrium

We have said that transient equilibrium occurs when $\lambda_1$ is smaller than $\lambda_2$. Now let us suppose that $\lambda_1$ is very small, and very much smaller than $\lambda_2$. (Scientists write $\lambda_1 \ll \lambda_2$ to mean $\lambda_1$ is very much smaller than $\lambda_2$.) This is actually a very common occurrence. For example, in the chain

$$\text{Ra}^{226} \rightarrow \text{Rn}^{222} \rightarrow \text{Po}^{218},$$

$\text{Ra}^{226}$ has a half life of about 1620 years. The decay constant for the first step is $\lambda_1 = 4.28 \times 10^{-4}$ yr$^{-1} = 1.17 \times 10^{-5}$ day$^{-1}$. In contrast, $\text{Rn}^{222}$ has a half life of 3.83 days, so that $\lambda_2 = 0.181$ day$^{-1} = 1.81 \times 10^{-1}$ day$^{-1}$.

We know that whenever $0 < \lambda_1 < \lambda_2$, $e^{-\lambda_1 t}$ decreases to zero more slowly than $e^{-\lambda_2 t}$. If $\lambda_1 \ll \lambda_2$, the difference in these rates is so
great that long after $e^{-\lambda_1 t}$ has become tiny enough to neglect in (9), we can still say $e^{-\lambda_1 t} = 1$. (When $t = 200$ days in the example given above, $e^{-\lambda_1 t} \approx 0.99977$ and $e^{-\lambda_2 t} \approx 2 \times 10^{-16}$.) Then we would have

$$N_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_0$$  \hspace{1cm} (15)

and

$$N_1(t) = N_0,$$  \hspace{1cm} (16)

both approximately constant.

Not only are $N_1(t)$ and $N_2(t)$ decreasing at the same rate, but this rate is so slow that they are in fact virtually constant. This situation is known as secular equilibrium. Again, as remarked in Section 4.1, we are stretching the term a bit, since the quantities involved are not really constant. Here there is yet another abuse of terminology, in that the "virtually constant" values of $N_1(t)$ and $N_2(t)$ are not their limiting values. Eventually (although perhaps none of us will live long enough to see it) these amounts will begin to decay noticeably, and ultimately they will approach zero.

Remember we also remarked in Section 4.1 that for all their abuse of the term, scientists do agree that it is wrong to apply the word "equilibrium" to a situation in which "nothing is happening" (there are no opposing forces or reactions). This is not a problem here. Plenty is happening. New $B$ nuclei are being formed, and old ones are decaying. The total number of $B$ nuclei remains the same, but they are not at all the same nuclei. (The total number of people in New York City is about the same as 40 years ago, but they are certainly not the very same people.)

### 4.5 More Comments on Approximation

One thing about Section 4.4 may puzzle you. Adding the approximations given in (16) and (15) we get

$$N_1(t) + N_2(t) = N_0 + \frac{\lambda_1}{\lambda_2 - \lambda_1} N_0 > N_0.$$  

But the total number of atoms, including those of $C$ and possibly later substances, must always equal $N_0$. How can this be?
What has happened is that \( N_i(t) \) has decreased by a certain amount while \( N_A(t) \) has increased by a lesser amount. But there was a lot of \( A \) to begin with, so the decrease is small compared to the original amount, and \( N_i(t)/N_0 = 1 \). On the other hand, there was none of \( B \) to begin with, and even at secular equilibrium there is very little. Compared to this amount, the increase is significant. It’s as if a large oil company (\( A \)) were to pay you (\( B \)) \$10,000 and you were to use \$10 of that money to bribe your math teacher (\( C \)). The oil company will still have essentially the same amount of money as before, and you will be much better off financially, even though the total of its money and your money, when calculated precisely, will be less than before.* The numerical calculations in Exercise 7 may help you see what is going on.

**Exercise 7**

a. Use Exercise 5 to show that the amount of \( Rn^{222} \) in the chain of Section 4.4 is greatest at about \( t = 66 \) days.

b. (Requires a calculator.) For the values of \( t \) given below, compute the precise amounts of \( Ra^{226} \) and \( Rn^{222} \) as given by (4) and (9), and also the sum of these amounts. Then compute the approximations given by (15) and (14), and also their sum. Tabulate and compare these results. Take \( N_0 = 10^9 \) atoms, and make all computations to the nearest integer. Use \( t \) (in days) = 1, 2, 3, 10, 66, 100, 1000.

## 5. Answers to Exercises

1. \( N_i(t^*) = N_0 e^{-\lambda_i t^*} = \frac{1}{2} N_0 \)

\[ e^{-\lambda_i t^*} = \frac{1}{2} \]

\[ -\lambda_i t^* = \ln \frac{1}{2} = -\ln 2 \]

\[ t^* = \frac{\ln 2}{\lambda_i} \]

