4 Systems of Linear Equations and Inequalities

Chapter Outline

Introduction

One of the most innovative and successful textbook authors of our time is Harold Jacobs. Mr. Jacobs taught mathematics, physics, and chemistry at Ulysses S. Grant High School in Los Angeles for 35 years. The page below is from the transparency masters that accompany his elementary algebra textbook. It is a page from a French mathematics book showing an example of how to solve a system of linear equations in two variables, which is one of the main topics of this chapter.

SYSTÈME DE DEUX ÉQUATIONS À DEUX INCONNUES

1° **Elimination d'une inconnue par addition.**

Exemple $\begin{cases} 2x + y = 1 & (1) \\ 3x - 4y = 29 & (2) \end{cases}$

Multiplions par 4 les deux membres de (1) :

$$8x + 4y = 4 \quad (3)$$

Ajoutons (2) et (3) :

$$11x = 33$$

$$x = \frac{33}{11} = 3.$$

Portons x = 3 dans (1) :

$$6 + y = 1$$

$$y = 1 - 6 = -5.$$

Règle pratique : par le jeu des multiplications, faire apparaître dans les équations deux coefficients opposés.

Based on W.H. Freeman and Company

After you have finished this chapter, come back to this introduction and see if you can understand the example written in French. I think you will be surprised how easy it is to understand the French words, when you understand the mathematics they are describing.

Getting Ready for Chapter 4

Simplify.

1. $0.09(6{,}000)$
2. $1.5(500)$
3. $3(11) - 5(7)$
4. $6 - 12$
5. $3(3) - 5$
6. $4(5) - 3(-3)$
7. $1 + 2(2) - 3(3)$
8. $-4(-1) + 1(2) + 2(6)$

Apply the distributive property, then simplify if possible.

9. $-3(2x - 3y)$
10. $3(4x + 5y)$
11. $6\left(\frac{1}{2}x - \frac{1}{3}y\right)$
12. $12\left(\frac{1}{4}x + \frac{2}{3}y\right)$
13. $10(0.3x + 0.7y)$
14. $100(0.06x + 0.07y)$

Solve.

15. $2(1) + y = 4$
16. $2x - 3(3x - 5) = -6$
17. $2(2y + 6) + 3y = 5$
18. $5\left(\frac{19}{15}\right) + 5y = 9$
19. $4x - 2x = 8$
20. $20x + 9{,}300 > 18{,}000$

Chapter Outline

4.1 Systems of Linear Equations in Two Variables

A Solve systems of linear equations in two variables by graphing.

B Solve systems of linear equations in two variables by the addition method.

C Solve systems of linear equations in two variables by the substitution method.

4.2 Systems of Linear Equations in Three Variables

A Solve systems of linear equations in three variables.

4.3 Applications of Linear Systems

A Solve application problems whose solutions are found through systems of linear equations.

4.4 Matrix Solutions to Linear Systems

A Solve a system of linear equations using an augmented matrix.

4.5 Systems of Linear Inequalities

A Graph the solution to a sytem of linear inequalities in two variables.

4.1 Systems of Linear Equations in Two Variables

OBJECTIVES

A Solve systems of linear equations in two variables by graphing.

B Solve systems of linear equations in two variables by the addition method.

C Solve systems of linear equations in two variables by the substitution method.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. How would you define a solution for a linear system of equations?
2. How would you use the addition method to solve a system of linear equations?
3. When would the substitution method be more efficient than the addition method in solving a system of linear equations?
4. Explain what an inconsistent system of linear equations looks like graphically and what would result algebraically when attempting to solve the system.

Suppose you and a friend want to order burritos and tacos for lunch. The restaurant's lighted menu, however, is broken and you can't read it. You order 2 burritos and 3 tacos and are charged a total of \$12. Your friend orders 1 burrito and 4 tacos for a total of \$11. How much is each burrito? How much is each taco? In this section, we will begin our work with systems of equations in two variables and use these systems to answer your lunch question.

A Solve Systems by Graphing

Previously, we found the graph of an equation of the form $ax + by = c$ to be a straight line. Since the graph is a straight line, the equation is said to be a linear equation. Two linear equations considered together form a *linear system* of equations. For example,

$$3x - 2y = 6$$
$$2x + 4y = 20$$

is a linear system. The solution set to the system is the set of all ordered pairs that satisfy both equations. If we graph each equation on the same set of axes, we can see the solution set (Figure 1).

