When writer Lewis Carroll took Alice on her journeys down the rabbit hole to Wonderland and through the looking glass, she had many fantastic encounters with the tea-sipping Mad Hatter, a hookah-smoking Caterpillar, the White Rabbit, the Cheshire Cat, the Red and White Queens, and Tweedledum and Tweedledee. On the surface, Carroll’s writings seem to be delightful nonsense and mere children’s entertainment. Many people are quite surprised to learn that Alice’s Adventures in Wonderland is much an exercise in logic as it is a fantasy and that Lewis Carroll was actually Charles Dodgson, an Oxford mathematician. Dodgson’s many writings include the whimsical The Game of Logic and the brilliant Symbolic Logic, in addition to Alice’s Adventures in Wonderland and Through the Looking Glass.

WHAT WE WILL DO IN THIS CHAPTER

WE’LL EXPLORE DIFFERENT TYPES OF LOGIC OR REASONING:
• Deductive reasoning involves the application of a general statement to a specific case; this type of logic is typified in the classic arguments of the renowned Greek logician Aristotle.
• Inductive reasoning involves generalizing after a pattern has been recognized and established; this type of logic is used in the solving of puzzles.

WE’LL ANALYZE AND EXPLORE VARIOUS TYPES OF STATEMENTS AND THE CONDITIONS UNDER WHICH THEY ARE TRUE:
• A statement is a simple sentence that is either true or false. Simple statements can be connected to form compound, or more complicated, statements.
• Symbolic representations reduce a compound statement to its basic form; phrases that appear to be different may actually have the same basic structure and meaning.
WHAT WE WILL DO IN THIS CHAPTER — continued

WE’LL ANALYZE AND EXPLORE CONDITIONAL, OR “IF . . . THEN . . .,” STATEMENTS:

• In everyday conversation, we often connect phrases by saying “if this, then that.” However, does “this” actually guarantee “that”? Is “this” in fact necessary for “that”?

• How does “if” compare with “only if”? What does “if and only if” really mean?

WE’LL DETERMINE THE VALIDITY OF AN ARGUMENT:

• What constitutes a valid argument? Can a valid argument yield a false conclusion?

• You may have used Venn diagrams to depict a solution set in an algebra class. We will use Venn diagrams to visualize and analyze an argument.

• Some of Lewis Carroll’s whimsical arguments are valid, and some are not. How can you tell?

Webster’s New World College Dictionary defines logic as “the science of correct reasoning: science which describes relationships among propositions in terms of implication, contradiction, contrariety, conversion, etc.” In addition to being flaunted in Mr. Spock’s claim that “your human emotions have drawn you to an illogical conclusion” and in Sherlock Holmes’s immortal phrase “elementary, my dear Watson,” logic is fundamental both to critical thinking and to problem solving. In today’s world of misleading commercial claims, innuendo, and political rhetoric, the ability to distinguish between valid and invalid arguments is important.

In this chapter, we will study the basic components of logic and its application. Mischievous, wild-eyed residents of Wonderland, eccentric, violin-playing detectives, and cold, emotionless Vulcans are not the only ones who can benefit from logic. Armed with the fundamentals of logic, we can surely join Spock and “live long and prosper!”
Logic is the science of correct reasoning. Webster’s New World College Dictionary defines reasoning as “the drawing of inferences or conclusions from known or assumed facts.” Reasoning is an integral part of our daily lives; we take appropriate actions based on our perceptions and experiences. For instance, if the sky is heavily overcast this morning, you might assume that it will rain today and take your umbrella when you leave the house.

Problem Solving

Logic and reasoning are associated with the phrases problem solving and critical thinking. If we are faced with a problem, puzzle, or dilemma, we attempt to reason through it in hopes of arriving at a solution.

The first step in solving any problem is to define the problem in a thorough and accurate manner. Although this might sound like an obvious step, it is often overlooked. Always ask yourself, “What am I being asked to do?” Before you can
solve a problem, you must understand the question. Once the problem has been defined, all known information that is relevant to it must be gathered, organized, and analyzed. This analysis should include a comparison of the present problem to previous ones. How is it similar? How is it different? Does a previous method of solution apply? If it seems appropriate, draw a picture of the problem; visual representations often provide insight into the interpretation of clues.

Before using any specific formula or method of solution, determine whether its use is valid for the situation at hand. A common error is to use a formula or method of solution when it does not apply. If a past formula or method of solution is appropriate, use it; if not, explore standard options and develop creative alternatives. Do not be afraid to try something different or out of the ordinary. “What if I try this...?” may lead to a unique solution.

**Deductive Reasoning**

Once a problem has been defined and analyzed, it might fall into a known category of problems, so a common method of solution may be applied. For instance, when one is asked to solve the equation \( x^2 = 2x + 1 \), realizing that it is a second-degree equation (that is, a quadratic equation) leads one to put it into the standard form \((x^2 - 2x - 1 = 0)\) and apply the Quadratic Formula.

**EXAMPLE 1**

**USING DEDUCTIVE REASONING TO SOLVE AN EQUATION** Solve the equation \( x^2 = 2x + 1 \).

The given equation is a second-degree equation in one variable. We know that all second-degree equations in one variable (in the form \( ax^2 + bx + c = 0 \)) can be solved by applying the Quadratic Formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
Therefore, \( x^2 = 2x + 1 \) can be solved by applying the Quadratic Formula:

\[
x^2 = 2x + 1
x^2 - 2x - 1 = 0
x = \frac{-(-2) \pm \sqrt{(-2)^2 - (4)(-1)}}{2(1)}
x = \frac{2 \pm \sqrt{4 + 4}}{2}
\]

\[
x = \frac{2 \pm \sqrt{8}}{2}
\]

\[
x = \frac{2 \pm 2\sqrt{2}}{2}
\]

\[
x = 1 \pm \sqrt{2}
\]

The solutions are \( x = 1 + \sqrt{2} \) and \( x = 1 - \sqrt{2} \).

In Example 1, we applied a general rule to a specific case; we reasoned that it was valid to apply the (general) Quadratic Formula to the (specific) equation \( x^2 = 2x + 1 \). This type of logic is known as deductive reasoning—that is, the application of a general statement to a specific instance.

Deductive reasoning and the formal structure of logic have been studied for thousands of years. One of the earliest logicians, and one of the most renowned, was Aristotle (384–322 B.C.). He was the student of the great philosopher Plato and the tutor of Alexander the Great, the conqueror of all the land from Greece to India. Aristotle’s philosophy is pervasive; it influenced Roman Catholic theology through St. Thomas Aquinas and continues to influence modern philosophy. For centuries, Aristotelian logic was part of the education of lawyers and politicians and was used to distinguish valid arguments from invalid ones.

For Aristotle, logic was the necessary tool for any inquiry, and the syllogism was the sequence followed by all logical thought. A syllogism is an argument composed of two statements, or premises (the major and minor premises), followed by a conclusion. For any given set of premises, if the conclusion of an argument is guaranteed (that is, if it is inescapable in all instances), the argument is valid. If the conclusion is not guaranteed (that is, if there is at least one instance in which it does not follow), the argument is invalid.

Perhaps the best known of Aristotle’s syllogisms is the following:

1. All men are mortal.  \hspace{1cm} \text{major premise}
2. Socrates is a man. \hspace{1cm} \text{minor premise}

Therefore, Socrates is mortal. \hspace{1cm} \text{conclusion}

When the major premise is applied to the minor premise, the conclusion is inescapable; the argument is valid.

Notice that the deductive reasoning used in the analysis of Example 1 has exactly the same structure as Aristotle’s syllogism concerning Socrates:

1. All second-degree equations in one variable can be solved by applying the Quadratic Formula. \hspace{1cm} \text{major premise}
2. \( x^2 = 2x + 1 \) is a second-degree equation in one variable. \hspace{1cm} \text{minor premise}

Therefore, \( x^2 = 2x + 1 \) can be solved by applying the Quadratic Formula.
Each of these syllogisms is of the following general form:

1. If $A$, then $B$.  
2. $x$ is $A$.  
Therefore, $x$ is $B$.

Historically, this valid pattern of deductive reasoning is known as *modus ponens*.

### Deductive Reasoning and Venn Diagrams

The validity of a deductive argument can be shown by use of a Venn diagram. A **Venn diagram** is a diagram consisting of various overlapping figures contained within a rectangle (called the “universe”). To depict a statement of the form “All $A$ are $B$” (or, equivalently, “If $A$, then $B$”), we draw two circles, one inside the other; the inner circle represents $A$, and the outer circle represents $B$. This relationship is shown in Figure 1.1.

Venn diagrams depicting “No $A$ are $B$” and “Some $A$ are $B$” are shown in Figures 1.2 and 1.3, respectively.

#### EXAMPLE 2

**ANALYZING A DEDUCTIVE ARGUMENT**  
Construct a Venn diagram to verify the validity of the following argument:

1. All men are mortal.  
2. Socrates is a man.  

**Therefore, Socrates is mortal.**

Premise 1 is of the form “All $A$ are $B$” and can be represented by a diagram like that shown in Figure 1.4.

Premise 2 refers to a specific man, namely, Socrates. If we let $x = $ Socrates, the statement “Socrates is a man” can then be represented by placing $x$ within the circle labeled “men,” as shown in Figure 1.5. Because we placed $x$ within the “men” circle, and all of the “men” circle is inside the “mortal” circle, the conclusion “Socrates is mortal” is inescapable; the argument is valid.
ARISTOTLE 384–322 B.C.

Aristotle was born in 384 B.C. in the small Macedonian town of Stagira, 200 miles north of Athens, on the shore of the Aegean Sea. Aristotle's father was the personal physician of King Amyntas II, ruler of Macedonia. When he was seventeen, Aristotle enrolled at the Academy in Athens and became a student of the famed Plato. Aristotle was one of Plato's brightest students; he frequently questioned Plato's teachings and openly disagreed with him. Whereas Plato emphasized the study of abstract ideas and mathematical truth, Aristotle was more interested in observing the "real world" around him. Plato often referred to Aristotle as "the brain" or "the mind of the school." Plato commented, "Where others need the spur, Aristotle needs the rein."

Aristotle stayed at the Academy for twenty years, until the death of Plato. Then the king of Macedonia invited Aristotle to supervise the education of his son Alexander, the future Alexander the Great. Aristotle accepted the invitation and taught Alexander until he succeeded his father as ruler. At that time, Aristotle founded a school known as the Lyceum, or Peripatetic School. The school had a large library with many maps, as well as botanical gardens containing an extensive collection of plants and animals. Aristotle and his students would walk about the grounds of the Lyceum while discussing various subjects (peripatetic is from the Greek word meaning "to walk").

Many consider Aristotle to be a founding father of the study of biology and of science in general; he observed and classified the behavior and anatomy of hundreds of living creatures. Alexander the Great, during his many military campaigns, had his troops gather specimens from distant places for Aristotle to study.

Aristotle was a prolific writer; some historians credit him with the writing of over 1,000 books. Most of his works have been lost or destroyed, but scholars have recreated some of his more influential works, including Organon.

**HISTORICAL NOTE**

**EXAMPLE 3**

**ANALYZING A DEDUCTIVE ARGUMENT**

Construct a Venn diagram to determine the validity of the following argument:

1. All doctors are men.
2. My mother is a doctor.

Therefore, my mother is a man.

Premise 1 is of the form “All A are B”; the argument is depicted in Figure 1.6. No matter where x is placed within the “doctors” circle, the conclusion “My mother is a man” is inescapable; the argument is valid.

*Saying that an argument is valid does not mean that the conclusion is true.* The argument given in Example 3 is valid, but the conclusion is false. One’s mother cannot be a man! Validity and truth do not mean the same thing. An argument is valid if the conclusion is inescapable, given the premises. Nothing is said about the truth of the premises. Thus, when examining the validity of an argument, we are not determining whether the conclusion is true or false. Saying that an argument is valid merely means that, given the premises, the reasoning used to obtain the conclusion is logical. However, if the premises of a valid argument are true, then the conclusion will also be true.
EXAMPLE 4

ANALYZING A DEDUCTIVE ARGUMENT

Construct a Venn diagram to determine the validity of the following argument:

1. All professional wrestlers are actors.
2. The Rock is an actor.

Therefore, The Rock is a professional wrestler.

SOLUTION

Premise 1 is of the form “All A are B”; the “circle of professional wrestlers” is contained within the “circle of actors.” If we let x represent The Rock, premise 2 simply requires that we place x somewhere within the actor circle; x could be placed in either of the two locations shown in Figures 1.7 and 1.8.

\[
\begin{align*}
\text{FIGURE 1.7} & \quad \text{FIGURE 1.8} \\
U & \quad U \\
\text{professional wrestlers} & \quad \text{professional wrestlers} \\
x & \quad x \\
\text{actors} & \quad \text{actors}
\end{align*}
\]

If x is placed as in Figure 1.7, the argument would appear to be valid; the figure supports the conclusion “The Rock is a professional wrestler.” However, the placement of x in Figure 1.8 does not support the conclusion; given the premises, we cannot logically deduce that “The Rock is a professional wrestler.” Since the conclusion is not inescapable, the argument is invalid.

Saying that an argument is invalid does not mean that the conclusion is false. Example 4 demonstrates that an invalid argument can have a true conclusion; even though The Rock is a professional wrestler, the argument used to obtain the conclusion is invalid. In logic, validity and truth do not have the same meaning. Validity refers to the process of reasoning used to obtain a conclusion; truth refers to conformity with fact or experience.

VENN DIAGRAMS AND INVALID ARGUMENTS

To show that an argument is invalid, you must construct a Venn diagram in which the premises are met yet the conclusion does not necessarily follow.

EXAMPLE 5

ANALYZING A DEDUCTIVE ARGUMENT

Construct a Venn diagram to determine the validity of the following argument:

1. Some plants are poisonous.
2. Broccoli is a plant.

Therefore, broccoli is poisonous.
Premise 1 is of the form “Some $A$ are $B$”; it can be represented by two overlapping circles (as in Figure 1.3). If we let $x$ represent broccoli, premise 2 requires that we place $x$ somewhere within the plant circle. If $x$ is placed as in Figure 1.9, the argument would appear to be valid. However, if $x$ is placed as in Figure 1.10, the conclusion does not follow. Because we can construct a Venn diagram in which the premises are met yet the conclusion does not follow (Figure 1.10), the argument is invalid.

When analyzing an argument via a Venn diagram, you might have to draw three or more circles, as in the next example.

**EXAMPLE 6**

**ANALYZING A DEDUCTIVE ARGUMENT** Construct a Venn diagram to determine the validity of the following argument:

1. No snake is warm-blooded.
2. All mammals are warm-blooded.

Therefore, snakes are not mammals.

**SOLUTION**

Premise 1 is of the form “No $A$ are $B$”; it is depicted in Figure 1.11. Premise 2 is of the form “All $A$ are $B$”; the “mammal circle” must be drawn within the “warm-blooded circle.” Both premises are depicted in Figure 1.12.

Because we placed $x$ (= snake) within the “snake” circle, and the “snake” circle is outside the “warm-blooded” circle, $x$ cannot be within the “mammal” circle (which is inside the “warm-blooded” circle). Given the premises, the conclusion “Snakes are not mammals” is inescapable; the argument is valid.
You might have encountered Venn diagrams when you studied sets in your algebra class. The academic fields of set theory and logic are historically intertwined; set theory was developed in the late nineteenth century as an aid in the study of logical arguments. Today, set theory and Venn diagrams are applied to areas other than the study of logical arguments; we will utilize Venn diagrams in our general study of set theory in Chapter 2.

**Inductive Reasoning**

The conclusion of a valid deductive argument (one that goes from general to specific) is guaranteed: Given true premises, a true conclusion must follow. However, there are arguments in which the conclusion is not guaranteed even though the premises are true. Consider the following:

1. Joe sneezed after petting Frako’s cat.
2. Joe sneezed after petting Paulette’s cat.

Therefore, Joe is allergic to cats.

Is the conclusion guaranteed? If the premises are true, they certainly support the conclusion, but we cannot say with 100% certainty that Joe is allergic to cats. The conclusion is not guaranteed. Maybe Joe is allergic to the flea powder that the cat owners used; maybe he is allergic to the dust that is trapped in the cats’ fur; or maybe he has a cold!

Reasoning of this type is called inductive reasoning. **Inductive reasoning** involves going from a series of specific cases to a general statement (see Figure 1.13). Although it may seem to follow and may in fact be true, the conclusion in an inductive argument is never guaranteed.

---

**EXAMPLE 7**

**INDUCTIVE REASONING AND PATTERN RECOGNITION** What is the next number in the sequence 1, 8, 15, 22, 29, . . . ?

Noticing that the difference between consecutive numbers in the sequence is 7, we may be tempted to say that the next term is $29 + 7 = 36$. Is this conclusion guaranteed? No! Another sequence in which numbers differ by 7 are dates of a given day of the week. For instance, the dates of the Saturdays in the year 2011 are (January) 1, 8, 15, 22, 29, (February) 5, 12, 19, 26, . . . . Therefore, the next number in the sequence 1, 8, 15, 22, 29, . . . might be 5. Without further information, we cannot determine the next number in the given sequence. We can only use inductive reasoning and give one or more possible answers.
Throughout history, people have always been attracted to puzzles, mazes, and brainteasers. Who can deny the inherent satisfaction of solving a seemingly unsolvable or perplexing riddle? A popular new addition to the world of puzzle solving is *sudoku*, a numbers puzzle. Loosely translated from Japanese, *sudoku* means “single number”; a *sudoku* puzzle simply involves placing the digits 1 through 9 in a grid containing 9 rows and 9 columns. In addition, the 9 by 9 grid of squares is subdivided into nine 3 by 3 grids, or “boxes,” as shown in Figure 1.14.