*You will note that for this analogy it does not matter whether your math teacher is stable or not.
2. Writing $\lambda = \lambda_1 - \lambda_2$:

$$v'(t) = \frac{\lambda N_0}{R} e^{(\lambda - \lambda)t} = \frac{\lambda N_0}{R}$$

$$v(t) = \frac{\lambda N_0}{R} t + K$$

$$N_2(t) = u(t) \cdot v(t) = R e^{-\lambda t} \left( \frac{\lambda N_0}{R} t + K \right)$$

$$= \lambda N_0 t e^{-\lambda t} + RK e^{-\lambda t}.$$

Since $N_2(0) = 0$, we have $RK = 0$ and

$$N_2(t) = \lambda N_0 t e^{-\lambda t}.$$

3. Set $D'(t) = 0$, where $D(t) = \lambda_1 N_1(t) + \lambda_2 N_2(t)$.

$$D'(t) = \lambda_1 N_1'(t) + \lambda_2 N_2'(t)$$

$$= -\lambda^2 N_0 e^{-\lambda t} + \frac{\lambda_1 \lambda_2 N_0}{\lambda_2 - \lambda_1} \left( -\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} \right)$$

$$= \left( -\lambda^2 N_0 - \frac{\lambda_1^2 \lambda_2 N_0}{\lambda_2 - \lambda_1} \right) e^{-\lambda t} + \frac{\lambda_1 \lambda_2^2 N_0}{\lambda_2 - \lambda_1} e^{-\lambda t}.$$

$$D'(t) = 0$$ when

$$\left( \lambda_1 + \frac{\lambda_1^2 \lambda_2}{\lambda_2 - \lambda_1} \right) e^{-\lambda_1 t} = \frac{\lambda_1 \lambda_2^2}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$

$$\lambda_1^2 (\lambda_2 - \lambda_1) + \lambda_1 \lambda_2^2$$

$$\frac{\lambda_1 (\lambda_2 - \lambda_1) + \lambda_1 \lambda_2}{\lambda_2^2} = e^{(\lambda_1 - \lambda_2) t}$$

$$e^{(\lambda_1 - \lambda_2) t} = \frac{\lambda_1 (\lambda_2 - \lambda_1) + \lambda_1 \lambda_2}{\lambda_2^2}$$

$$= \frac{\lambda_1}{\lambda_2^2} (2 \lambda_2 - \lambda_1)$$

$$t = \frac{1}{\lambda_1 - \lambda_2} \ln \left[ \frac{\lambda_1}{\lambda_2^2} (2 \lambda_2 - \lambda_1) \right]$$

$$= \frac{\ln \lambda_1 + \ln(2 \lambda_2 - \lambda_1) - 2 \ln \lambda_2}{\lambda_1 - \lambda_2}.$$

4. a. Use $e^{-\lambda_1 t} \to 0$ and $e^{-\lambda_2 t} \to 0$.  

13
b. 

\[ N_2'(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} (-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}) = 0 \text{ when} \]

\[ \lambda_1 e^{-\lambda_1 t} = \lambda_2 e^{-\lambda_2 t} \]

\[ \frac{\lambda_1}{\lambda_2} = e^{(\lambda_1 - \lambda_2)t} \]

\[ (\lambda_1 - \lambda_2)t = \ln \frac{\lambda_1}{\lambda_2} = \ln \lambda_1 - \ln \lambda_2 \]

\[ t = \frac{\ln \lambda_1 - \ln \lambda_2}{\lambda_1 - \lambda_2} \].

Since \( N_0(0) = 0 \), \( N_0(t_0) > 0 \) (from physical considerations), and \( \lim_{t \to \infty} N_0(t) = 0 \), the desired inequalities for \( N_2'(t) = 0 \) follow.

c. 

\[ N_2''(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} (\lambda_1^2 e^{-\lambda_1 t} - \lambda_2^2 e^{-\lambda_2 t}) = 0 \text{ when} \]

\[ \lambda_1^2 e^{-\lambda_1 t} = \lambda_2^2 e^{-\lambda_2 t} \]

\[ \left( \frac{\lambda_1}{\lambda_2} \right)^2 = e^{(\lambda_1 - \lambda_2)t}, \text{ etc.} \]

(Use the fact that \( \ln \left( \frac{\lambda_1}{\lambda_2} \right)^2 = 2 \ln \frac{\lambda_1}{\lambda_2} \)).

5. This occurs when \( N_2'(t) = 0 \). That is, at \( t_0 = \frac{\ln \lambda_1 - \ln \lambda_2}{\lambda_1 - \lambda_2} \), by (10).

6. a. By Exercise 5,

\[ t_0 = \frac{\ln(1.37 \times 10^{-1}) - \ln(5.1 \times 10^{-3})}{1.37 \times 10^{-1} - 5.1 \times 10^{-3}} = 24.95 \text{ days}. \]

b. 

\[ N_2(t_0) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}) \]

\[ = \frac{(1.37 \times 10^{-1}) \times 10^{-8}}{5.1 \times 10^{-3} - 1.37 \times 10^{-1}} \]

\[ \times (e^{-1.37 \times 10^{-1} \times 24.95} - e^{-5.1 \times 10^{-3} \times 24.95}) \]

\[ = 8.81 \times 10^{-9} \text{ grams}. \]
7. Column $A$ gives the exact amount of $\text{Ra}^{226}$.
   Column $B$ gives the exact amount of $\text{Rn}^{222}$.
   Column $C$ gives the sum of columns $A$ and $B$.
   Column $D$ gives the approximation of the amount of $\text{Ra}^{226}$.
   Column $E$ gives the approximation of the amount of $\text{Rn}^{222}$.
   Column $F$ gives the sum of columns $D$ and $E$.

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<th>$C$</th>
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Table 1

Abbreviations and atomic numbers of chemical elements mentioned

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References


About the Author

Brindell Horelick taught for 30 years at various universities and colleges in the northeastern United States, primarily at the University of Maryland Baltimore County. He is now happily retired and busily pursuing the rest of his life.