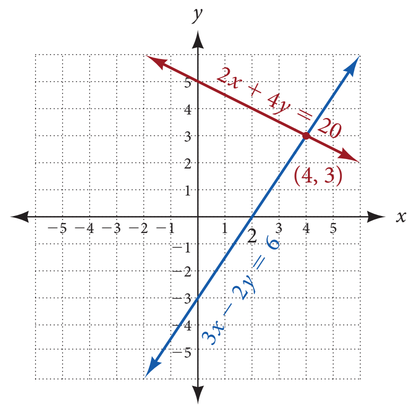


FIGURE 1

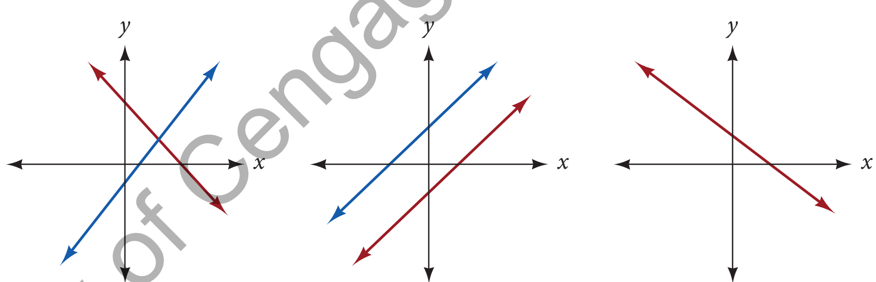
NOTE
It is important that you write solutions to systems of equations in two variables as ordered pairs, like (4, 3); that is, you should always enclose the ordered pairs in parentheses.

The point (4, 3) lies on both lines and therefore must satisfy both equations. It is obvious from the graph that it is the only point that does so. The solution set for the system is {(4, 3)}.

More generally, if $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ are linear equations, then the solution set for the system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

can be illustrated through one of the graphs in Figure 2.



Case I
(One solution)

Case II
(No solutions)

Case III
(Infinite number of solutions)

FIGURE 2

Case I The two lines intersect at one and only one point. The coordinates of the point give the solution to the system. This is what usually happens.

Case II The lines are parallel and therefore have no points in common. The solution set to the system is the empty set, ∅. In this case, we say the system is *inconsistent.*

Case III The lines coincide; that is, their graphs represent the same line. The solution set consists of all ordered pairs that satisfy either equation. In this case, the equations are said to be *dependent.*

In the beginning of this section we found the solution set for the system

$$3x - 2y = 6$$
$$2x + 4y = 20$$

by graphing each equation and then reading the solution set from the graph. Solving a system of linear equations by graphing is the least accurate method. If the coordinates of the point of intersection are not integers, it can be very difficult to read the solution set from the graph. There is another method of solving a linear system that does not depend on the graph. It is called the *addition method.*

B The Addition Method

EXAMPLE 1 Solve the system.

$$4x + 3y = 10$$
$$2x + y = 4$$

SOLUTION If we multiply the bottom equation by -3, the coefficients of y in the resulting equation and the top equation will be opposites.

$$\begin{aligned} 4x + 3y &= 10 \xrightarrow{\text{No change}} & 4x + 3y &= 10 \\ 2x + y &= 4 \xrightarrow[\text{Multiply by } -3]{} & -6x - 3y &= -12 \end{aligned}$$

Adding the left and right sides of the resulting equations, we have

$$\begin{array}{r} 4x + 3y = 10 \\ \underline{-6x - 3y = -12} \\ -2x = -2 \end{array}$$

The result is a linear equation in one variable. We have eliminated the variable y from the equations by addition. (It is for this reason we call this method of solving a linear system the *addition method*.) Solving $-2x = -2$ for x, we have

$$x = 1$$

This is the x-coordinate of the solution to our system. To find the y-coordinate, we substitute $x = 1$ into any of the equations containing both the variables x and y. Let's try the second equation in our original system.

$$2(1) + y = 4$$
$$2 + y = 4$$
$$y = 2$$

This is the y-coordinate of the solution to our system. The ordered pair $(1, 2)$ is the solution to the system.

CHECKING SOLUTIONS We can check our solution by substituting it into both of our equations.

Substituting $x = 1$ and $y = 2$ into $4x + 3y = 10$, we have

$$4(1) + 3(2) \stackrel{?}{=} 10$$
$$4 + 6 \stackrel{?}{=} 10$$
$$10 = 10 \quad \text{A true statement}$$

Substituting $x = 1$ and $y = 2$ into $2x + y = 4$, we have

$$2(1) + 2 \stackrel{?}{=} 4$$
$$2 + 2 \stackrel{?}{=} 4$$
$$4 = 4 \quad \text{A true statement}$$

Our solution satisfies both equations; therefore, it is a solution to our system of equations.