The rules of *sudoku* are quite simple: Each row, each column, and each box must contain the digits 1 through 9; and no row, column, or box can contain 2 squares with the same number. Consequently, *sudoku* does not require any arithmetic or mathematical skill; *sudoku* requires logic only. In solving a puzzle, a common thought is “What happens if I put this number here?”

Like *crossword* puzzles, *sudoku* puzzles are printed daily in many newspapers across the country and around the world. Web sites containing *sudoku* puzzles and strategies provide an endless source of new puzzles and help. See Exercise 62 to find links to popular sites.

**EXAMPLE 8**

Solve the *sudoku* puzzle given in Figure 1.15.

![A blank *sudoku* grid.](FIGURE 1.14)

![A *sudoku* puzzle.](FIGURE 1.15)
Recall that each 3 by 3 grid is referred to as a box. For convenience, the boxes are numbered 1 through 9, starting in the upper left-hand corner and moving from left to right, and each square can be assigned coordinates \((x, y)\) based on its row number \(x\) and column number \(y\) as shown in Figure 1.16.

For example, the digit 2 in Figure 1.16 is in box 1 and has coordinates \((1, 3)\), the digit 8 is in box 3 and has coordinates \((1, 7)\), the digit 6 is in box 4 and has coordinates \((5, 1)\) and the digit 7 is in box 9 and has coordinates \((9, 7)\).

When you are first solving a sudoku puzzle, concentrate on only a few boxes rather than the puzzle as a whole. For instance, looking at boxes 1, 4, and 7, we see that boxes 4 and 7 each contain the digit 6, whereas box 1 does not. Consequently, the 6 in box 1 must be placed in column 3 because (shaded) columns 1 and 2 already have a 6. However, (shaded) row 2 already has a 6, so we can deduce that 6 must be placed in row 3, column 3, that is, in square \((3, 3)\) as shown in Figure 1.17.

Examining boxes 1, 2, and 3, we see that boxes 2 and 3 each contain the digit 5, whereas box 1 does not. We deduce that 5 must be placed in square \((2, 3)\) because rows 1 and 3 already have a 5. In a similar fashion, square \((1, 4)\) must contain 3. See Figure 1.18.
Because we have placed two new digits in box 1, we might wish to focus on the remainder of (shaded) box 1. Notice that the digit 4 can be placed only in square (3, 1), as row 1 and column 2 already have a 4 in each of them; likewise, the digit 9 can be placed only in square (1, 1) because column 2 already has a 9. Finally, either of the digits 1 or 7 can be placed in square (1, 2) or (3, 2) as shown in Figure 1.19. At some point later in the solution, we will be able to determine the exact values of squares (1, 2) and (3, 2), that is, which square receives a 1 and which receives a 7.

Using this strategy of analyzing the contents of three consecutive boxes, we deduce the following placement of digits: 1 must go in (2, 5), 6 must go in (6, 7), 6 must go in (8, 6), 7 must go in (8, 3), 3 must go in (8, 5), 8 must go in (9, 6), and 5 must go in (7, 4). At this point, box 8 is complete as shown in Figure 1.20. (Remember, each box must contain each of the digits 1 through 9.)

Once again, we use the three consecutive box strategy and deduce the following placement of digits: 5 must go in (8, 7), 5 must go in (9, 1), 8 must go in (7, 1), 2 must go in (8, 1), and 1 must go in (7, 3). At this point, box 7 is complete as shown in Figure 1.21.
We now focus on box 4 and deduce the following placement of digits: 1 must go in (4, 1), 9 must go in (5, 3), 4 must go in (4, 3), 5 must go in (6, 2), 3 must go in (4, 2), and 2 must go in (5, 2). At this point, box 4 is complete. In addition, we deduce that 1 must go in (9, 8), and 3 must go in (5, 6) as shown in Figure 1.22.
Once again, we use the three consecutive box strategy and deduce the following placement of digits: 3 must go in (6, 9), 3 must go in (7, 7), 9 must go in (2, 4), and 9 must go in (6, 5). Now, to finish row 6, we place 4 in (6, 8) and 2 in (6, 4) as shown in Figure 1.23. (Remember, each row must contain each of the digits 1 through 9.)

After we place 7 in (5, 4), column 4 is complete. (Remember, each column must contain each of the digits 1 through 9.) This leads to placing 4 in (4, 5), thus completing box 5; row 5 is finalized by placing 8 in (5, 8) as shown in Figure 1.24.

Now column 7 is completed by placing 4 in (2, 7) and 2 in (3, 7); placing 2 in (2, 6) and 7 in (3, 6) completes column 6 as shown in Figure 1.25.

At this point, we deduce that the digit in (3, 2) must be 1 because row 3 cannot have two 7’s. This in turn reveals that 7 must go in (1, 2), and box 1 is now complete. To complete row 7, we place 4 in (7, 9) and 2 in (7, 8); row 4 is finished with 2 in (4, 9) and 7 in (4, 8). See Figure 1.26.
As a final check, we scrutinize each box, row, and column to verify that no box, row, or column contains the same digit twice. Congratulations, the puzzle has been solved!

To finish rows 1, 2, and 3, 1 must go in (1, 9), 7 must go in (2, 9), and 9 must go in (3, 9). The puzzle is now complete as shown in Figure 1.27.

Box 1, row 7, and row 4 are complete.

Columns 7 and 6 are complete.

To finish rows 1, 2, and 3, 1 must go in (1, 9), 7 must go in (2, 9), and 9 must go in (3, 9). The puzzle is now complete as shown in Figure 1.27.

As a final check, we scrutinize each box, row, and column to verify that no box, row, or column contains the same digit twice. Congratulations, the puzzle has been solved!
1.1 **EXERCISES**

In Exercises 1–20, construct a Venn diagram to determine the validity of the given argument.

1. a. 1. All master photographers are artists.
2. Ansel Adams is a master photographer.
   Therefore, Ansel Adams is an artist.

b. 1. All master photographers are artists.
2. Ansel Adams is an artist.
   Therefore, Ansel Adams is a master photographer.

2. a. 1. All Olympic gold medal winners are role models.
2. Michael Phelps is an Olympic gold medal winner.
   Therefore, Michael Phelps is a role model.

b. 1. All Olympic gold medal winners are role models.
2. Michael Phelps is a role model.
   Therefore, Michael Phelps is an Olympic gold medal winner.

3. a. 1. All homeless people are unemployed.
2. Bill Gates is not a homeless person.
   Therefore, Bill Gates is not unemployed.

b. 1. All homeless people are unemployed.
2. Bill Gates is not a homeless person.
   Therefore, Bill Gates is not an unemployed person.

4. a. 1. All professional wrestlers are actors.
2. Ralph Nader is not an actor.
   Therefore, Ralph Nader is not a professional wrestler.

b. 1. All professional wrestlers are actors.
2. Ralph Nader is not a professional wrestler.
   Therefore, Ralph Nader is not an actor.

5. 1. All pesticides are harmful to the environment.
2. No fertilizer is a pesticide.
   Therefore, no fertilizer is harmful to the environment.

6. 1. No one who can afford health insurance is unemployed.
2. All politicians can afford health insurance.
   Therefore, no politician is unemployed.

7. 1. No vegetarian owns a gun.
2. All policemen own guns.
   Therefore, no policeman is a vegetarian.

8. 1. No professor is a millionaire.
2. No millionaire is illiterate.
   Therefore, no professor is illiterate.

9. 1. All poets are loners.
2. All loners are taxi drivers.
   Therefore, all poets are taxi drivers.

10. 1. All forest rangers are environmentalists.
2. All forest rangers are storytellers.
   Therefore, all environmentalists are storytellers.

11. 1. Real men don’t eat quiche.
2. Clint Eastwood is a real man.
   Therefore, Clint Eastwood doesn’t eat quiche.

12. 1. Real men don’t eat quiche.
2. Oscar Meyer eats quiche.
   Therefore, Oscar Meyer isn’t a real man.

13. 1. All roads lead to Rome.
2. Route 66 is a road.
   Therefore, Route 66 leads to Rome.

14. 1. All smiling cats talk.
2. The Cheshire Cat smiles.
   Therefore, the Cheshire Cat talks.

15. 1. Some animals are dangerous.
2. A tiger is an animal.
   Therefore, a tiger is dangerous.

16. 1. Some professors wear glasses.
2. Mr. Einstein wears glasses.
   Therefore, Mr. Einstein is a professor.

17. 1. Some women are police officers.
2. Some police officers ride motorcycles.
   Therefore, some women ride motorcycles.

18. 1. All poets are eloquent.
2. Some poets are wine connoisseurs.
   Therefore, some wine connoisseurs are eloquent.

19. 1. All squares are rectangles.
2. Some quadrilaterals are squares.
   Therefore, some quadrilaterals are rectangles.

20. 1. All squares are rectangles.
2. Some quadrilaterals are rectangles.
   Therefore, some quadrilaterals are squares.

21. Classify each argument as deductive or inductive.
   a. 1. My television set did not work two nights ago.
2. My television set did not work last night.
   Therefore, my television set is broken.

   b. 1. All electronic devices give their owners grief.
2. My television set is an electronic device.
   Therefore, my television set gives me grief.
22. Classify each argument as deductive or inductive.
   a. 1. I ate a chili dog at Joe’s and got indigestion.
      2. I ate a chili dog at Ruby’s and got indigestion.
      Therefore, chili dogs give me indigestion.
   b. 1. All spicy foods give me indigestion.
      2. Chili dogs are spicy food.
      Therefore, chili dogs give me indigestion.

In Exercises 23–32, fill in the blank with what is most likely to be the next number. Explain (using complete sentences) the pattern generated by your answer.

23. 3, 8, 13, 18, ______
24. 10, 11, 13, 16, ______
25. 0, 2, 6, 12, ______
26. 1, 2, 5, 10, ______
27. 1, 4, 9, 16, ______
28. 1, 8, 27, 64, ______
29. 2, 3, 5, 7, 11, ______
30. 1, 1, 2, 3, 5, ______
31. 5, 8, 11, 2, ______
32. 12, 5, 10, 3, ______

In Exercises 33–36, fill in the blanks with what are most likely to be the next letters. Explain (using complete sentences) the pattern generated by each of your answers.

33. O, T, T, F, ______, ______
34. T, F, S, E, ______, ______
35. F, S, S, M, ______, ______

In Exercises 37–42, explain the general rule or pattern used to assign the given letter to the given word. Fill in the blank with the letter that fits the pattern.

37. circle square trapezoid octagon rectangle
   c s t o ______

38. circle square trapezoid octagon rectangle
   i u a o ______

39. circle square trapezoid octagon rectangle
   j v b p ______

40. circle square trapezoid octagon rectangle
   c r p g ______

41. banana strawberry asparagus eggplant orange
   b z t u ______

42. banana strawberry asparagus eggplant orange
   y r g p ______

43. Find two different numbers that could be used to fill in the blank.
   1, 4, 7, 10, ______
   Explain the pattern generated by each of your answers.

44. Find five different numbers that could be used to fill in the blank.
   7, 14, 21, 28, ______
   Explain the pattern generated by each of your answers.

45. Example 1 utilized the Quadratic Formula. Verify that
   \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
   is a solution of the equation \( ax^2 + bx + c = 0 \).
   \( \text{HINT: Substitute the fraction for } x \text{ in } ax^2 + bx + c \)
   and simplify.

46. Example 1 utilized the Quadratic Formula. Verify that
   \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
   is a solution of the equation \( ax^2 + bx + c = 0 \).
   \( \text{HINT: Substitute the fraction for } x \text{ in } ax^2 + bx + c \)
   and simplify.

47. As a review of algebra, use the Quadratic Formula to solve
   \( x^2 - 6x + 7 = 0 \)

48. As a review of algebra, use the Quadratic Formula to solve
   \( x^2 - 2x - 4 = 0 \)

Solve the sudoku puzzles in Exercises 49–54.

49. 
   
   5 7 3 4 9
   6 1 8 9 2 1 5 6
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   4 3 8 4 1
1.1 Exercises

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Answer the following questions using complete sentences and your own words.

- **CONCEPT QUESTIONS**

55. Explain the difference between deductive reasoning and inductive reasoning.
56. Explain the difference between truth and validity.
57. What is a syllogism? Give an example of a syllogism that relates to your life.

- **HISTORY QUESTIONS**

58. From the days of the ancient Greeks, the study of logic has been mandatory in what two professions? Why?
59. Who developed a formal system of deductive logic based on arguments?
60. What was the name of the school Aristotle founded? What does it mean?
61. How did Aristotle’s school of thought differ from Plato’s?

- **WEB PROJECT**

62. Obtain a sudoku puzzle and its solution from a popular web site. Some useful links for this web project are listed on the text web site:

www.cengage.com/math/johnson
1.2 Symbolic Logic

OBJECTIVES

- Identify simple statements
- Express a compound statement in symbolic form
- Create the negation of a statement
- Express a conditional statement in terms of necessary and sufficient conditions

The syllogism ruled the study of logic for nearly 2,000 years and was not supplanted until the development of symbolic logic in the late seventeenth century. As its name implies, symbolic logic involves the use of symbols and algebraic manipulations in logic.

Statements

All logical reasoning is based on statements. A statement is a sentence that is either true or false.

EXAMPLE 1

IDENTIFYING STATEMENTS Which of the following are statements? Why or why not?

a. Apple manufactures computers.
b. Apple manufactures the world’s best computers.
c. Did you buy a Dell?
d. A $2,000 computer that is discounted 25% will cost $1,000.
e. I am telling a lie.

SOLUTION

a. The sentence “Apple manufactures computers” is true; therefore, it is a statement.
b. The sentence “Apple manufactures the world’s best computers” is an opinion, and as such, it is neither true nor false. It is true for some people and false for others. Therefore, it is not a statement.
c. The sentence “Did you buy a Dell?” is a question. As such, it is neither true nor false; it is not a statement.
d. The sentence “A $2,000 computer that is discounted 25% will cost $1,000” is false; therefore, it is a statement. (A $2,000 computer that is discounted 25% would cost $1,500.)
e. The sentence “I am telling a lie” is a self-contradiction, or paradox. If it were true, the speaker would be telling a lie, but in telling the truth, the speaker would be contradicting the statement that he or she was lying; if it were false, the speaker would not be telling a lie, but in not telling a lie, the speaker would be contradicting the statement that he or she was lying. The sentence is not a statement.

By tradition, symbolic logic uses lowercase letters as labels for statements. The most frequently used letters are \( p, q, r, s \), and \( t \). We can label the statement “It is snowing” as statement \( p \) in the following manner:

\[ p: \text{It is snowing.} \]

If it is snowing, \( p \) is labeled true, whereas if it is not snowing, \( p \) is labeled false.
**Compound Statements and Logical Connectives**

It is easy to determine whether a statement such as “Charles donated blood” is true or false; either he did or he didn’t. However, not all statements are so simple; some are more involved. For example, the truth of “Charles donated blood and did not wash his car, or he went to the library,” depends on the truth of the individual pieces that make up the larger, compound statement. A compound statement is a statement that contains one or more simpler statements. A compound statement can be formed by inserting the word not into a simpler statement or by joining two or more statements with connective words such as and, or, if . . . then . . ., only if, and if and only if. The compound statement “Charles did not wash his car” is formed from the simpler statement “Charles did wash his car.” The compound statement “Charles donated blood and did not wash his car, or he went to the library” consists of three statements, each of which may be true or false.

Figure 1.28 diagrams two equivalent compound statements.

When is a compound statement true? Before we can answer this question, we must first examine the various ways in which statements can be connected. Depending on how the statements are connected, the resulting compound statement can be a negation, a conjunction, a disjunction, a conditional, or any combination thereof.

**The Negation ~p**

The negation of a statement is the denial of the statement and is represented by the symbol ~. The negation is frequently formed by inserting the word not. For example, given the statement “p: It is snowing,” the negation would be “~p: It is not snowing.” If it is snowing, p is true and ~p is false. Similarly, if it is not snowing, p is false and ~p is true. A statement and its negation always have opposite truth values; when one is true, the other is false. Because the truth of the negation depends on the truth of the original statement, a negation is classified as a compound statement.

**EXAMPLE 2**  **WRITING A NEGATION**  Write a sentence that represents the negation of each statement:

a. The senator is a Democrat.
b. The senator is not a Democrat.
c. Some senators are Republicans.
d. All senators are Republicans.
e. No senator is a Republican.
The compound statement "Norma Rae is a union member and she is not a Democrat" can be expressed as a conjunction, symbolized as $p \land q$.

**Example 3**

**Translating Words into Symbols**
Using the symbolic representations

- $p$: Norma Rae is a union member.
- $q$: Norma Rae is a Democrat.

express the following compound statements in symbolic form:

- a. Norma Rae is a union member and she is a Democrat.
- b. Norma Rae is a union member and she is not a Democrat.

<table>
<thead>
<tr>
<th>$p$: Norma Rae is a union member.</th>
<th>$q$: Norma Rae is a Democrat.</th>
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<tbody>
<tr>
<td>$p \land q$: Norma Rae is a union member and she is a Democrat.</td>
<td>$p \land \neg q$: Norma Rae is a union member and she is not a Democrat.</td>
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</table>
The Disjunction \( p \lor q \)

When statements are connected by the word or, a disjunction is formed. We use the symbol \( \lor \) to represent the word or. Thus, the disjunction “\( p \lor q \)” represents the compound statement “\( p \) or \( q \).” We can interpret the word or in two ways. Consider the statements

\[
p: \text{Kaitlin is a registered Republican.} \\
q: \text{Paki is a registered Republican.}
\]

The statement “Kaitlin is a registered Republican or Paki is a registered Republican” can be symbolized as \( p \lor q \). Notice that it is possible that both Kaitlin and Paki are registered Republicans. In this example, or includes the possibility that both things may happen. In this case, we are working with the inclusive or.

Now consider the statements

\[
p: \text{Kaitlin is a registered Republican.} \\
q: \text{Kaitlin is a registered Democrat.}
\]

The statement “Kaitlin is a registered Republican or Kaitlin is a registered Democrat” does not include the possibility that both may happen; one statement excludes the other. When this happens, we are working with the exclusive or. In our study of symbolic logic (as in most mathematics), we will always use the inclusive or. Therefore, “\( p \) or \( q \)” means “\( p \) or \( q \) or both.”

**Example 4**

**Translating Symbols into Words**  Using the symbolic representations

\[
p: \text{Juanita is a college graduate.} \\
q: \text{Juanita is employed.}
\]

express the following compound statements in words:

\[
a. p \lor q \\
b. p \land q \\
c. p \lor \neg q \\
d. \neg p \land q
\]

**Solution**

a. \( p \lor q \) represents the statement “Juanita is a college graduate or Juanita is employed (or both).”

b. \( p \land q \) represents the statement “Juanita is a college graduate and Juanita is employed.”

c. \( p \lor \neg q \) represents the statement “Juanita is a college graduate or Juanita is not employed.”

d. \( \neg p \land q \) represents the statement “Juanita is not a college graduate and Juanita is employed.”

The Conditional \( p \to q \)

Consider the statement “If it is raining, then the streets are wet.” This is a compound statement because it connects two statements, namely, “it is raining” and “the streets are wet.” Notice that the statements are connected with “if . . . then . . .” phrasing. Any statement of the form “if \( p \) then \( q \)” is called a conditional (or an implication); \( p \) is called the hypothesis (or premise) of the conditional, and \( q \) is called the conclusion of the conditional. The conditional “if \( p \) then \( q \)” is represented by the symbols “\( p \to q \)” (\( p \) implies \( q \)). When people use conditionals in
CHAPTER 1 Logic

GOTTFRID WILHELM LEIBNIZ 1646–1716

Leibniz’s affinity for logic was characterized by his search for a characteristica universalis, or “universal character.” Leibniz believed that by combining logic and mathematics, a general symbolic language could be created in which all scientific problems could be solved with a minimum of effort. In this universal language, statements and the logical relationships between them would be represented by letters and symbols. In Leibniz’s words, “All truths of reason would be reduced to a kind of calculus, and the errors would only be errors of computation.” In essence, Leibniz believed that once a problem had been translated into this universal language of symbolic logic, it would be solved automatically by simply applying the mathematical rules that governed the manipulation of the symbols.

Leibniz’s work in the field of symbolic logic did not arouse much academic curiosity; many say that it was too far ahead of its time. The study of symbolic logic was not systematically investigated again until the nineteenth century.

In addition to cofounding calculus (see Chapter 13), the German-born Gottfried Wilhelm Leibniz contributed much to the development of symbolic logic. A precocious child, Leibniz was self-taught in many areas. He taught himself Latin at the age of eight and began the study of Greek when he was twelve. In the process, he was exposed to the writings of Aristotle and became intrigued by formalized logic.

At the age of fifteen, Leibniz entered the University of Leipzig to study law. He received his bachelor’s degree two years later, earned his master’s degree the following year, and then transferred to the University of Nuremberg.

Leibniz received his doctorate in law within a year and was immediately offered a professorship but refused it, saying that he had “other things in mind.” Besides law, these “other things” included politics, religion, history, literature, metaphysics, philosophy, logic, and mathematics. Thereafter, Leibniz worked under the sponsorship of the courts of various nobles, serving as lawyer, historian, and librarian to the elite. At one point, Leibniz was offered the position of librarian at the Vatican but declined the offer.

Everyday speech, they often omit the word then, as in “If it is raining, the streets are wet.” Alternatively, the conditional “if p then q” may be phrased as “q if p” (“The streets are wet if it is raining”).

EXAMPLE 5 TRANSLATING WORDS INTO SYMBOLS Using the symbolic representations

\[ p: \text{I am healthy.} \]
\[ q: \text{I eat junk food.} \]
\[ r: \text{I exercise regularly.} \]

express the following compound statements in symbolic form:

a. I am healthy if I exercise regularly.

b. If I eat junk food and do not exercise, then I am not healthy.
1.2 Symbolic Logic

SOLUTION

a. “I am healthy if I exercise regularly” is a conditional (if . . . then . . .) and can be rephrased as follows:

“If I exercise regularly, then I am healthy.”

Statement \( r \) is the premise. Statement \( p \) is the conclusion.

The given compound statement can be expressed as \( r \rightarrow p \).

b. “If I eat junk food and do not exercise, then I am not healthy” is a conditional (if . . . then . . .) that contains a conjunction (and) and two negations (not):

“If I eat junk food and do not exercise, then I am not healthy.”

The premise contains a conjunction and a negation. The conclusion contains a negation.

The premise of the conditional can be represented by \( q \land \sim r \), while the conclusion can be represented by \( \sim p \). Thus, the given compound statement has the symbolic form \((q \land \sim r) \rightarrow \sim p\).

EXAMPLE 6

TRANSLATING WORDS INTO SYMBOLS  Express the following statements in symbolic form:

a. All mammals are warm-blooded.

b. No snake is warm-blooded.

SOLUTION

a. The statement “All mammals are warm-blooded” can be rephrased as “If it is a mammal, then it is warm-blooded.” Therefore, we define two simple statements \( p \) and \( q \) as:

\( p \): It is a mammal.

\( q \): It is warm-blooded.

The statement now has the form

“If it is a mammal, then it is warm-blooded.”

Statement \( p \) is the premise. Statement \( q \) is the conclusion.

and can be expressed as \( p \rightarrow q \). In general, any statement of the form “All \( p \) are \( q \)” can be symbolized as \( p \rightarrow q \).

b. The statement “No snake is warm-blooded” can be rephrased as “If it is a snake, then it is not warm-blooded.” Therefore, we define two simple statements \( p \) and \( q \) as:

\( p \): It is a snake.

\( q \): It is warm-blooded.

The statement now has the form

“If it is a snake, then it is not warm-blooded.”

Statement \( p \) is the premise. The negation of statement \( q \) is the conclusion.

and can be expressed as \( p \rightarrow \sim q \). In general, any statement of the form “No \( p \) is \( q \)” can be symbolized as \( p \rightarrow \sim q \).
As Example 6 shows, conditionals are not always expressed in the form “if $p$ then $q$.” In addition to “all $p$ are $q$,” other standard forms of a conditional include statements that contain the word *sufficient* or *necessary*.

Consider the statement “Being a mammal is sufficient for being warm-blooded.” One definition of the word *sufficient* is “adequate.” Therefore, “being a mammal” is an adequate condition for “being warm-blooded”; hence, “being a mammal” implies “being warm-blooded.” Logically, the statement “Being a mammal is sufficient for being warm-blooded” is equivalent to saying “If it is a mammal, then it is warm-blooded.” Consequently, the general statement “$p$ is sufficient for $q$” is an alternative form of the conditional “if $p$ then $q$” and can be symbolized as $p \rightarrow q$.

“Being a mammal” is a sufficient (adequate) condition for “being warm-blooded,” but is it a necessary condition? Of course not: some animals are warm-blooded but are not mammals (chickens, for example). One definition of the word *necessary* is “required.” Therefore, “being a mammal” is not required for “being warm-blooded.” However, is “being warm-blooded” a necessary (required) condition for “being a mammal”? Of course it is: all mammals are warm-blooded (that is, there are no cold-blooded mammals). Logically, the statement “being warm-blooded is necessary for being a mammal” is equivalent to saying “If it is a mammal, then it is warm-blooded.” Consequently, the general statement “$q$ is necessary for $p$” is an alternative form of the conditional “if $p$ then $q$” and can be symbolized as $p \rightarrow q$.

In summary, a sufficient condition is the hypothesis or premise of a conditional statement, whereas a necessary condition is the conclusion of a conditional statement.

### Example 7

**Translating Words into Symbols** Using the symbolic representations

- $p$: A person obeys the law.
- $q$: A person is arrested.

express the following compound statements in symbolic form:

a. Being arrested is necessary for not obeying the law.

b. Obeying the law is sufficient for not being arrested.

**Solution**

a. “Being arrested” is a necessary condition; hence, “a person is arrested” is the conclusion of a conditional statement.

A person does not obey the law. \[\rightarrow\] A person is arrested.

A necessary condition.

Therefore, the statement “Being arrested is necessary for not obeying the law” can be rephrased as follows:

“If a person does not obey the law, then the person is arrested.”

The negation of statement $p$ is the premise. \[\neg\]

Statement $q$ is the conclusion.

The given compound statement can be expressed as $\neg p \rightarrow q$. 
b. “Obeying the law” is a sufficient condition; hence, “A person obeys the law” is the premise of a conditional statement.

A sufficient condition.

Therefore, the statement “Obeying the law is sufficient for not being arrested” can be rephrased as follows:

“If a person obeys the law, then the person is not arrested.”

The given compound statement can be expressed as $p \rightarrow \neg q$.

We have seen that a statement is a sentence that is either true or false and that connecting two or more statements forms a compound statement. Figure 1.30 summarizes the logical connectives and symbols that were introduced in this section. The various connectives have been defined; we can now proceed in our analysis of the conditions under which a compound statement is true. This analysis is carried out in the next section.

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<th>Symbol</th>
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<td>$\neg p$</td>
<td>not $p$</td>
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<tr>
<td>conjunction</td>
<td>$p \land q$</td>
<td>$p$ and $q$</td>
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<tr>
<td>disjunction</td>
<td>$p \lor q$</td>
<td>$p$ or $q$</td>
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<tr>
<td>conditional (implication)</td>
<td>$p \rightarrow q$</td>
<td>if $p$, then $q$</td>
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<td>$p$ is sufficient for $q$</td>
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<td></td>
<td>$q$ is necessary for $p$</td>
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Figure 1.30 Logical connectives.

1. Which of the following are statements? Why or why not?
   a. George Washington was the first president of the United States.
   b. Abraham Lincoln was the second president of the United States.
   c. Who was the first vice president of the United States?
   d. Abraham Lincoln was the best president.

2. Which of the following are statements? Why or why not?
   a. $3 + 5 = 6$
   b. Solve the equation $2x + 5 = 3$.
   c. $x^2 + 1 = 0$ has no solution.
   d. $x^2 - 1 = (x + 1)(x - 1)$
   e. Is $\sqrt{2}$ a rational number?

3. Determine which pairs of statements are negations of each other.
   a. All of the fruits are red.
   b. None of the fruits is red.
   c. Some of the fruits are red.
   d. Some of the fruits are not red.

4. Determine which pairs of statements are negations of each other.
   a. Some of the beverages contain caffeine.
   b. Some of the beverages do not contain caffeine.
   c. None of the beverages contain caffeine.
   d. All of the beverages contain caffeine.

5. Write a sentence that represents the negation of each statement.
   a. Her dress is not red.
   b. Some computers are priced under $100.
c. All dogs are four-legged animals.
d. No sleeping bag is waterproof.

6. Write a sentence that represents the negation of each statement.
   a. She is not a vegetarian.
   b. Some elephants are pink.
   c. All candy promotes tooth decay.
   d. No lunch is free.

7. Using the symbolic representations
   \( p \): The lyrics are controversial.
   \( q \): The performance is banned.
   express the following compound statements in symbolic form.
   a. The lyrics are controversial, and the performance is banned.
   b. If the lyrics are not controversial, the performance is not banned.
   c. It is not the case that the lyrics are controversial or the performance is banned.
   d. The lyrics are controversial, and the performance is not banned.
   e. Having controversial lyrics is sufficient for banning a performance.
   f. Noncontroversial lyrics are necessary for not banning a performance.

8. Using the symbolic representations
   \( p \): The food is spicy.
   \( q \): The food is aromatic.
   express the following compound statements in symbolic form.
   a. The food is aromatic and spicy.
   b. If the food isn’t spicy, it isn’t aromatic.
   c. The food is spicy, and it isn’t aromatic.
   d. The food isn’t spicy or aromatic.
   e. Being nonaromatic is sufficient for food to be nonspicy.
   f. Being spicy is necessary for food to be aromatic.

9. Using the symbolic representations
   \( p \): A person plays the guitar.
   \( q \): A person rides a motorcycle.
   \( r \): A person wears a leather jacket.
   express the following compound statements in symbolic form.
   a. If a person plays the guitar or rides a motorcycle, then the person wears a leather jacket.
   b. A person plays the guitar, rides a motorcycle, and wears a leather jacket.
   c. A person wears a leather jacket and doesn’t play the guitar or ride a motorcycle.
   d. All motorcycle riders wear leather jackets.
   e. Not wearing a leather jacket is sufficient for not playing the guitar or riding a motorcycle.
   f. Riding a motorcycle or playing the guitar is necessary for wearing a leather jacket.

10. Using the symbolic representations
    \( p \): The car costs $70,000.
    \( q \): The car goes 140 mph.
    \( r \): The car is red.
    express the following compound statements in symbolic form.
    a. All red cars go 140 mph.
    b. The car is red, goes 140 mph, and does not cost $70,000.
    c. If the car does not cost $70,000, it does not go 140 mph.
    d. The car is red and it does not go 140 mph or cost $70,000.
    e. Being able to go 140 mph is sufficient for a car to cost $70,000 or be red.
    f. Not being red is necessary for a car to cost $70,000 and not go 140 mph.

In Exercises 11–34, translate the sentence into symbolic form. Be sure to define each letter you use. (More than one answer is possible.)

11. All squares are rectangles.
12. All people born in the United States are American citizens.
13. No square is a triangle.
14. No convicted felon is eligible to vote.
15. All whole numbers are even or odd.
16. All muscle cars from the Sixties are polluters.
17. No whole number is greater than 3 and less than 4.
18. No electric-powered car is a polluter.
19. Being an orthodontist is sufficient for being a dentist.
20. Being an author is sufficient for being literate.
21. Knowing Morse code is necessary for operating a telegraph.
22. Knowing CPR is necessary for being a paramedic.
23. Being a monkey is sufficient for not being an ape.
24. Being a chimpanzee is sufficient for not being a monkey.
25. Not being a monkey is necessary for being an ape.
26. Not being a chimpanzee is necessary for being a monkey.
27. I do not sleep soundly if I drink coffee or eat chocolate.
28. I sleep soundly if I do not drink coffee or eat chocolate.
29. Your check is not accepted if you do not have a driver’s license or a credit card.
30. Your check is accepted if you have a driver’s license or a credit card.
31. If you drink and drive, you are fined or you go to jail.
32. If you are rich and famous, you have many friends and enemies.
33. You get a refund or a store credit if the product is defective.
34. The streets are slippery if it is raining or snowing.
35. Using the symbolic representations
   
   \[ p: \text{I am an environmentalist.} \]
   \[ q: \text{I recycle my aluminum cans.} \]
   \[ r: \text{I recycle my newspapers.} \]
   \[ s: \text{I go to jail.} \]

   express the following in words.

   a. \( p \land q \)
   b. \( p \rightarrow q \)
   c. \( p \land q \rightarrow r \)
   d. \( p \rightarrow r \)

36. Using the symbolic representations
   
   \[ p: \text{I am innocent.} \]
   \[ q: \text{I have an alibi.} \]
   \[ r: \text{I recycle my aluminum cans.} \]
   \[ s: \text{I go to jail.} \]

   express the following in words.

   a. \( p \land q \)
   b. \( p \rightarrow q \)
   c. \( (p \land q) \rightarrow r \)
   d. \( q \rightarrow r \)

37. Using the symbolic representations
   
   \[ p: \text{I am an environmentalist.} \]
   \[ q: \text{I recycle my aluminum cans.} \]
   \[ r: \text{I recycle my newspapers.} \]
   \[ s: \text{I go to jail.} \]

   express the following in words.

   a. \( (q \land r) \rightarrow p \)
   b. \( (p \land q) \rightarrow r \)
   c. \( (q \land r) \rightarrow s \)
   d. \( (p \land r) \rightarrow q \)

38. Using the symbolic representations
   
   \[ p: \text{I am innocent.} \]
   \[ q: \text{I have an alibi.} \]
   \[ r: \text{I recycle my newspapers.} \]
   \[ s: \text{I go to jail.} \]

   express the following in words.

   a. \( (p \lor q) \rightarrow r \)
   b. \( (p \land q) \rightarrow r \)
   c. \( (p \land q) \lor r \)
   d. \( (p \land q) \rightarrow r \)

39. Which statement, \#1 or \#2, is more appropriate? Explain why.
   
   Statement \#1: “Cold weather is necessary for it to snow.”
   Statement \#2: “Cold weather is sufficient for it to snow.”

40. Which statement, \#1 or \#2, is more appropriate? Explain why.
   
   Statement \#1: “Being cloudy is necessary for it to rain.”
   Statement \#2: “Being cloudy is sufficient for it to rain.”

41. Which statement, \#1 or \#2, is more appropriate? Explain why.
   
   Statement \#1: “Having 31 days in a month is necessary for it not to be February.”
   Statement \#2: “Having 31 days in a month is sufficient for it not to be February.”

42. Which statement, \#1 or \#2, is more appropriate? Explain why.
   
   Statement \#1: “Being the Fourth of July is necessary for the U.S. Post Office to be closed.”
   Statement \#2: “Being the Fourth of July is sufficient for the U.S. Post Office to be closed.”

**CONCEPT QUESTIONS**

43. What is a negation? 44. What is a conjunction?
45. What is a disjunction? 46. What is a conditional?
47. What is a sufficient condition?
48. What is a necessary condition?
49. What is the difference between the inclusive or and the exclusive or?
50. Create a sentence that is a self-contradiction, or paradox, as in part (e) of Example 1.