EXAMPLE 2 Solve the system.

$$3x - 5y = -2$$
$$2x - 3y = 1$$

SOLUTION We can eliminate either variable. Let's decide to eliminate the variable x. We can do so by multiplying the top equation by 2 and the bottom equation by -3, and then adding the left and right sides of the resulting equations.

$$\begin{array}{rcr}
3x - 5y = -2 & \xrightarrow{\text{Multiply by 2}} & 6x - 10y = -4 \\
2x - 3y = 1 & \xrightarrow[\text{Multiply by } -3]{} & -6x + 9y = -3 \\ \hline
 & & -y = -7 \\
 & & y = 7
\end{array}$$

The y-coordinate of the solution to the system is 7. Substituting this value of y into any of the equations with both x- and y-variables gives $x = 11$. The solution to the system is (11, 7). It is the only ordered pair that satisfies both equations.

CHECKING SOLUTIONS Checking (11, 7) in each equation looks like this:

Substituting $x = 11$ and $y = 7$ into $3x - 5y = -2$, we have

$$\begin{aligned}
3(11) - 5(7) &\stackrel{?}{=} -2 \\
33 - 35 &\stackrel{?}{=} -2 \\
-2 &= -2 \quad \text{A true statement}
\end{aligned}$$

Substituting $x = 11$ and $y = 7$ into $2x - 3y = 1$, we have

$$\begin{aligned}
2(11) - 3(7) &\stackrel{?}{=} 1 \\
22 - 21 &\stackrel{?}{=} 1 \\
1 &= 1 \quad \text{A true statement}
\end{aligned}$$

Our solution satisfies both equations; therefore, (11, 7) is a solution to our system.

EXAMPLE 3 Solve the system.

$$\begin{aligned}
2x - 3y &= 4 \\
4x + 5y &= 3
\end{aligned}$$

SOLUTION We can eliminate x by multiplying the top equation by -2 and adding it to the bottom equation.

$$\begin{array}{rcr}
2x - 3y = 4 & \xrightarrow{\text{Multiply by } -2} & -4x + 6y = -8 \\
4x + 5y = 3 & \xrightarrow[\text{No change}]{} & 4x + 5y = 3 \\ \hline
 & & 11y = -5 \\
 & & y = -\dfrac{5}{11}
\end{array}$$

The y-coordinate of our solution is $-\frac{5}{11}$. If we were to substitute this value of y back into either of our original equations, we would find the arithmetic necessary to solve for x cumbersome. For this reason, it is probably best to go back to the original system and solve it a second time—for x instead of y. Here is how we do that:

$$\begin{array}{rcr}
2x - 3y = 4 & \xrightarrow{\text{Multiply by 5}} & 10x - 15y = 20 \\
4x + 5y = 3 & \xrightarrow[\text{Multiply by 3}]{} & 12x + 15y = 9 \\ \hline
 & & 22x \qquad\quad = 29 \\
 & & x = \dfrac{29}{22}
\end{array}$$

The solution to our system is $\left(\dfrac{29}{22}, -\dfrac{5}{11}\right)$.

The main idea in solving a system of linear equations by the addition method is to use the multiplication property of equality on one or both of the original equations, if necessary, to make the coefficients of either variable opposites. The following box shows some steps to follow when solving a system of linear equations by the addition method.

Strategy Solving a System of Linear Equations by the Addition Method

Step 1: Decide which variable to eliminate. (In some cases, one variable will be easier to eliminate than the other. With some practice, you will notice which one it is.)

Step 2: Use the multiplication property of equality on each equation separately to make the coefficients of the variable that is to be eliminated opposites.

Step 3: Add the respective left and right sides of the system together.

Step 4: Solve for the remaining variable.

Step 5: Substitute the value of the variable from step 4 into an equation containing both variables and solve for the other variable. (Or repeat steps 2–4 to eliminate the other variable.)

Step 6: Check your solution in both equations, if necessary.

EXAMPLE 4 Solve the system.

$$\begin{aligned} 5x - 2y &= 5 \\ -10x + 4y &= 15 \end{aligned}$$

SOLUTION We can eliminate y by multiplying the first equation by 2 and adding the result to the second equation.

$$\begin{array}{rcl} 5x - 2y = 5 & \xrightarrow{\text{Multiply by 2}} & 10x - 4y = 10 \\ -10x + 4y = 15 & \xrightarrow[\text{No change}]{} & -10x + 4y = 15 \\ & & \hline 0 = 25 \end{array}$$

The result is the false statement $0 = 25$, which indicates there is no solution to the system. If we were to graph the two lines, we would find that they are parallel. In a case like this, we say the system is *inconsistent.* Whenever both variables have been eliminated and the resulting statement is false, the solution set for the system will be the empty set, $\varnothing$.