**HISTORY QUESTIONS**

51. In what academic field did Gottfried Leibniz receive his degrees? Why is the study of logic important in this field?
52. Who developed a formal system of logic based on syllogistic arguments?
53. What is meant by *characteristica universalis*? Who proposed this theory?

**THE NEXT LEVEL**

If a person wants to pursue an advanced degree (something beyond a bachelor’s or four-year degree), chances are the person must take a standardized exam to gain admission to a graduate school or to be admitted into a specific program. These exams are intended to measure verbal, quantitative, and analytical skills that have developed throughout a person’s life. Many classes and study guides are available to help people prepare for the exams. The following questions are typical of those found in the study guides.

Exercises 54–58 refer to the following: A culinary institute has a small restaurant in which the students prepare various dishes. The menu changes daily, and during a specific week, the following dishes are to be prepared: moussaka, pilaf, quiche, ratatouille, stroganoff, and teriyaki. During the week, the restaurant does not prepare any other kind of dish. The selection of dishes the restaurant offers is consistent with the following conditions:

- If the restaurant offers pilaf, then it does not offer ratatouille.
- If the restaurant does not offer stroganoff, then it offers pilaf.
- If the restaurant offers quiche, then it offers both ratatouille and teriyaki.
- If the restaurant offers teriyaki, then it offers moussaka or stroganoff or both.

54. Which one of the following could be a complete and accurate list of the dishes the restaurant offers on a specific day?
   a. pilaf, quiche, ratatouille, teriyaki
   b. quiche, stroganoff, teriyaki
c. quiche, ratatouille, teriyaki
d. ratatouille, stroganoff
e. quiche, ratatouille

55. Which one of the following cannot be a complete and accurate list of the dishes the restaurant offers on a specific day?
   a. moussaka, pilaf, quiche, ratatouille, teriyaki
   b. quiche, ratatouille, stroganoff, teriyaki
   c. moussaka, pilaf, teriyaki
   d. stroganoff, teriyaki
   e. pilaf, stroganoff

56. Which one of the following could be the only kind of dish the restaurant offers on a specific day?
   a. teriyaki
   b. stroganoff
c. ratatouille
d. quiche
e. moussaka

57. If the restaurant does not offer teriyaki, then which one of the following must be true?
   a. The restaurant offers pilaf.
   b. The restaurant offers at most three different dishes.
   c. The restaurant offers at least two different dishes.
   d. The restaurant offers neither quiche nor ratatouille.
   e. The restaurant offers neither quiche nor pilaf.

58. If the restaurant offers teriyaki, then which one of the following must be false?
   a. The restaurant does not offer moussaka.
   b. The restaurant does not offer ratatouille.
   c. The restaurant does not offer stroganoff.
   d. The restaurant offers ratatouille but not quiche.
   e. The restaurant offers ratatouille but not stroganoff.
The Negation \( \sim p \)

The negation of a statement is the denial, or opposite, of the statement. (As was stated in the previous section, because the truth value of the negation depends on the truth value of the original statement, a negation can be classified as a compound statement.) To construct the truth table for the negation of a statement, we must first examine the original statement. A statement \( p \) may be true or false, as shown in Figure 1.31. If the statement \( p \) is true, the negation \( \sim p \) is false; if \( p \) is false, \( \sim p \) is true. The truth table for the compound statement \( \sim p \) is given in Figure 1.32. Row 1 of the table is read “\( \sim p \) is false when \( p \) is true.” Row 2 is read “\( \sim p \) is true when \( p \) is false.”

The Conjunction \( p \land q \)

A conjunction is the joining of two statements with the word and. The compound statement “Maria is a doctor and a Republican” is a conjunction with the following symbolic representation:

- \( p \): Maria is a doctor.
- \( q \): Maria is a Republican.
- \( p \land q \): Maria is a doctor and a Republican.

The truth value of a compound statement depends on the truth values of the individual statements that make it up. How many rows will the truth table for the conjunction \( p \land q \) contain? Because \( p \) has two possible truth values (T or F) and \( q \) has two possible truth values (T or F), we need four \( (2 \cdot 2) \) rows in order to list all possible combinations of Ts and Fs, as shown in Figure 1.33.

For the conjunction \( p \land q \) to be true, the components \( p \) and \( q \) must both be true; the conjunction is false otherwise. The completed truth table for the conjunction \( p \land q \) is given in Figure 1.34. The symbols \( p \) and \( q \) can be replaced by any statements. The table gives the truth value of the statement “\( p \) and \( q \)” dependent upon the truth values of the individual statements “\( p \)” and “\( q \).” For instance, row 3 is read “The conjunction \( p \land q \) is false when \( p \) is false and \( q \) is true.” The other rows are read in a similar manner.

The Disjunction \( p \lor q \)

A disjunction is the joining of two statements with the word or. The compound statement “Maria is a doctor or a Republican” is a disjunction (the inclusive or) with the following symbolic representation:

- \( p \): Maria is a doctor.
- \( q \): Maria is a Republican.
- \( p \lor q \): Maria is a doctor or a Republican.

Even though your friend Maria the doctor is not a Republican, the disjunction “Maria is a doctor or a Republican” is true. For a disjunction to be true, at least one of the components must be true. A disjunction is false only when both components are false. The truth table for the disjunction \( p \lor q \) is given in Figure 1.35.
EXAMPLE 1

CONSTRUCTING A TRUTH TABLE  Under what specific conditions is the following compound statement true? “I have a high school diploma, or I have a full-time job and no high school diploma.”

SOLUTION

First, we translate the statement into symbolic form, and then we construct the truth table for the symbolic expression. Define $p$ and $q$ as

- $p$: I have a high school diploma.
- $q$: I have a full-time job.

The given statement has the symbolic representation $p \lor (q \land \neg p)$.

Because there are two letters, we need $2 \cdot 2 = 4$ rows. We need to insert a column for each connective in the symbolic expression $p \lor (q \land \neg p)$. As in algebra, we start inside any grouping symbols and work our way out. Therefore, we need a column for $\neg p$, a column for $q \land \neg p$, and a column for the entire expression $p \lor (q \land \neg p)$, as shown in Figure 1.36.

![Figure 1.36](image)

The completed truth table is shown in Figure 1.37.

![Figure 1.37](image)

In the $\neg p$ column, fill in truth values that are opposite those for $p$. Next, the conjunction $q \land \neg p$ is true only when both components are true; enter a T in row 3 and Fs elsewhere. Finally, the disjunction $p \lor (q \land \neg p)$ is false only when both components $p$ and $(q \land \neg p)$ are false; enter an F in row 4 and Ts elsewhere. The completed truth table is shown in Figure 1.37.

As is indicated in the truth table, the symbolic expression $p \lor (q \land \neg p)$ is true under all conditions except one: row 4; the expression is false when both $p$ and $q$ are false. Therefore, the statement “I have a high school diploma, or I have a full-time job and no high school diploma” is true in every case except when the speaker has no high school diploma and no full-time job.

If the symbolic representation of a compound statement consists of two different letters, its truth table will have $2 \cdot 2 = 4$ rows. How many rows are required if a compound statement consists of three letters—say, $p$, $q$, and $r$? Because each statement has two possible truth values (T and F), the truth table must contain $2 \cdot 2 \cdot 2 = 8$ rows. In general, each time a new statement is added, the number of rows doubles.
EXAMPLE 2

CONSTRUCTING A TRUTH TABLE  
Under what specific conditions is the following compound statement true? “I own a handgun, and it is not the case that I am a criminal or police officer.”

SOLUTION

First, we translate the statement into symbolic form, and then we construct the truth table for the symbolic expression. Define the three simple statements as follows:

\[ p: \text{I own a handgun.} \]
\[ q: \text{I am a criminal.} \]
\[ r: \text{I am a police officer.} \]

The given statement has the symbolic representation \( p \land \neg(q \lor r) \). Since there are three letters, we need \( 2^3 = 8 \) rows. We start with three columns, one for each letter. To account for all possible combinations of \( p, q, \) and \( r \) as true or false, proceed as follows:

1. Fill the first half (four rows) of column 1 with Ts and the rest with Fs, as shown in Figure 1.38(a).
2. In the next column, split each half into halves, the first half receiving Ts and the second Fs. In other words, alternate two Ts and two Fs in column 2, as shown in Figure 1.38(b).
3. Again, split each half into halves; the first half receives Ts, and the second half receives Fs. Because we are dealing with the third (last) column, the Ts and Fs will alternate, as shown in Figure 1.38(c).

\[
\begin{array}{c|c|c|c}
\hline
p & q & r \\
\hline
1. & T & & \\
2. & T & & \\
3. & T & & \\
4. & T & & \\
5. & F & & \\
6. & F & & \\
7. & F & & \\
8. & F & & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\hline
p & q & r \\
\hline
1. & T & T & \\
2. & T & T & \\
3. & T & F & \\
4. & T & F & \\
5. & F & T & \\
6. & F & T & \\
7. & F & F & \\
8. & F & F & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\hline
p & q & r \\
\hline
1. & T & T & T \\
2. & T & T & F \\
3. & T & F & T \\
4. & T & F & F \\
5. & F & T & T \\
6. & F & T & F \\
7. & F & F & T \\
8. & F & F & F \\
\hline
\end{array}
\]

Figures 1.38(a), (b), (c)

Truth values for three statements.

(This process of filling the first half of the first column with Ts and the second half with Fs and then splitting each half into halves with blocks of Ts and Fs applies to all truth tables.)

We need to insert a column for each connective in the symbolic expression \( p \land \neg(q \lor r) \), as shown in Figure 1.39.

NUMBER OF ROWS

If a compound statement consists of \( n \) individual statements, each represented by a different letter, the number of rows required in its truth table is \( 2^n \).

1.3 Truth Tables

33
### Required columns in the truth table.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>1.</td>
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<td>T</td>
<td></td>
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<tr>
<td>4.</td>
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<td>5.</td>
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<td>F</td>
<td>T</td>
<td>F</td>
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</tr>
<tr>
<td>7.</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

**FIGURE 1.39** Required columns in the truth table.

Now fill in the appropriate symbol in the column under \( q \lor r \). Enter F if *both* \( q \) and \( r \) are false; enter T otherwise (that is, if at least one is true). In the \( \neg(q \lor r) \) column, fill in truth values that are opposite those for \( q \lor r \), as in Figure 1.40.

### Truth values of the expressions \( q \lor r \) and \( \neg(q \lor r) \).

<p>| | | | | |</p>
<table>
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<td>1.</td>
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<td>7.</td>
<td>F</td>
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<td>T</td>
<td>T</td>
</tr>
<tr>
<td>8.</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**FIGURE 1.40** Truth values of the expressions \( q \lor r \) and \( \neg(q \lor r) \).

The conjunction \( p \land \neg(q \lor r) \) is true only when *both* \( p \) and \( \neg(q \lor r) \) are true; enter a T in row 4 and Fs elsewhere. The truth table is shown in Figure 1.41.

### Truth table for \( p \land \neg(q \lor r) \).

<p>| | | | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1.</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>2.</td>
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<tr>
<td>3.</td>
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<td>T</td>
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<tr>
<td>8.</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**FIGURE 1.41** Truth table for \( p \land \neg(q \lor r) \).
As indicated in the truth table, the expression \( p \land \neg (q \lor r) \) is true only when \( p \) is true and both \( q \) and \( r \) are false. Therefore, the statement “I own a handgun, and it is not the case that I am a criminal or police officer” is true only when the speaker owns a handgun, is not a criminal, and is not a police officer—in other words, the speaker is a law-abiding citizen who owns a handgun.

**The Conditional \( p \rightarrow q \)**

A **conditional** is a compound statement of the form “If \( p \), then \( q \)” and is symbolized \( p \rightarrow q \). Under what circumstances is a conditional true, and when is it false? Consider the following (compound) statement: “If you give me $50, then I will give you a ticket to the ballet.” This statement is a conditional and has the following representation:

\[
p: \text{You give me $50.} \\
q: \text{I give you a ticket to the ballet.} \\
p \rightarrow q: \text{If you give me $50, then I will give you a ticket to the ballet.}
\]

The conditional can be viewed as a promise: *If you give me $50, then I will give you a ticket to the ballet.* Suppose you give me $50; that is, suppose \( p \) is true. I have two options: Either I give you a ticket to the ballet (\( q \) is true), or I do not (\( q \) is false). If I do give you the ticket, the conditional \( p \rightarrow q \) is true (I have kept my promise); if I do not give you the ticket, the conditional \( p \rightarrow q \) is false (I have not kept my promise). These situations are shown in rows 1 and 2 of the truth table in Figure 1.42. Rows 3 and 4 require further analysis.

Suppose you do not give me $50; that is, suppose \( p \) is false. Whether or not I give you a ticket, you cannot say that I broke my promise; that is, you cannot say that the conditional \( p \rightarrow q \) is false. Consequently, since a statement is either true or false, the conditional is labeled true (by default). In other words, when the premise \( p \) of a conditional is false, it does not matter whether the conclusion \( q \) is true or false. In both cases, the conditional \( p \rightarrow q \) is automatically labeled true, because it is not false.

The completed truth table for a conditional is given in Figure 1.43. Notice that the only circumstance under which a conditional is false is when the premise \( p \) is true and the conclusion \( q \) is false, as shown in row 2.

**Example 3**

**Solution**

**CONSTRUCTING A TRUTH TABLE** Under what conditions is the symbolic expression \( q \rightarrow \neg p \) true?

Our truth table has \( 2^2 = 4 \) rows and contains a column for \( p \), \( q \), \( \neg p \), and \( q \rightarrow \neg p \), as shown in Figure 1.44.
EXAMPLE 4

CONSTRUCTING A TRUTH TABLE

In the \( \sim p \) column, fill in truth values that are opposite those for \( p \). Now, a conditional is false only when its premise (in this case, \( q \)) is true and its conclusion (in this case, \( \sim p \)) is false. Therefore, \( q \rightarrow \sim p \) is false only in row 1; the conditional \( q \rightarrow \sim p \) is true under all conditions except the condition that both \( p \) and \( q \) are true. The completed truth table is shown in Figure 1.45.

SOLUTION

Rewriting the statement so the word \( if \) is first, we have “If I want to exercise or (if) the elevator isn’t working, then I walk up the stairs.”

Now we must translate the statement into symbols and construct a truth table. Define the following:

- \( p \): I want to exercise.
- \( q \): The elevator is working.
- \( r \): I walk up the stairs.

The statement now has the symbolic representation \( (p \lor \sim q) \rightarrow r \). Because we have three letters, our table must have \( 2^3 = 8 \) rows. Inserting a column for each letter and a column for each connective, we have the initial setup shown in Figure 1.46.

| \( p \) | \( q \) | \( r \) | \( \sim q \) | \( p \lor \sim q \) | \( (p \lor \sim q) \rightarrow r \) |
|-------|-------|-------|-----------|----------------||------------------|
| T     | T     | T     | F         | T             | T                |
| T     | T     | F     | F         | F             | F                |
| T     | F     | T     | T         | T             | T                |
| T     | F     | F     | F         | F             | F                |
| F     | T     | T     | F         | F             | T                |
| F     | T     | F     | F         | F             | F                |
| F     | F     | T     | T         | T             | T                |
| F     | F     | F     | F         | F             | F                |

FIGURE 1.46 Required columns in the truth table.

In the column labeled \( \sim q \), enter truth values that are the opposite of those of \( q \). Next, enter the truth values of the disjunction \( p \lor \sim q \) in column 5. Recall that a disjunction is false only when both components are false and is true otherwise. Consequently, enter Fs in rows 5 and 6 (since both \( p \) and \( \sim q \) are false) and Ts in the remaining rows, as shown in Figure 1.47.

The last column involves a conditional; it is false only when its premise is true and its conclusion is false. Therefore, enter Fs in rows 2, 4, and 8 (since \( p \lor \sim q \) is true and \( r \) is false) and Ts in the remaining rows. The truth table is shown in Figure 1.48.

As Figure 1.48 shows, the statement “I walk up the stairs if I want to exercise or if the elevator isn’t working” is true in all situations except those listed in rows 2, 4, and 8. For instance, the statement is false (row 8) when the speaker does not want to exercise, the elevator is not working, and the speaker does not walk up the stairs—in other words, the speaker stays on the ground floor of the building when the elevator is broken.
1.3 Truth Tables

When you purchase a car, the car is either new or used. If a salesperson told you, “It is not the case that the car is not new,” what condition would the car be in? This compound statement consists of one individual statement (“p: The car is new”) and two negations: “It is not the case that the car is not new.”

Does this mean that the car is new? To answer this question, we will construct a truth table for the symbolic expression \((p \lor \neg q)\) and compare its truth values with those of the original \(p\). Because there is only one letter, we need \(2^1 = 2\) rows, as shown in Figure 1.49.

We must insert a column for \(\neg p\) and a column for \(\neg(\neg p)\). Now, \(\neg p\) has truth values that are opposite those of \(p\), and \(\neg(\neg p)\) has truth values that are opposite those of \(\neg p\), as shown in Figure 1.50.
Notice that the values in the column labeled \( \sim(\sim p) \) are identical to those in the column labeled \( p \). Whenever this happens, the expressions are said to be equivalent and may be used interchangeably. Therefore, the statement “It is not the case that the car is not new” is equivalent in meaning to the statement “The car is new.”

Equivalent expressions are symbolic expressions that have identical truth values in each corresponding entry. The expression \( p \equiv q \) is read “\( p \) is equivalent to \( q \)” or “\( p \) and \( q \) are equivalent.” As we can see in Figure 1.50, an expression and its double negation are logically equivalent. This relationship can be expressed as \( p \equiv \sim(\sim p) \).

**Example 5**

**Determining Whether Statements Are Equivalent**

Are the statements “If I am a homeowner, then I pay property taxes” and “I am a homeowner, and I do not pay property taxes” equivalent?

We begin by defining the statements:

\( p: \) I am a homeowner.
\( q: \) I pay property taxes.

\( p \rightarrow q: \) If I am a homeowner, then I pay property taxes.

\( p \land \sim q: \) I am a homeowner, and I do not pay property taxes.

The truth table contains \( 2^2 = 4 \) rows, and the initial setup is shown in Figure 1.51.

Now enter the appropriate truth values under \( \sim q \) (the opposite of \( q \)). Because the conjunction \( p \land \sim q \) is true only when both \( p \) and \( \sim q \) are true, enter a T in row 2 and Fs elsewhere. The conditional \( p \rightarrow q \) is false only when \( p \) is true and \( q \) is false; therefore, enter an F in row 2 and Ts elsewhere. The completed truth table is shown in Figure 1.52.

Because the entries in the columns labeled \( p \land \sim q \) and \( p \rightarrow q \) are not the same, the statements are not equivalent. “If I am a homeowner, then I pay property taxes” is *not* equivalent to “I am a homeowner and I do not pay property taxes.”

Notice that the truth values in the columns under \( p \land \sim q \) and \( p \rightarrow q \) in Figure 1.52 are exact opposites; when one is T, the other is F. Whenever this happens, one statement is the negation of the other. Consequently, \( p \land \sim q \) is the negation of \( p \rightarrow q \) (and vice versa). This can be expressed as \( p \land \sim q \equiv \sim(p \rightarrow q) \). The negation of a conditional is logically equivalent to the conjunction of the premise and the negation of the conclusion.

Statements that look or sound different may in fact have the same meaning. For example, “It is not the case that the car is not new” really means the same as “The car is new,” and “It is not the case that if I am a homeowner, then I pay property taxes”
Boole’s most influential work, An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities, was published in 1854. In it, he wrote, “There exist certain general principles founded in the very nature of language and logic that exhibit laws as identical in form as with the laws of the general symbols of algebra.” With this insight, Boole had taken a big step into the world of logical reasoning and abstract mathematical analysis.

Perhaps because of his lack of formal training, Boole challenged the status quo, including the Aristotelian assumption that all logical arguments could be reduced to syllogistic arguments. In doing so, he employed symbols to represent concepts, as did Leibniz. But he also developed systems of algebraic manipulation to accompany these symbols. Thus, Boole’s creation is a marriage of logic and mathematics. However, as is the case with almost all new theories, Boole’s symbolic logic was not met with total approval. In particular, one staunch opponent of his work was Georg Cantor, whose work on the origins of set theory and the magnitude of infinity will be investigated in Chapter 2.

De Morgan’s Laws

Earlier in this section, we saw that the negation of a negation is equivalent to the original statement; that is, \( \neg(\neg p) \equiv p \). Another negation “formula” that we discovered was \( \neg(p \rightarrow q) \equiv p \land \neg q \), that is, the negation of a conditional. Can we find similar “formulas” for the negations of the other basic connectives, namely, the conjunction and the disjunction? The answer is yes, and the results are credited to the English mathematician and logician Augustus De Morgan.
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DE MORGAN’S LAWS
The negation of the conjunction \( p \land q \) is given by \( \neg(p \land q) = \neg p \lor \neg q \).
“Not \( p \) and \( q \)” is equivalent to “not \( p \) or not \( q \).”
The negation of the disjunction \( p \lor q \) is given by \( \neg(p \lor q) = \neg p \land \neg q \).
“Not \( p \) or \( q \)” is equivalent to “not \( p \) and not \( q \).”

De Morgan’s Laws are easily verified through the use of truth tables and will be addressed in the exercises (see Exercises 55 and 56).

EXAMPLE 6 APPL YING DE MORGAN’S LAWS Using De Morgan’s Laws, find the negation of each of the following:

a. It is Friday and I receive a paycheck.
   b. You are correct or I am crazy.

SOLUTION
   a. The symbolic representation of “It is Friday and I receive a paycheck” is
      \( p: \) It is Friday.
      \( q: \) I receive a paycheck.
      \( p \land q: \) It is Friday and I receive a paycheck.
      Therefore, the negation is \( \neg(p \land q) = \neg p \lor \neg q \), that is, “It is not Friday or I do not receive a paycheck.”

   b. The symbolic representation of “You are correct or I am crazy” is
      \( p: \) You are correct.
      \( q: \) I am crazy.
      \( p \lor q: \) You are correct or I am crazy.
      Therefore, the negation is \( \neg(p \lor q) = \neg p \land \neg q \), that is, “You are not correct and I am not crazy.”

As we have seen, the truth value of a compound statement depends on the truth values of the individual statements that make it up. The truth tables of the basic connectives are summarized in Figure 1.53.

Equivalent statements are statements that have the same meaning. Equivalent statements for the negations of the basic connectives are given in Figure 1.54.

FIGURE 1.53 Truth tables for the basic connectives.
1.3 Exercises

In Exercises 1–20, construct a truth table for the symbolic expressions.

1. \( p \lor \neg q \)
2. \( p \land \neg q \)
3. \( p \lor \neg p \)
4. \( p \land \neg p \)
5. \( p \rightarrow \neg q \)
6. \( \neg p \rightarrow q \)
7. \( \neg q \rightarrow \neg p \)
8. \( \neg p \rightarrow \neg q \)
9. \( (p \lor q) \rightarrow \neg p \)
10. \( (p \land q) \rightarrow \neg q \)
11. \( (p \lor q) \rightarrow (p \land q) \)
12. \( (p \land q) \rightarrow (p \lor q) \)
13. \( p \land \neg (q \lor r) \)
14. \( p \lor \neg (q \lor r) \)
15. \( p \lor \neg (q \land r) \)
16. \( \neg p \lor \neg (q \land r) \)
17. \( (\neg r \lor p) \rightarrow (q \land p) \)
18. \( (q \land p) \rightarrow (\neg r \lor p) \)
19. \( (p \lor r) \rightarrow (q \land \neg r) \)
20. \( (p \land r) \rightarrow (q \lor \neg r) \)

In Exercises 21–40, translate the compound statement into symbolic form and then construct the truth table for the expression.

21. If it is raining, then the streets are wet.
22. If the lyrics are not controversial, the performance is not banned.
23. The water supply is rationed if it does not rain.
24. The country is in trouble if he is elected.
25. All squares are rectangles.
26. All muscle cars from the Sixties are polluters.
27. No square is a triangle.
28. No electric-powered car is a polluter.
29. Being a monkey is sufficient for not being an ape.
30. Being a chimpanzee is sufficient for not being a monkey.
31. Not being a monkey is necessary for being an ape.
32. Not being a chimpanzee is necessary for being a monkey.
33. Your check is accepted if you have a driver’s license or a credit card.
34. You get a refund or a store credit if the product is defective.
35. If leaded gasoline is used, the catalytic converter is damaged and the air is polluted.
36. If he does not go to jail, he is innocent or has an alibi.
37. I have a college degree and I do not have a job or own a house.
38. I surf the Internet and I make purchases and do not pay sales tax.
39. If Proposition A passes and Proposition B does not, jobs are lost or new taxes are imposed.
40. If Proposition A does not pass and the legislature raises taxes, the quality of education is lowered and unemployment rises.

In Exercises 41–50, construct a truth table to determine whether the statements in each pair are equivalent.

41. The streets are wet or it is not raining.
   If it is raining, then the streets are wet.
42. The streets are wet or it is not raining.
   If the streets are not wet, then it is not raining.
43. He has a high school diploma or he is unemployed.
   If he does not have a high school diploma, then he is unemployed.
44. She is unemployed or she does not have a high school diploma.
   If she is employed, then she does not have a high school diploma.
45. If handguns are outlawed, then outlaws have handguns.
   If outlaws have handguns, then handguns are outlawed.
46. If interest rates continue to fall, then I can afford to buy a house.
   If interest rates do not continue to fall, then I cannot afford to buy a house.
47. If the spotted owl is on the endangered species list, then lumber jobs are lost.
CHAPTER 1 Logic

42 If lumber jobs are not lost, then the spotted owl is not on the endangered species list.

48. If I drink decaffeinated coffee, then I do not stay awake.
   If I do stay awake, then I do not drink decaffeinated coffee.

49. The plaintiff is innocent or the insurance company does not settle out of court.
   The insurance company settles out of court and the plaintiff is not innocent.

50. The plaintiff is not innocent and the insurance company settles out of court.
   It is not the case that the plaintiff is innocent or the insurance company does not settle out of court.

In Exercises 51–54, construct truth tables to determine which pairs of statements are equivalent.

51. i. Knowing Morse code is sufficient for operating a telegraph.
   ii. Knowing Morse code is necessary for operating a telegraph.
   iii. Not knowing Morse code is sufficient for not operating a telegraph.
   iv. Not knowing Morse code is necessary for not operating a telegraph.

52. i. Knowing CPR is necessary for being a paramedic.
   ii. Knowing CPR is sufficient for being a paramedic.
   iii. Not knowing CPR is necessary for not being a paramedic.
   iv. Not knowing CPR is sufficient for not being a paramedic.

53. i. The water being cold is necessary for not going swimming.
   ii. The water not being cold is necessary for going swimming.
   iii. The water being cold is sufficient for not going swimming.
   iv. The water not being cold is sufficient for going swimming.

54. i. The sky not being clear is sufficient for it to be raining.
   ii. The sky being clear is sufficient for it not to be raining.
   iii. The sky not being clear is necessary for it to be raining.
   iv. The sky being clear is necessary for it not to be raining.

55. Using truth tables, verify De Morgan’s Law
   \[ \sim(p \land q) \equiv \sim p \lor \sim q. \]

56. Using truth tables, verify De Morgan’s Law
   \[ \sim(p \lor q) \equiv \sim p \land \sim q. \]

In Exercises 57–68, write the statement in symbolic form, construct the negation of the expression (in simplified symbolic form), and express the negation in words.

57. I have a college degree and I am not employed.
58. It is snowing and classes are canceled.
59. The television set is broken or there is a power outage.
60. The freeway is under construction or I do not ride the bus.
61. If the building contains asbestos, the original contractor is responsible.
62. If the legislation is approved, the public is uninformed.
63. The First Amendment has been violated if the lyrics are censored.
64. Your driver’s license is taken away if you do not obey the laws.
65. Rainy weather is sufficient for not washing my car.
66. Drinking caffeinated coffee is sufficient for not sleeping.
67. Not talking is necessary for listening.
68. Not eating dessert is necessary for being on a diet.

Answer the following questions using complete sentences and your own words.

• CONCEPT QUESTIONS

89. a. Under what conditions is a disjunction true?
   b. Under what conditions is a disjunction false?
70. a. Under what conditions is a conjunction true?
   b. Under what conditions is a conjunction false?
71. a. Under what conditions is a conditional true?
   b. Under what conditions is a conditional false?
72. a. Under what conditions is a negation true?
   b. Under what conditions is a negation false?
73. What are equivalent expressions?
74. What is a truth table?
75. When constructing a truth table, how do you determine how many rows to create?

• HISTORY QUESTIONS

76. Who is considered “the father of symbolic logic”?
77. Boolean algebra is a combination of logic and mathematics. What is it used for?
1.4 More on Conditionals

OBJECTIVES
- Create the converse, inverse, and contrapositive of a conditional statement
- Determine equivalent variations of a conditional statement
- Interpret “only if” statements
- Interpret a biconditional statement

Conditionals differ from conjunctions and disjunctions with regard to the possibility of changing the order of the statements. In algebra, the sum $x + y$ is equal to the sum $y + x$; that is, addition is commutative. In everyday language, one realtor might say, “The house is perfect and the lot is priceless,” while another says, “The lot is priceless and the house is perfect.” Logically, their meanings are the same, since $(p \land q) \equiv (q \land p)$. The order of the components in a conjunction or disjunction makes no difference in regard to the truth value of the statement. This is not so with conditionals.

Variations of a Conditional

Given two statements $p$ and $q$, various “if . . . then . . .” statements can be formed.

Example 1

**TRANSLATING SYMBOLS INTO WORDS**  Using the statements

$p$: You are compassionate.
$q$: You contribute to charities.

write an “if . . . then . . .” sentence represented by each of the following:

a. $p \rightarrow q$  

b. $q \rightarrow p$  

c. $\sim p \rightarrow \sim q$  

d. $\sim q \rightarrow \sim p$

**SOLUTION**

a. $p \rightarrow q$: If you are compassionate, then you contribute to charities.

b. $q \rightarrow p$: If you contribute to charities, then you are compassionate.

c. $\sim p \rightarrow \sim q$: If you are not compassionate, then you do not contribute to charities.

d. $\sim q \rightarrow \sim p$: If you do not contribute to charities, then you are not compassionate.

Each part of Example 1 contains an “if . . . then . . .” statement and is called a conditional. Any given conditional has three variations: a converse, an inverse, and a contrapositive. The converse of the conditional “if $p$ then $q$” is the compound statement “if $q$ then $p$.” That is, we form the converse of the conditional by interchanging the premise and the conclusion; $q \rightarrow p$ is the converse of $p \rightarrow q$. The statement in part (b) of Example 1 is the converse of the statement in part (a).

The inverse of the conditional “if $p$ then $q$” is the compound statement “if not $p$ then not $q$.” We form the inverse of the conditional by negating both the premise and the conclusion; $\sim p \rightarrow \sim q$ is the inverse of $p \rightarrow q$. The statement in part (c) of Example 1 is the inverse of the statement in part (a).

The contrapositive of the conditional “if $p$ then $q$” is the compound statement “if not $q$ then not $p$.” We form the contrapositive of the conditional by
negating and interchanging both the premise and the conclusion; \( \sim q \rightarrow \sim p \) is the contrapositive of \( p \rightarrow q \). The statement in part (d) of Example 1 is the contrapositive of the statement in part (a). The variations of a given conditional are summarized in Figure 1.55. As we will see, some of these variations are equivalent, and some are not. Unfortunately, many people incorrectly treat them all as equivalent.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbolic Form</th>
<th>Read As...</th>
</tr>
</thead>
<tbody>
<tr>
<td>a (given) conditional</td>
<td>( p \rightarrow q )</td>
<td>If ( p ), then ( q ).</td>
</tr>
<tr>
<td>the converse (of ( p \rightarrow q ))</td>
<td>( q \rightarrow p )</td>
<td>If ( q ), then ( p ).</td>
</tr>
<tr>
<td>the inverse (of ( p \rightarrow q ))</td>
<td>( \sim p \rightarrow \sim q )</td>
<td>If not ( p ), then not ( q ).</td>
</tr>
<tr>
<td>the contrapositive (of ( p \rightarrow q ))</td>
<td>( \sim q \rightarrow \sim p )</td>
<td>If not ( q ), then not ( p ).</td>
</tr>
</tbody>
</table>

**FIGURE 1.55** Variations of a conditional.

**EXAMPLE 2**

**CREATING VARIATIONS OF A CONDITIONAL STATEMENT** Given the conditional “You did not receive the proper refund if you prepared your own income tax form,” write the sentence that represents each of the following.

a. the converse of the conditional
b. the inverse of the conditional
c. the contrapositive of the conditional

**SOLUTION**

a. Rewriting the statement in the standard “if... then...” form, we have the conditional “If you prepared your own income tax form, then you did not receive the proper refund.” The converse is formed by interchanging the premise and the conclusion. Thus, the converse is written as “If you did not prepare your own income tax form, then you received the proper refund.”

b. The inverse is formed by negating both the premise and the conclusion. Thus, the inverse is written as “If you did not prepare your own income tax form, then you received the proper refund.”

c. The contrapositive is formed by negating and interchanging the premise and the conclusion. Thus, the contrapositive is written as “If you received the proper refund, then you did not prepare your own income tax form.”

**Equivalent Conditionals**

We have seen that the conditional \( p \rightarrow q \) has three variations: the converse \( q \rightarrow p \), the inverse \( \sim p \rightarrow \sim q \), and the contrapositive \( \sim q \rightarrow \sim p \). Do any of these “if... then...” statements convey the same meaning? In other words, are any of these compound statements equivalent?

**EXAMPLE 3**

**DETERMINING EQUIVALENT STATEMENTS** Determine which (if any) of the following are equivalent: a conditional \( p \rightarrow q \), the converse \( q \rightarrow p \), the inverse \( \sim p \rightarrow \sim q \), and the contrapositive \( \sim q \rightarrow \sim p \).

**SOLUTION**

To investigate the possible equivalencies, we must construct a truth table that contains all the statements. Because there are two letters, we need \( 2^2 = 4 \) rows. The table must have a column for \( \sim p \), one for \( \sim q \), one for the conditional \( p \rightarrow q \), and one for each variation of the conditional. The truth values of the negations \( \sim p \) and \( \sim q \) are readily entered, as shown in Figure 1.56.
An “if . . . then . . .” statement is false only when the premise is true and the conclusion is false. Consequently, \( p \rightarrow q \) is false only when \( p \) is T and \( q \) is F; enter an F in row 2 and Ts elsewhere in the column under \( p \rightarrow q \).

Likewise, the converse \( q \rightarrow p \) is false only when \( q \) is T and \( p \) is F; enter an F in row 3 and Ts elsewhere.

In a similar manner, the inverse \( \sim p \rightarrow \sim q \) is false only when \( \sim p \) is T and \( \sim q \) is F; enter an F in row 3 and Ts elsewhere.

Finally, the contrapositive \( \sim q \rightarrow \sim p \) is false only when \( \sim q \) is T and \( \sim p \) is F; enter an F in row 2 and Ts elsewhere.

The completed truth table is shown in Figure 1.57. Examining the entries in Figure 1.57, we can see that the columns under \( p \rightarrow q \) and \( \sim q \rightarrow \sim p \) are identical; each has an F in row 2 and Ts elsewhere. Consequently, a conditional and its contrapositive are equivalent: \( p \rightarrow q \equiv \sim q \rightarrow \sim p \).