EXAMPLE 5 Solve the system.

$$\begin{aligned} 4x + 3y &= 2 \\ 8x + 6y &= 4 \end{aligned}$$

SOLUTION Multiplying the top equation by -2 and adding, we can eliminate the variable x.

$$\begin{array}{rcl} 4x + 3y = 2 & \xrightarrow{\text{Multiply by } -2} & -8x - 6y = -4 \\ 8x + 6y = 4 & \xrightarrow[\text{No change}]{} & 8x + 6y = 4 \\ & & \hline 0 = 0 \end{array}$$

Both variables have been eliminated and the resulting statement $0 = 0$ is true. In this case, the lines coincide and the equations are said to be *dependent.* The solution set consists of all ordered pairs that satisfy either equation. We can write the solution set as $\{(x, y) \mid 4x + 3y = 2\}$ or $\{(x, y) \mid 8x + 6y = 4\}$.

Special Cases

The previous two examples illustrate the two special cases in which the graphs of the equations in the system either coincide or are parallel. In both cases, the left-hand sides of the equations were multiples of each other. In the case of the dependent equations, the right-hand sides were also multiples. We can generalize these observations for the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

Inconsistent System

What Happens	*Geometric Interpretation*	*Algebraic Interpretation*
Both variables are eliminated, and the resulting statement is false.	The lines are parallel, and there is no solution to the system.	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Dependent Equations

What Happens	*Geometric Interpretation*	*Algebraic Interpretation*
Both variables are eliminated, and the resulting statement is true.	The lines coincide, and there are an infinite number of solutions to the system.	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

EXAMPLE 6 Solve the system.

$$\begin{aligned} \frac{1}{2}x - \frac{1}{3}y &= 2 \\ \frac{1}{4}x + \frac{2}{3}y &= 6 \end{aligned}$$

SOLUTION Although we could solve this system without clearing the equations of fractions, there is probably less chance for error if we have only integer coefficients to work with. So let's begin by multiplying both sides of the top equation by 6 and both sides of the bottom equation by 12 to clear each equation of fractions.

$$\frac{1}{2}x - \frac{1}{3}y = 2 \xrightarrow{\text{Multiply by 6}} 3x - 2y = 12$$

$$\frac{1}{4}x + \frac{2}{3}y = 6 \xrightarrow[\text{Multiply by 12}]{} 3x + 8y = 72$$

Now we can eliminate x by multiplying the top equation by -1 and leaving the bottom equation unchanged.

$$3x - 2y = 12 \xrightarrow{\text{Multiply by } -1} -3x + 2y = -12$$

$$3x + 8y = 72 \xrightarrow[\text{No change}]{} \underline{3x + 8y = 72}$$

$$10y = 60$$

$$y = 6$$

We can substitute $y = 6$ into any equation that contains both x and y. Let's use $3x - 2y = 12$.

$$\begin{aligned} 3x - 2(6) &= 12 \\ 3x - 12 &= 12 \end{aligned}$$

$$3x = 24$$
$$x = 8$$

The solution to the system is (8, 6).

C The Substitution Method

EXAMPLE 7 Solve the system.

$$2x - 3y = -6$$
$$y = 3x - 5$$

SOLUTION The second equation tells us y is $3x - 5$. Substituting the expression $3x - 5$ for y in the first equation, we have

$$2x - 3(3x - 5) = -6$$

The result of the substitution is the elimination of the variable y. Solving the resulting linear equation in x as usual, we have

$$2x - 9x + 15 = -6$$
$$-7x + 15 = -6$$
$$-7x = -21$$
$$x = 3$$

Putting $x = 3$ into the second equation in the original system, we have

$$y = 3(3) - 5$$
$$= 9 - 5$$
$$= 4$$

The solution to the system is (3, 4).

CHECKING SOLUTIONS Checking (3, 4) in each equation looks like this:

Substituting $x = 3$ and $y = 4$ into $2x - 3y = -6$, we have

$$2(3) - 3(4) \stackrel{?}{=} -6$$
$$6 - 12 \stackrel{?}{=} -6$$
$$-6 = -6 \quad \text{A true statement}$$

Substituting $x = 3$ and $y = 4$ into $y = 3x - 5$, we have

$$4 \stackrel{?}{=} 3(3) - 5$$
$$4 \stackrel{?}{=} 9 - 5$$
$$4 = 4 \quad \text{A true statement}$$

Our solution satisfies both equations; therefore, (3, 4) is a solution to our system.