Likewise, we notice that \( q \rightarrow p \) and \( \sim p \rightarrow \sim q \) have identical truth values; each has an F in row 3 and Ts elsewhere. Thus, the converse and the inverse of a conditional are equivalent: \( q \rightarrow p \equiv \sim p \rightarrow \sim q \).

We have seen that different “if . . . then . . .” statements can convey the same meaning—that is, that certain variations of a conditional are equivalent (see Figure 1.58). For example, the compound statements “If you are compassionate, then you contribute to charities” and “If you do not contribute to charities, then you are not compassionate” convey the same meaning. (The second conditional is the contrapositive of the first.) Regardless of its specific contents \((p, q, \sim p, \text{ or } \sim q)\), every “if . . . then . . .” statement has an equivalent variation formed by negating and interchanging the premise and the conclusion of the given conditional statement.
EXAMPLE 4

CREATING A CONTRAPOSITIVE  Given the statement “Being a doctor is necessary for being a surgeon,” express the contrapositive in terms of the following:

a. a sufficient condition
b. a necessary condition

d. Recalling that a necessary condition is the conclusion of a conditional, we can rephrase the statement “Being a doctor is necessary for being a surgeon” as follows:

“If a person is a surgeon, then the person is a doctor.”

The premise. A necessary condition is the conclusion.

Therefore, by negating and interchanging the premise and conclusion, the contrapositive is

“If a person is not a doctor, then the person is not a surgeon.”

The negation of the conclusion. The negation of the premise.

Recalling that a sufficient condition is the premise of a conditional, we can phrase the contrapositive of the original statement as “Not being a doctor is sufficient for not being a surgeon.”

b. From part (a), the contrapositive of the original statement is the conditional statement

“If a person is not a doctor, then the person is not a surgeon.”

The premise of the contrapositive. The conclusion of the contrapositive.

Because a necessary condition is the conclusion of a conditional, the contrapositive of the (original) statement “Being a doctor is necessary for being a surgeon” can be expressed as “Not being a surgeon is necessary for not being a doctor.”

The “Only If” Connective

Consider the statement “A prisoner is paroled only if the prisoner obeys the rules.” What is the premise, and what is the conclusion? Rather than using p and q (which might bias our investigation), we define

\[ r: \text{A prisoner is paroled.} \]
\[ s: \text{A prisoner obeys the rules.} \]

The given statement is represented by “r only if s.” Now, “r only if s” means that r can happen only if s happens. In other words, if s does not happen, then r does not happen, or \( \sim s \rightarrow \sim r \). We have seen that \( \sim s \rightarrow \sim r \) is equivalent to \( r \rightarrow s \). Consequently, “r only if s” is equivalent to the conditional \( r \rightarrow s \). The premise of the statement “A prisoner is paroled only if the prisoner obeys the rules” is “A prisoner is paroled,” and the conclusion is “The prisoner obeys the rules.”

The conditional \( p \rightarrow q \) can be phrased “p only if q.” Even though the word if precedes \( q \), \( q \) is not the premise. Whatever follows the connective “only if” is the conclusion of the conditional.
### Example 5

**Analyzing an “Only If” Statement**

For the compound statement “You receive a federal grant only if your artwork is not obscene,” do the following:

a. Determine the premise and the conclusion.

b. Rewrite the compound statement in the standard “if . . . then . . .” form.

c. Interpret the conditions that make the statement false.

a. Because the compound statement contains an “only if” connective, the statement that follows “only if” is the conclusion of the conditional. The premise is “You receive a federal grant.” The conclusion is “Your artwork is not obscene.”

b. The given compound statement can be rewritten as “If you receive a grant, then your artwork is not obscene.”

c. First we define the symbols.

\[ p: \text{You receive a federal grant.} \]
\[ q: \text{Your artwork is obscene.} \]

Then the statement has the symbolic representation \( p \rightarrow \sim q \). The truth table for \( p \rightarrow \sim q \) is given in Figure 1.59.

The expression \( p \rightarrow q \) is false under the conditions listed in row 1 (when \( p \) and \( q \) are both true). Therefore, the statement “You receive a federal grant only if your artwork is not obscene” is false when an artist does receive a federal grant and the artist’s artwork is obscene.

### The Biconditional \( p \leftrightarrow q \)

What do the words bicyclic, binomial, and bilingual have in common? Each word begins with the prefix bi, meaning “two.” Just as the word bilingual means “two languages,” the word biconditional means “two conditionals.”

In everyday speech, conditionals often get “hooked together” in a circular fashion. For instance, someone might say, “If I am rich, then I am happy, and if I am happy, then I am rich.” Notice that this compound statement is actually the conjunction (and) of a conditional (if rich, then happy) and its converse (if happy, then rich). Such a statement is referred to as a biconditional. A biconditional is a statement of the form \( (p \rightarrow q) \land (q \rightarrow p) \) and is symbolized as \( p \leftrightarrow q \). The symbol \( p \leftrightarrow q \) is read “\( p \) if and only if \( q \)” and is frequently abbreviated “\( p \) iff \( q \).” A biconditional is equivalent to the conjunction of two conversely related conditionals: \( p \leftrightarrow q = [(p \rightarrow q) \land (q \rightarrow p)] \).

In addition to the phrase “if and only if,” a biconditional can also be expressed by using “necessary” and “sufficient” terminology. The statement “\( p \) is sufficient for \( q \)” can be rephrased as “\( p \) then \( q \)” (and symbolized as \( p \rightarrow q \)), whereas the statement “\( p \) is necessary for \( q \)” can be rephrased as “\( q \) then \( p \)” (and symbolized as \( q \rightarrow p \)). Therefore, the biconditional “\( p \) if and only if \( q \)” can also be phrased as “\( p \) is necessary and sufficient for \( q \).”

### Example 6

**Analyzing a Biconditional Statement**

Express the biconditional “A citizen is eligible to vote if and only if the citizen is at least eighteen years old” as the conjunction of two conditionals.

The given biconditional is equivalent to “If a citizen is eligible to vote, then the citizen is at least eighteen years old, and if a citizen is at least eighteen years old, then the citizen is eligible to vote.”
Under what circumstances is the biconditional \( p \leftrightarrow q \) true, and when is it false? To find the answer, we must construct a truth table. Utilizing the equivalence \( p \leftrightarrow q = [(p \rightarrow q) \land (q \rightarrow p)] \), we get the completed table shown in Figure 1.60. (Recall that a conditional is false only when its premise is true and its conclusion is false and that a conjunction is true only when both components are true.) We can see that a biconditional is true only when the two components \( p \) and \( q \) have the same truth value—that is, when \( p \) and \( q \) are both true or when \( p \) and \( q \) are both false. On the other hand, a biconditional is false when the two components \( p \) and \( q \) have opposite truth value—that is, when \( p \) is true and \( q \) is false or vice versa.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
<th>( q \rightarrow p )</th>
<th>( (p \rightarrow q) \land (q \rightarrow p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

**FIGURE 1.60** Truth table for a biconditional \( p \leftrightarrow q \).

Many theorems in mathematics can be expressed as biconditionals. For example, when solving a quadratic equation, we have the following: “The equation \( ax^2 + bx + c = 0 \) has exactly one solution if and only if the discriminant \( b^2 - 4ac = 0 \).” Recall that the solutions of a quadratic equation are

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

This biconditional is equivalent to “If the equation \( ax^2 + bx + c = 0 \) has exactly one solution, then the discriminant \( b^2 - 4ac = 0 \), and if the discriminant \( b^2 - 4ac = 0 \), then the equation \( ax^2 + bx + c = 0 \) has exactly one solution”—that is, one condition implies the other.

### 1.4 Exercises

**In Exercises 1–2, using the given statements, write the sentence represented by each of the following.**

a. \( p \rightarrow q \)

b. \( q \rightarrow p \)

c. \( \sim p \rightarrow \sim q \)

d. \( \sim q \rightarrow \sim p \)

e. Which of parts (a)–(d) are equivalent? Why?

1. \( p \): She is a police officer.
   \( q \): She carries a gun.
2. \( p \): I am a multimillion-dollar lottery winner.
   \( q \): I am a world traveler.

**In Exercises 3–4, using the given statements, write the sentence represented by each of the following.**

a. \( p \rightarrow \sim q \)

b. \( \sim q \rightarrow p \)

c. \( \sim p \rightarrow q \)

d. \( q \rightarrow \sim p \)

e. Which of parts (a)–(d) are equivalent? Why?

3. \( p \): I watch television.
   \( q \): I do my homework.
4. \( p \): He is an artist.
   \( q \): He is a conformist.

**In Exercises 5–10, form (a) the inverse, (b) the converse, and (c) the contrapositive of the given conditional.**

5. If you pass this mathematics course, then you fulfill a graduation requirement.
6. If you have the necessary tools, assembly time is less than thirty minutes.
7. The television set does not work if the electricity is turned off.
8. You do not win if you do not buy a lottery ticket.
In Exercises 11–14, express the contrapositive of the given biconditional in terms of (a) a sufficient condition and (b) a necessary condition.

11. Being an orthodontist is sufficient for being a dentist.
12. Being an author is sufficient for being literate.
13. Knowing Morse code is necessary for operating a telegraph.
14. Knowing CPR is necessary for being a paramedic.

In Exercises 15–20, (a) determine the premise and conclusion, (b) rewrite the compound statement in the standard “if . . . then . . .” form, and (c) interpret the conditions that make the statement false.

15. I take public transportation only if it is convenient.
16. I eat raw fish only if I am in a Japanese restaurant.
17. I buy foreign products only if domestic products are not available.
18. I ride my bicycle only if it is not raining.
19. You may become a U.S. senator only if you are at least thirty years old and have been a citizen for nine years.
20. You may become the president of the United States only if you are at least thirty-five years old and were born a citizen of the United States.

In Exercises 21–28, express the given biconditional as the conjunction of two conditionals.

21. You obtain a refund if and only if you have a receipt.
22. We eat at Burger World if and only if Ju Ju’s Kitsch-Inn is closed.
23. The quadratic equation $ax^2 + bx + c = 0$ has two distinct real solutions if and only if $b^2 - 4ac > 0$.
24. The quadratic equation $ax^2 + bx + c = 0$ has complex solutions iff $b^2 - 4ac < 0$.
25. A polygon is a triangle if and only if the polygon has three sides.
26. A triangle is isosceles if and only if the triangle has two equal sides.
27. A triangle having a 90° angle is necessary and sufficient for $a^2 + b^2 = c^2$.
28. A triangle having three equal sides is necessary and sufficient for a triangle having three equal angles.

In Exercises 29–36, translate the two statements into symbolic form and use truth tables to determine whether the statements are equivalent.

29. I cannot have surgery if I do not have health insurance. If I can have surgery, then I do have health insurance.
30. If I am illiterate, I cannot fill out an application form. I can fill out an application form if I am not illiterate.
31. If you earn less than $12,000 per year, you are eligible for assistance. If you are not eligible for assistance, then you earn at least $12,000 per year.
32. If you earn less than $12,000 per year, you are eligible for assistance. If you earn at least $12,000 per year, you are not eligible for assistance.
33. I watch television only if the program is educational. I do not watch television if the program is not educational.
34. I buy seafood only if the seafood is fresh. If I do not buy seafood, the seafood is not fresh.
35. Being an automobile that is American-made is sufficient for an automobile having hardware that is not metric. Being an automobile that is not American-made is necessary for an automobile having hardware that is metric.
36. Being an automobile having metric hardware is sufficient for being an automobile that is not American-made. Being an automobile not having metric hardware is necessary for being an automobile that is American-made.

In Exercises 37–46, write an equivalent variation of the given conditional.

37. If it is not raining, I walk to work.
38. If it makes a buzzing noise, it is not working properly.
39. It is snowing only if it is cold.
40. You are a criminal only if you do not obey the law.
41. You are not a vegetarian if you eat meat.
42. You are not an artist if you are not creative.
43. All policemen own guns.
44. All college students are sleep deprived.
45. No convicted felon is eligible to vote.
46. No man asks for directions.

In Exercises 47–52, determine which pairs of statements are equivalent.

47. i. If Proposition 111 passes, freeways are improved.
   ii. If Proposition 111 is defeated, freeways are not improved.
   iii. If the freeways are improved, Proposition 111 passes.
   iv. If the freeways are not improved, Proposition 111 does not pass.
48. i. If the Giants win, then I am happy.
   ii. If I am happy, then the Giants win.
If the Giants lose, then I am unhappy.
iv. If I am unhappy, then the Giants lose.

49. i. I go to church if it is Sunday.
ii. I go to church only if it is Sunday.
iii. If I do not go to church, it is not Sunday.
iv. If it is not Sunday, I do not go to church.

50. i. I am a rebel if I do not have a cause.
ii. I am a rebel only if I do not have a cause.
iii. I am not a rebel if I have a cause.
iv. If I am not a rebel, I have a cause.

51. i. If line 34 is greater than line 29, I use Schedule X.
ii. If I use Schedule X, then line 34 is greater than line 29.
iii. If I do not use Schedule X, then line 34 is not greater than line 29.
iv. If line 34 is not greater than line 29, then I do not use Schedule X.

52. i. If you answer yes to all of the above, then you complete Part II.
ii. If you answer no to any of the above, then you do not complete Part II.
iii. If you completed Part II, then you answered yes to all of the above.
iv. If you did not complete Part II, then you answered no to at least one of the above.

Answer the following questions using complete sentences and your own words.

• CONCEPT QUESTIONS

53. What is a contrapositive?
54. What is a converse?
55. What is an inverse?
56. What is a biconditional?
57. How is an “if . . . then . . .” statement related to an “only if” statement?

Exercises 58–62 refer to the following: Assuming that a movie’s popularity is measured by its gross box office receipts, six recently released movies—M, N, O, P, Q, and R—are ranked from most popular (first) to least popular (sixth). There are no ties. The ranking is consistent with the following conditions:

• O is more popular than R.
• If N is more popular than O, then neither Q nor R is more popular than P.
• If O is more popular than N, then neither P nor R is more popular than Q.
• M is more popular than N, or else M is more popular than O, but not both.

58. Which one of the following could be the ranking of the movies, from most popular to least popular?
   a. N, M, O, R, P, Q
   b. P, O, M, Q, N, R
   c. Q, P, O, M, N
   d. O, Q, M, P, N, R
   e. P, Q, N, Q, R, M

59. If N is the second most popular movie, then which one of the following could be true?
   a. O is more popular than M.
   b. Q is more popular than M.
   c. R is more popular than M.
   d. Q is more popular than P.
   e. O is more popular than N.

60. Which one of the following cannot be the most popular movie?
   a. M
   b. N
   c. O
   d. P
   e. Q

61. If R is more popular than M, then which one of the following could be true?
   a. M is more popular than O.
   b. M is more popular than Q.
   c. N is more popular than P.
   d. N is more popular than O.
   e. N is more popular than R.

62. If O is more popular than P and less popular than Q, then which one of the following could be true?
   a. M is more popular than O.
   b. N is more popular than M.
   c. N is more popular than O.
   d. R is more popular than Q.
   e. P is more popular than R.
Analyzing Arguments

1.5

OBJECTIVES

- Identify a tautology
- Use a truth table to analyze an argument

Lewis Carroll’s Cheshire Cat told Alice that he was mad (crazy). Alice then asked, "'And how do you know that you’re mad?' ‘To begin with,’ said the cat, ‘a dog’s not mad. You grant that?’ ‘I suppose so,’ said Alice. ‘Well, then,’ the cat went on, ‘you see a dog growls when it’s angry, and wags its tail when it’s pleased. Now I growl when I’m pleased, and wag my tail when I’m angry. Therefore I’m mad!’”

Does the Cheshire Cat have a valid deductive argument? Does the conclusion follow logically from the hypotheses? To answer this question, and others like it, we will utilize symbolic logic and truth tables to account for all possible combinations of the individual statements as true or false.

Valid Arguments

When someone makes a sequence of statements and draws some conclusion from them, he or she is presenting an argument. An argument consists of two components: the initial statements, or hypotheses, and the final statement, or conclusion. When presented with an argument, a listener or reader may ask, “Does this person have a logical argument? Does his or her conclusion necessarily follow from the given statements?”

An argument is valid if the conclusion of the argument is guaranteed under its given set of hypotheses. (That is, the conclusion is inescapable in all instances.) For example, we used Venn diagrams in Section 1.1 to show the argument

\[
\begin{align*}
\text{All men are mortal.} \\
\text{Socrates is a man.} \\
\text{Therefore, Socrates is mortal.}
\end{align*}
\]

is a valid argument. Given the hypotheses, the conclusion is guaranteed. The term valid does not mean that all the statements are true but merely that the conclusion was reached via a proper deductive process. As shown in Example 3 of Section 1.1, the argument

\[
\begin{align*}
\text{All doctors are men.} \\
\text{My mother is a doctor.} \\
\text{Therefore, my mother is a man.}
\end{align*}
\]

is also a valid argument. Even though the conclusion is obviously false, the conclusion is guaranteed, given the hypotheses.

The hypotheses in a given logical argument may consist of several interrelated statements, each containing negations, conjunctions, disjunctions, and conditionals. By joining all the hypotheses in the form of a conjunction, we can form a single conditional that represents the entire argument. That is, if an argument has \( n \) hypotheses \( (h_1, h_2, \ldots, h_n) \) and conclusion \( c \), the argument will have the form “if \( (h_1 \land h_2 \land \ldots \land h_n) \), then \( c \).”
CHAPTER 1 Logic

Using a logical argument, Lewis Carroll’s Cheshire Cat tried to convince Alice that he was crazy. Was his argument valid?

If the conditional representation of an argument is always true (regardless of the actual truthfulness of the individual statements), the argument is valid. If there is at least one instance in which the conditional is false, the argument is invalid.

EXAMPLE 1

USING A TRUTH TABLE TO ANALYZE AN ARGUMENT

Determine whether the following argument is valid:

“If he is illiterate, he cannot fill out the application.
He can fill out the application.
Therefore, he is not illiterate.”

SOLUTION

First, number the hypotheses and separate them from the conclusion with a line:

1. If he is illiterate, he cannot fill out the application.
2. He can fill out the application.
Therefore, he is not illiterate.

Now use symbols to represent each different component in the statements:

\[ p: \text{He is illiterate.} \]
\[ q: \text{He can fill out the application.} \]
1.5 Analyzing Arguments

CHURCH CARVING MAY BE ORIGINAL ‘CHeshire Cat’

London—Devotees of writer Lewis Carroll believe they have found what inspired his grinning Cheshire Cat, made famous in his book “Alice’s Adventures in Wonderland.”

Members of the Lewis Carroll Society made the discovery over the weekend in a church at which the author’s father was once rector in the Yorkshire village of Croft in northern England.

It is a rough-hewn carving of a cat’s head smiling near an altar, probably dating to the 10th century. Seen from below and from the perspective of a small boy, all that can be seen is the grinning mouth.

Carroll’s Alice watched the Cheshire Cat disappear “ending with the grin, which remained for some time after the rest of the head had gone.”

Alice mused: “I have often seen a cat without a grin, but not a grin without a cat. It is the most curious thing I have seen in all my life.”

We could have defined \( q \) as “He cannot fill out the application” (as stated in premise 1), but it is customary to define the symbols with a positive sense. Symbolically, the argument has the form

\[
\begin{align*}
1. & \quad p \rightarrow \sim q \\
2. & \quad q \\
\therefore & \quad \sim p
\end{align*}
\]

and is represented by the conditional \( [(p \rightarrow \sim q) \land q] \rightarrow \sim p \). The symbol \( \therefore \) is read “therefore.”

To construct a truth table for this conditional, we need \( 2^2 = 4 \) rows. A column is required for the following: each negation, each hypothesis, the conjunction of the hypotheses, the conclusion, and the conditional representation of the argument. The initial setup is shown in Figure 1.61.

Fill in the truth table as follows:

\( \sim q \): A negation has the opposite truth values; enter a T in rows 2 and 4 and an F in rows 1 and 3.

**Hypothesis 1**: A conditional is false only when its premise is true and its conclusion is false; enter an F in row 1 and Ts elsewhere.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Hypothesis 1</th>
<th>Hypothesis 2</th>
<th>Column Representing All the Hypotheses</th>
<th>Conclusion</th>
<th>Conditional Representation of the Argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>1 \land 2</td>
<td>\sim q</td>
<td>\lnot (1 \land 2) \rightarrow c</td>
</tr>
<tr>
<td>2.</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>1 \land 2</td>
<td>\sim q</td>
<td>\lnot (1 \land 2) \rightarrow c</td>
</tr>
<tr>
<td>3.</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>1 \land 2</td>
<td>\sim q</td>
<td>\lnot (1 \land 2) \rightarrow c</td>
</tr>
<tr>
<td>4.</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>1 \land 2</td>
<td>\sim q</td>
<td>\lnot (1 \land 2) \rightarrow c</td>
</tr>
</tbody>
</table>

*Figure 1.61* Required columns in the truth table.
Hypothesis 2: Recopy the $q$ column.

1 $\land$ 2: A conjunction is true only when both components are true; enter a T in row 3 and Fs elsewhere.

Conclusion c: A negation has the opposite truth values; enter an F in rows 1 and 2 and a T in rows 3 and 4.

At this point, all that remains is the final column (see Figure 1.62).

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$q$</td>
<td>$\sim q$</td>
<td>$p \rightarrow q$</td>
<td>$q$</td>
<td>$1 \land 2$</td>
<td>$\sim p$</td>
</tr>
<tr>
<td>1.</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>2.</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>3.</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>4.</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**FIGURE 1.62** Truth values of the expressions.

The last column in the truth table is the conditional that represents the entire argument. A conditional is false only when its premise is true and its conclusion is false. The only instance in which the premise $(1 \land 2)$ is true is row 3. Corresponding to this entry, the conclusion $\sim p$ is also true. Consequently, the conditional $(1 \land 2) \rightarrow c$ is true in row 3. Because the premise $(1 \land 2)$ is false in rows 1, 2, and 4, the conditional $(1 \land 2) \rightarrow c$ is automatically true in those rows as well. The completed truth table is shown in Figure 1.63.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$q$</td>
<td>$\sim q$</td>
<td>$p \rightarrow q$</td>
<td>$q$</td>
<td>$1 \land 2$</td>
<td>$\sim p$</td>
<td>$(1 \land 2) \rightarrow c$</td>
</tr>
<tr>
<td>1.</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>2.</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>3.</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>4.</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**FIGURE 1.63** Truth table for the argument $[(p \rightarrow \sim q) \land q] \rightarrow \sim p$.

The completed truth table shows that the conditional $[(p \rightarrow \sim q) \land q] \rightarrow \sim p$ is always true. The conditional represents the argument “If he is illiterate, he cannot fill out the application. He can fill out the application. Therefore, he is not illiterate.” Thus, the argument is valid.

### Tautologies

A tautology is a statement that is always true. For example, the statement

$$(a + b)^2 = a^2 + 2ab + b^2$$

is a tautology.
DETERMINING WHETHER A STATEMENT IS A TAUTOLOGY

Determine whether the statement \( (p \land q) \rightarrow (p \lor q) \) is a tautology.

We need to construct a truth table for the statement. Because there are two letters, the table must have \( 2^2 = 4 \) rows. We need a column for \( (p \land q) \), one for \( (p \lor q) \), and one for \( (p \land q) \rightarrow (p \lor q) \). The completed truth table is shown in Figure 1.64.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( p \land q )</th>
<th>( p \lor q )</th>
<th>( (p \land q) \rightarrow (p \lor q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>2.</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>3.</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>4.</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

FIGURE 1.64 Truth table for the statement \( (p \land q) \rightarrow (p \lor q) \).

Because \( (p \land q) \rightarrow (p \lor q) \) is always true, it is a tautology.

As we have seen, an argument can be represented by a single conditional. If this conditional is always true, the argument is valid (and vice versa).

VALIDITY OF AN ARGUMENT

An argument having \( n \) hypotheses \( h_1, h_2, \ldots, h_n \) and conclusion \( c \) is valid if and only if the conditional \( [h_1 \land h_2 \land \ldots \land h_n] \rightarrow c \) is a tautology.

EXAMPLE 3

USING A TRUTH TABLE TO ANALYZE AN ARGUMENT

Determine whether the following argument is valid:

“If the defendant is innocent, the defendant does not go to jail. The defendant does not go to jail. Therefore, the defendant is innocent.”

Separating the hypotheses from the conclusion, we have

1. If the defendant is innocent, the defendant does not go to jail.
2. The defendant does not go to jail.

Therefore, the defendant is innocent.

Now we define symbols to represent the various components of the statements:

\( p \): The defendant is innocent.
\( q \): The defendant goes to jail.

Symbolically, the argument has the form

1. \( p \rightarrow \neg q \)
2. \( \neg q \)
\[ \therefore p \]

and is represented by the conditional \( [(p \rightarrow \neg q) \land \neg q] \rightarrow p \).
Now we construct a truth table with four rows, along with the necessary columns. The completed table is shown in Figure 1.65.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>1</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>q</td>
<td>~q</td>
<td>p → ~q</td>
</tr>
<tr>
<td>1</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>2</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>3</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>4</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**Figure 1.65** Truth table for the argument \([p → ~q] ∧ ~q → p\).

The column representing the argument has an \(F\) in row 4; therefore, the conditional representation of the argument is not a tautology. In particular, the conclusion does not logically follow the hypotheses when both \(p\) and \(q\) are false (row 4). The argument is not valid. Let us interpret the circumstances expressed in row 4, the row in which the argument breaks down. Both \(p\) and \(q\) are false—that is, the defendant is guilty and the defendant does not go to jail. Unfortunately, this situation can occur in the real world; guilty people do not always go to jail! As long as it is possible for a guilty person to avoid jail, the argument is invalid.

The following argument was presented as Example 6 in Section 1.1. In that section, we constructed a Venn diagram to show that the argument was in fact valid. We now show an alternative method; that is, we construct a truth table to determine whether the argument is valid.

**Example 4**

**Using a Truth Table to Analyze an Argument** Determine whether the following argument is valid: “No snake is warm-blooded. All mammals are warm-blooded. Therefore, snakes are not mammals.”

**Solution**

Separating the hypotheses from the conclusion, we have

1. No snake is warm-blooded.
2. All mammals are warm-blooded.
   Therefore, snakes are not mammals.

These statements can be rephrased as follows:

1. If it is a snake, then it is not warm-blooded.
2. If it is a mammal, then it is warm-blooded.
   Therefore, if it is a snake, then it is not a mammal.

Now we define symbols to represent the various components of the statements:

\(p\): It is a snake.
\(q\): It is warm-blooded.
\(r\): It is a mammal.
Symbolically, the argument has the form

1. \( p \rightarrow \sim q \)
2. \( r \rightarrow q \)
\[ \therefore p \rightarrow \sim r \]

and is represented by the conditional \([(p \rightarrow \sim q) \land (r \rightarrow q)] \rightarrow (p \rightarrow \sim r)\).

Now we construct a truth table with eight rows \((2^3 = 8)\), along with the necessary columns. The completed table is shown in Figure 1.66.

<table>
<thead>
<tr>
<th></th>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( \sim q )</th>
<th>( \sim r )</th>
<th>( p \rightarrow \sim q )</th>
<th>( r \rightarrow q )</th>
<th>( 1 \land 2 )</th>
<th>( p \rightarrow \sim r )</th>
<th>( (1 \land 2) \rightarrow \sim r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>2</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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**FIGURE 1.66** Truth table for the argument \([(p \rightarrow \sim q) \land (r \rightarrow q)] \rightarrow (p \rightarrow \sim r).\)

The last column of the truth table represents the argument and contains all T’s. Consequently, the conditional \([(p \rightarrow \sim q) \land (r \rightarrow q)] \rightarrow (p \rightarrow \sim r)\) is a tautology; the argument is valid.

The preceding examples contained relatively simple arguments, each consisting of only two hypotheses and two simple statements (letters). In such cases, many people try to employ “common sense” to confirm the validity of the argument. For instance, the argument “If it is raining, the streets are wet. It is raining. Therefore, the streets are wet” is obviously valid. However, it might not be so simple to determine the validity of an argument that contains several hypotheses and many simple statements. Indeed, in such cases, the argument’s truth table might become quite lengthy, as in the next example.

**EXAMPLE 5**

**USING A TRUTH TABLE TO ANALYZE AN ARGUMENT** The following whimsical argument was written by Lewis Carroll and appeared in his 1896 book *Symbolic Logic*:

“No ducks waltz. No officers ever decline to waltz. All my poultry are ducks. Therefore, my poultry are not officers.”

Construct a truth table to determine whether the argument is valid.
His obvious interest in telling stories, the fantastic stories manifested itself in much attraction to entertaining children with siblings with elaborate games, poems, children, Charles amused his younger photographers of the Victorian era. Recognized as one of the leading portrait if that were not enough, Dodgson is now teaching mathematics and logic. And as resident at the University at Oxford, eighteen to his death) was a permanent prised to learn that Dodgson (from age Glass.

Charles Lutwidge Dodgson, 1832–1898

To those who assume that it is impossible for a person to excel both in the creative worlds of art and literature and in the disciplined worlds of mathematics and logic, the life of Charles Lutwidge Dodgson is a wondrous counterexample. Known the world over as Lewis Carroll, Dodgson penned the nonsensical classics Alice’s Adventures in Wonderland and Through the Looking-Glass. However, many people are surprised to learn that Dodgson (from age eighteen to his death) was a permanent resident at the University at Oxford, teaching mathematics and logic. And as if that were not enough, Dodgson is now recognized as one of the leading portrait photographers of the Victorian era.

The eldest son in a family of eleven children, Charles amused his younger siblings with elaborate games, poems, stories, and humorous drawings. This attraction to entertaining children with fantastic stories manifested itself in much of his later work as Lewis Carroll. Besides his obvious interest in telling stories, the young Dodgson was also intrigued by mathematics. At the age of eight, Charles asked his father to explain a book on logarithms. When told that he was too young to understand, Charles persisted, “But please, explain!”

The Dodgson family had a strong ecclesiastical tradition; Charles’s father, great-grandfather, and great-great-grandfather were all clergymen. Following in his father’s footsteps, Charles attended Christ Church, the largest and most celebrated of all the Oxford colleges. After graduating in 1854, Charles remained at Oxford, accepting the position of mathematical lecturer in 1855. However, appointment to his position was conditional upon his taking Holy Orders in the Anglican church and upon his remaining celibate. Dodgson complied and was named a deacon in 1861.

The year 1856 was filled with events that had lasting effects on Dodgson. Charles Lutwidge created his pseudonym by translating his first and middle names into Latin (Carolus Ludovicus), reversing their order (Ludovic Carolus), and translating them back into English (Lewis Carroll). In this same year, Dodgson began his “hobby” of photography. He is considered by many to have been an artistic pioneer in this new field (photography was invented in 1839). Most of Dodgson’s work consists of portraits that chronicle the Victorian era, and over 700 photographs taken by Dodgson have been preserved. His favorite subjects were children, especially young girls.

Dodgson’s affinity for children brought about a meeting in 1856 that would eventually establish his place in the history of literature. Early in the year, Dodgson met the four children of the dean of Christ Church: Harry, Lorina, Edith, and Alice Liddell. He began seeing the children on a regular basis, amusing them with stories and photographing them. Although he had a wonderful relationship with all four, Alice received his special attention.

On July 4, 1862, while rowing and picnicking with Alice and her sisters, Dodgson entertained the Liddell girls with a fantastic story of a little girl named Alice who fell into a rabbit hole. Captivated by

SOLUTION

Separating the hypotheses from the conclusion, we have

1. No ducks waltz.
2. No officers ever decline to waltz.
3. All my poultry are ducks.

Therefore, my poultry are not officers.

These statements can be rephrased as

1. If it is a duck, then it does not waltz.
2. If it is an officer, then it does not decline to waltz.
   (Equivalently, “If it is an officer, then it will waltz.”)
3. If it is my poultry, then it is a duck.

Therefore, if it is my poultry, then it is not an officer.
1.5 Analyzing Arguments

Young Alice Liddell inspired Lewis Carroll to write Alice’s Adventures in Wonderland. This photo is one of the many Carroll took of Alice.

Though the book appeared to be a whimsical excursion into chaotic nonsense, Dodgson’s masterpiece contained many exercises in logic and metaphor. The book was a success, and in 1871, a sequel, Through the Looking Glass, was printed. When asked to comment on the meaning of his writings, Dodgson replied, “I’m very much afraid I didn’t mean anything but nonsense! Still, you know, words mean more than we mean to express when we use them; so a whole book ought to mean a great deal more than the writer means. So, whatever good meanings are in the book, I’m glad to accept as the meaning of the book.”

In addition to writing “children’s stories,” Dodgson wrote numerous mathematics essays and texts, including The Fifth Book of Euclid Proved Algebraically, Formulae of Plane Trigonometry, A Guide to the Mathematical Student, and Euclid and His Modern Rivals. In the field of formal logic, Dodgson’s books The Game of Logic (1887) and Symbolic Logic (1896) are still used as sources of inspiration in numerous schools worldwide.

Now we define symbols to represent the various components of the statements:

- \( p \): It is a duck.
- \( q \): It will waltz.
- \( r \): It is an officer.
- \( s \): It is my poultry.

Symbolically, the argument has the form

1. \( p \rightarrow \sim q \)
2. \( r \rightarrow q \)
3. \( s \rightarrow p \)
\[ \therefore s \rightarrow \sim r \]
In Exercises 1–10, use the given symbols to rewrite the argument in symbolic form.

1. \( p \): It is raining.
   \( q \): The streets are wet.
   1. If it is raining, then the streets are wet.
   2. It is raining.
   Therefore, the streets are wet.

2. \( p \): I have a college degree.
   \( q \): I am lazy.
   1. If I have a college degree, I am not lazy.
   2. I do not have a college degree.
   Therefore, I am lazy.

3. \( p \): It is Tuesday.
   \( q \): The tour group is in Belgium.
   1. If it is Tuesday, then the tour group is in Belgium.
   2. The tour group is not in Belgium.
   Therefore, it is not Tuesday.

4. \( p \): You are a gambler.
   \( q \): You have financial security.
   1. You do not have financial security if you are a gambler.
   2. You do not have financial security.
   Therefore, you are a gambler.

Now we construct a truth table with sixteen rows (\(2^4 = 16\)), along with the necessary columns. The completed table is shown in Figure 1.67. The last column of the truth table represents the argument and contains all T’s. Consequently, the conditional \([p \rightarrow \neg q] \land [r \rightarrow q] \land (s \rightarrow p) \rightarrow (s \rightarrow \neg r)\) is a tautology; the argument is valid.
5. \( p \): You exercise regularly. \( q \): You are healthy.
1. You exercise regularly only if you are healthy.
2. You do not exercise regularly.

Therefore, you are not healthy.

6. \( p \): The senator supports new taxes. \( q \): The senator is reelected.
1. The senator is not reelected if she supports new taxes.
2. The senator does not support new taxes.

Therefore, the senator is reelected.

7. \( p \): A person knows Morse code. \( q \): A person operates a telegraph. \( r \): A person is Nikola Tesla.
1. Knowing Morse code is necessary for operating a telegraph.
2. Nikola Tesla knows Morse code.

Therefore, Nikola Tesla operates a telegraph.

HINT: Hypothesis 2 can be symbolized as \( r \land p \).

8. \( p \): A person knows CPR. \( q \): A person is a paramedic. \( r \): A person is David Lee Roth.
1. Knowing CPR is necessary for being a paramedic.
2. David Lee Roth is a paramedic.

Therefore, David Lee Roth knows CPR.

HINT: Hypothesis 2 can be symbolized as \( r \land q \).

9. \( p \): It is a monkey. \( q \): It is an ape. \( r \): It is King Kong.
1. Being a monkey is sufficient for not being an ape.
2. King Kong is an ape.

Therefore, King Kong is not a monkey.

10. \( p \): It is warm-blooded. \( q \): It is a reptile. \( r \): It is Godzilla.
1. Being warm-blooded is sufficient for not being a reptile.
2. Godzilla is not warm-blooded.

Therefore, Godzilla is a reptile.

In Exercises 11–20, use a truth table to determine the validity of the argument specified. If the argument is invalid, interpret the specific circumstances that cause it to be invalid.

11. the argument in Exercise 1
12. the argument in Exercise 2
13. the argument in Exercise 3
14. the argument in Exercise 4
15. the argument in Exercise 5
16. the argument in Exercise 6

17. the argument in Exercise 7
18. the argument in Exercise 8
19. the argument in Exercise 9
20. the argument in Exercise 10

In Exercises 21–42, define the necessary symbols, rewrite the argument in symbolic form, and use a truth table to determine whether the argument is valid. If the argument is invalid, interpret the specific circumstances that cause the argument to be invalid.

21. 1. If the Democrats have a majority, Smith is appointed and student loans are funded.
2. Smith is appointed or student loans are not funded.

Therefore, the Democrats do not have a majority.

22. 1. If you watch television, you do not read books.
2. If you read books, you are wise.

Therefore, you are not wise if you watch television.

23. 1. If you argue with a police officer, you get a ticket.
2. If you do not break the speed limit, you do not get a ticket.

Therefore, if you break the speed limit, you argue with a police officer.

24. 1. If you do not recycle newspapers, you are not an environmentalist.
2. If you recycle newspapers, you save trees.

Therefore, you are an environmentalist only if you save trees.

25. 1. All pesticides are harmful to the environment.
2. No fertilizer is a pesticide.

Therefore, no fertilizer is harmful to the environment.

26. 1. No one who can afford health insurance is unemployed.
2. All politicians can afford health insurance.

Therefore, no politician is unemployed.

27. 1. All poets are loners.
2. All loners are taxi drivers.

Therefore, all poets are taxi drivers.

28. 1. All forest rangers are environmentalists.
2. All forest rangers are storytellers.

Therefore, all environmentalists are storytellers.

29. 1. No professor is a millionaire.
2. No millionaire is illiterate.

Therefore, no professor is illiterate.

30. 1. No artist is a lawyer.
2. No lawyer is a musician.

Therefore, no artist is a musician.

31. 1. All lawyers study logic.
2. You study logic only if you are a scholar.
3. You are not a scholar.

Therefore, you are not a lawyer.
32. 1. All licensed drivers have insurance.
2. You obey the law if you have insurance.
3. You obey the law.
Therefore, you are a licensed driver.
33. 1. Drinking espresso is sufficient for not sleeping.
2. Not eating dessert is necessary for being on a diet.
3. Not eating dessert is sufficient for drinking espresso.
Therefore, not being on a diet is necessary for sleeping.
34. 1. Not being eligible to vote is sufficient for ignoring politics.
2. Not being a convicted felon is necessary for being eligible to vote.
3. Ignoring politics is sufficient for being naive.
Therefore, being naive is necessary being a convicted felon.
35. If the defendant is innocent, he does not go to jail. The defendant goes to jail. Therefore, the defendant is guilty.
36. If the defendant is innocent, he does not go to jail. The defendant is guilty. Therefore, the defendant goes to jail.
37. If you are not in a hurry, you eat at Lulu’s Diner. If you are in a hurry, you do not eat good food. You eat at Lulu’s. Therefore, you eat good food.
38. If you give me a hamburger today, I pay you tomorrow. If you are a sensitive person, you give me a hamburger today. You are not a sensitive person. Therefore, I do not pay you tomorrow.
39. If you listen to rock and roll, you do not go to heaven. If you are a moral person, you go to heaven. Therefore, you are not a moral person if you listen to rock and roll.
40. If you follow the rules, you have no trouble. If you are not clever, you have trouble. You are clever. Therefore, you do not follow the rules.
41. The water not being cold is sufficient for going swimming. Having goggles is necessary for going swimming. I have no goggles. Therefore, the water is cold.
42. I wash my car only if the sky is clear. The sky not being clear is necessary for it to rain. I do not wash my car. Therefore, it is raining.

The arguments given in Exercises 43–50 were written by Lewis Carroll and appeared in his 1896 book Symbolic Logic. For each argument, define the necessary symbols, rewrite the argument in symbolic form, and use a truth table to determine whether the argument is valid.

43. 1. All medicine is nasty.
   2. Senna is a medicine.
   Therefore, senna is nasty.
   
   NOTE: Senna is a laxative extracted from the dried leaves of cassia plants.

44. 1. All pigs are fat.
   2. Nothing that is fed on barley-water is fat.
   Therefore, pigs are not fed on barley-water.

45. 1. Nothing intelligible ever puzzles me.
   2. Logic puzzles me.
   Therefore, logic is unintelligible.

46. 1. No misers are unselfish.
   2. None but misers save eggshells.
   Therefore, no unselfish people save eggshells.

47. 1. No Frenchmen like plum pudding.
   2. All Englishmen like plum pudding.
   Therefore, Englishmen are not Frenchmen.

48. 1. A prudent man shuns hyenas.
   2. No banker is imprudent.
   Therefore, no banker fails to shun hyenas.

49. 1. All wasps are unfriendly.
   2. No puppies are unfriendly.
   Therefore, puppies are not wasps.

50. 1. Improbable stories are not easily believed.
   2. None of his stories are probable.
   Therefore, none of his stories are easily believed.

**CONCEPT QUESTIONS**

51. What is a tautology?
52. What is the conditional representation of an argument?
53. Find a “logical” argument in a newspaper article, an advertisement, or elsewhere in the media. Analyze that argument and discuss the implications.

**HISTORY QUESTIONS**

54. What was Charles Dodgson’s pseudonym? How did he get it? What classic “children’s stories” did he write?
55. What did Charles Dodgson contribute to the study of formal logic?
56. Charles Dodgson was a pioneer in what artistic field?
57. Who was Alice Liddell?

**WEB PROJECT**

58. Write a research paper on any historical topic referred to in this chapter or a related topic. Below is a partial list of topics.
   - Aristotle
   - George Boole
   - Augustus De Morgan
   - Charles Dodgson/Lewis Carroll
   - Gottfried Wilhelm Leibniz

Some useful links for this web project are listed on the text web site: [www.cengage.com/math/johnson](http://www.cengage.com/math/johnson)
CHAPTER REVIEW

TERMS
argument  converse
bicongditional  deductive reasoning
compound statement  disjunction
conclusion  equivalent expressions
conditional  exclusive or
conjunction  hypothesis
contrapositive  implication

invalid argument  inverse
logic  necessary
negation  premise
statement  quantifier

sudoku  sufficient
syllogism  tautology
type of reasoning  truth table
truth value  valid argument
Venn diagram

REVIEW EXERCISES

1. Classify each argument as deductive or inductive.
   a. 1. Hitchcock’s “Psycho” is a suspenseful movie.
      2. Hitchcock’s “The Birds” is a suspenseful movie.
      Therefore, all Hitchcock movies are suspenseful.
   b. 1. All Hitchcock movies are suspenseful.
      2. “Psycho” is a Hitchcock movie.
      Therefore, “Psycho” is suspenseful.
2. Explain the general rule or pattern used to assign the given letter to the given word. Fill in the blank with the letter that fits the pattern.
   Day: y, R, t, a, ______
3. Fill in the blank with what is most likely to be the next number. Explain the pattern generated by your answer.
   1, 6, 11, 4, ______
In Exercises 4–9, construct a Venn diagram to determine the validity of the given argument.
4. 1. All truck drivers are union members.
    2. Rocky is a truck driver.
    Therefore, Rocky is a union member.
5. 1. All truck drivers are union members.
    2. Rocky is not a truck driver.
    Therefore, Rocky is not a union member.
6. 1. All mechanics are engineers.
    2. Casey Jones is an engineer.
    Therefore, Casey Jones is a mechanic.
7. 1. All mechanics are engineers.
    2. Casey Jones is not an engineer.
    Therefore, Casey Jones is not a mechanic.
8. 1. Some animals are dangerous.
    2. A gun is not an animal.
    Therefore, a gun is not dangerous.
9. 1. Some contractors are electricians.
    2. All contractors are carpenters.
    Therefore, some electricians are carpenters.
10. Solve the following sudoku puzzle.

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   |   |   |   |

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---|---|---|---|
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11. Explain why each of the following is or is not a statement.
   a. The Golden Gate Bridge spans Chesapeake Bay.
   b. The capital of Delaware is Dover.
   c. Where are you spending your vacation?
   d. Hawaii is the best place to spend a vacation.
12. Determine which pairs of statements are negations of each other.
   a. All of the lawyers are ethical.
   b. Some of the lawyers are ethical.
   c. None of the lawyers is ethical.
   d. Some of the lawyers are not ethical.
13. Write a sentence that represents the negation of each statement.
   a. His car is not new.
   b. Some buildings are earthquake proof.
   c. All children eat candy.
   d. I never cry in a movie theater.

14. Using the symbolic representations
   
   \( p \): The television program is educational.
   \( q \): The television program is controversial.

   express the following compound statements in symbolic form.
   a. The television program is educational and controversial.
   b. If the television program isn’t controversial, it isn’t educational.
   c. The television program is educational and it isn’t controversial.
   d. The television program isn’t educational or controversial.
   e. Not being controversial is necessary for a television program to be educational.
   f. Being controversial is sufficient for a television program not to be educational.

15. Using the symbolic representations
   
   \( p \): The advertisement is effective.
   \( q \): The advertisement is misleading.
   \( r \): The advertisement is outdated.

   express the following compound statements in symbolic form.
   a. All misleading advertisements are effective.
   b. It is a current, honest, effective advertisement.
   c. If an advertisement is outdated, it isn’t effective.
   d. The advertisement is effective and it isn’t misleading or outdated.
   e. Not being outdated or misleading is necessary for an advertisement to be effective.
   f. Being outdated and misleading is sufficient for an advertisement not to be educational.

16. Using the symbolic representations
   
   \( p \): It is expensive.
   \( q \): It is undesirable.

   express the following in words.
   a. \( p \to \neg q \)
   b. \( q \leftrightarrow \neg p \)
   c. \( \neg(p \lor q) \)
   d. \( (p \land \neg q) \lor (\neg p \land q) \)

17. Using the symbolic representations
   
   \( p \): The movie is critically acclaimed.
   \( q \): The movie is a box office hit.
   \( r \): The movie is available on DVD.

   express the following in words.
   a. \( (p \lor q) \to r \)
   b. \( (p \land \neg q) \to \neg r \)
   c. \( \neg(p \lor q) \land r \)
   d. \( \neg r \to (\neg p \land \neg q) \)

18. \( p \lor \neg q \)
19. \( p \land \neg q \)
20. \( \neg p \to q \)
21. \( (p \land q) \to \neg q \)
22. \( q \lor \neg(q \lor r) \)
23. \( \neg p \to (q \lor r) \)
24. \( (q \land p) \to (\neg r \lor p) \)
25. \( (p \lor r) \to (q \land \neg r) \)

In Exercises 18–25, construct a truth table for the compound statement.

26. The car is unreliable or expensive.
   If the car is reliable, then it is expensive.
27. If I get a raise, I will buy a new car.
   If I do not get a raise, I will not buy a new car.
28. She is a Democrat or she did not vote.
   She is not a Democrat and she did vote.
29. The raise is not unjustified and the management opposes it.
   It is not the case that the raise is unjustified or the management does not oppose it.
30. Walking on the beach is sufficient for not wearing shoes.
   Wearing shoes is necessary for not walking on the beach.

In Exercises 31–38, write a sentence that represents the negation of each statement.
31. Jesse had a party and nobody came.
32. You do not go to jail if you pay the fine.
33. I am the winner or you are blind.
34. He is unemployed and he did not apply for financial assistance.
35. The selection procedure has been violated if his application is ignored.
36. The jackpot is at least $1 million.
37. Drinking espresso is sufficient for not sleeping.
38. Not eating dessert is necessary for being on a diet.
39. Given the statements
   \( p \): You are an avid jogger.
   \( q \): You are healthy.
   write the sentence represented by each of the following.
   a. \( p \to q \)
   b. \( q \rightarrow p \)
   c. \( \neg p \to \neg q \)
   d. \( \neg q \to \neg p \)
   e. \( p \leftrightarrow q \)

40. Form (a) the inverse, (b) the converse, and (c) the contrapositive of the conditional “If he is elected, the country is in big trouble.”

In Exercises 41 and 42, express the contrapositive of the given conditional in terms of (a) a sufficient condition, and (b) a necessary condition.
41. Having a map is sufficient for not being lost.
42. Having syrup is necessary for eating pancakes.
In Exercises 43–48, (a) determine the premise and conclusion and (b) rewrite the compound statement in the standard “if . . . then . . .” form.

43. The economy improves only if unemployment goes down.

44. The economy improves if unemployment goes down.

45. No computer is unrepairable.

46. All gemstones are valuable.

47. Being the fourth Thursday in November is sufficient for the U.S. Post Office to be closed.

48. Having diesel fuel is necessary for the vehicle to operate.

In Exercises 49 and 50, translate the two statements into symbolic form and use truth tables to determine whether the statements are equivalent.

49. If you are allergic to dairy products, you cannot eat cheese.

If you cannot eat cheese, then you are allergic to dairy products.

50. You are a fool if you listen to me.

You are not a fool only if you do not listen to me.

In Exercises 51–57, define the necessary symbols, rewrite the argument in symbolic form, and use a truth table to determine whether the argument is valid.

51. Which pairs of statements are equivalent?

i. If it is not raining, I ride my bicycle to work.

ii. If I ride my bicycle to work, it is not raining

iii. If I do not ride my bicycle to work, it is raining.

iv. If it is raining, I do not ride my bicycle to work.

52. 1. If you do not make your loan payment, your car is repossessed.

2. Your car is repossessed.

Therefore, you did not make your loan payment.

53. 1. If you do not pay attention, you do not learn the new method.

2. You do learn the new method.

Therefore, you do pay attention.

54. 1. If you rent DVD, you will not go to the movie theater.

2. If you go to the movie theater, you pay attention to the movie.

Therefore, you do not pay attention to the movie if you rent DVDs.

55. 1. If the Republicans have a majority, Farnsworth is appointed and no new taxes are imposed.

2. New taxes are imposed.

Therefore, the Republicans do not have a majority or Farnsworth is not appointed.

56. 1. Practicing is sufficient for making no mistakes.

2. Making a mistake is necessary for not receiving an award.

3. You receive an award.

Therefore, you practice.

57. 1. Practicing is sufficient for making no mistakes.

2. Making a mistake is necessary for not receiving an award.

3. You do not receive an award.

Therefore, you do not practice.

In Exercises 58–66, define the necessary symbols, rewrite the argument in symbolic form, and use a truth table to determine whether the argument is valid.

58. If the defendant is guilty, he goes to jail. The defendant does not go to jail. Therefore, the defendant is not guilty.

59. I will go to the concert only if you buy me a ticket. You bought me a ticket. Therefore, I will go to the concert.

60. If tuition is raised, students take out loans or drop out. If students do not take out loans, they drop out. Students do drop out. Therefore, tuition is raised.

61. If our oil supply is cut off, our economy collapses. If we go to war, our economy doesn’t collapse. Therefore, if our oil supply isn’t cut off, we do not go to war.

62. No professor is uneducated. No monkey is educated. Therefore, no professor is a monkey.

63. No professor is uneducated. No monkey is a professor. Therefore, no monkey is educated.

64. Vehicles stop if the traffic light is red. There is no accident if vehicles stop. There is an accident. Therefore, the traffic light is not red.

65. Not investing money in the stock market is necessary for invested money to be guaranteed. Invested money not being guaranteed is sufficient for not retiring at an early age. Therefore, if your money is not invested in the stock market, you retire at an early age.

66. Not investing money in the stock market is necessary for invested money to be guaranteed. Invested money not being guaranteed is sufficient for not retiring at an early age. You do not invest in the stock market. Therefore, you retire at an early age.

Determine the validity of the arguments in Exercises 67 and 68 by constructing a

a. Venn diagram and a

b. truth table.

c. How do the answers to parts (a) and (b) compare? Why?

67. 1. If you own a hybrid vehicle, then you are an environmentalist.

2. You are not an environmentalist.

Therefore, you do not own a hybrid vehicle.
66. 1. If you own a hybrid vehicle, then you are an environmentalist.
2. You are an environmentalist.

Therefore, you own a hybrid vehicle.

71. a. What is a sufficient condition?
    b. What is a necessary condition?

72. What is a tautology?

73. When constructing a truth table, how do you determine how many rows to create?

**HISTORY QUESTIONS**

74. What role did the following people play in the development of formalized logic?
   - Aristotle
   - George Boole
   - Augustus De Morgan
   - Charles Dodgson
   - Gottfried Wilhelm Leibniz

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**CONCEPT QUESTIONS**

69. What is a statement?

70. a. What is a disjunction? Under what conditions is a disjunction true?
    b. What is a conjunction? Under what conditions is a conjunction true?
    c. What is a conditional? Under what conditions is a conditional true?
    d. What is a negation? Under what conditions is a negation true?