Strategy Solving a System of Equations by the Substitution Method

Step 1: Solve either one of the equations for x or y. (This step is not necessary if one of the equations is already in the correct form, as in Example 7.)

Step 2: Substitute the expression for the variable obtained in step 1 into the other equation and solve it.

Step 3: Substitute the solution for step 2 into any equation in the system that contains both variables and solve it.

Step 4: Check your results, if necessary.

EXAMPLE 8 Solve by substitution.

$$2x + 3y = 5$$
$$x - 2y = 6$$

NOTE
Both the substitution method and the addition method can be used to solve any system of linear equations in two variables. Systems like the one in Example 7, however, are easier to solve using the substitution method because one of the variables is already written in terms of the other. A system like the one in Example 2 is easier to solve using the addition method because solving for one of the variables would lead to an expression involving fractions. The system in Example 8 could be solved easily by either method because solving the second equation for x is a one-step process.

SOLUTION To use the substitution method, we must solve one of the two equations for x or y. We can solve for x in the second equation by adding $2y$ to both sides.

$$x - 2y = 6$$
$$x = 2y + 6 \quad \text{Add } 2y \text{ to both sides}$$

Substituting the expression $2y + 6$ for x in the first equation of our system, we have

$$2(2y + 6) + 3y = 5$$
$$4y + 12 + 3y = 5$$
$$7y + 12 = 5$$
$$7y = -7$$
$$y = -1$$

Using $y = -1$ in either equation in the original system, we get $x = 4$. The solution is $(4, -1)$.

USING TECHNOLOGY

Graphing Calculators: Solving a System That Intersects at Exactly One Point

A graphing calculator can be used to solve a system of equations in two variables if the equations intersect at exactly one point. To solve the system shown in Example 3, we first solve each equation for y. Here is the result:

$$2x - 3y = 4 \quad \text{becomes} \quad y = \frac{4 - 2x}{-3}$$

$$4x + 5y = 3 \quad \text{becomes} \quad y = \frac{3 - 4x}{5}$$

Graphing these two functions on the calculator gives a diagram similar to the one in Figure 3.

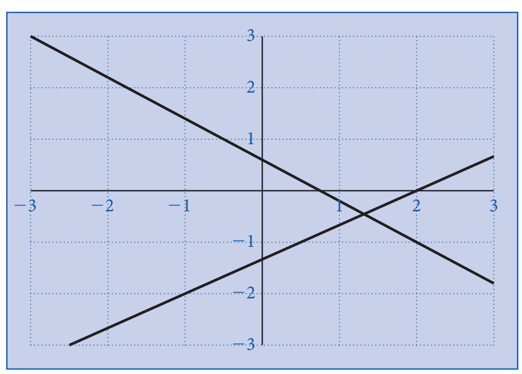


FIGURE 3

Using the Trace and Zoom features, we find that the two lines intersect at $x = 1.32$ and $y = -0.45$, which are the decimal equivalents (accurate to the nearest hundredth) of the fractions found in Example 3.

Problem Set 4.1

Moving Toward Success

"When one has a great deal to put into it, a day has a hundred pockets."

—Friedrich Nietzsche, 1844–1900, German philosopher

1. How would keeping a calendar of your daily schedule help you lay out a successful study plan?
2. Why should you keep a daily to do list and write quiz and exam dates on your calendar?

A Solve each system by graphing both equations on the same set of axes and then reading the solution from the graph.

1. $3x - 2y = 6$
$x - y = 1$

2. $5x - 2y = 10$
$x - y = -1$

3. $y = \frac{3}{5}x - 3$
$2x - y = -4$

4. $y = \frac{1}{2}x - 2$
$2x - y = -1$

5. $y = \frac{1}{2}x$
$y = -\frac{3}{4}x + 5$

6. $y = \frac{2}{3}x$
$y = -\frac{1}{3}x + 6$

7. $3x + 3y = -2$
$y = -x + 4$

8. $2x - 2y = 6$
$y = x - 3$

9. $2x - y = 5$
$y = 2x - 5$

10. $x + 2y = 5$
$y = -\frac{1}{2}x + 3$

B Solve each of the following systems by the addition method. [Examples 1–6]

11. $x + y = 5$
$3x - y = 3$

12. $x - y = 4$
$-x + 2y = -3$

13. $3x + y = 4$
$4x + y = 5$

14. $6x - 2y = -10$
$6x + 3y = -15$

15. $3x - 2y = 6$
$6x - 4y = 12$

16. $4x + 5y = -3$
$-8x - 10y = 3$

17. $x + 2y = 0$
$2x - 6y = 5$

18. $x + 3y = 3$
$2x - 9y = 1$

19. $2x - 5y = 16$
$4x - 3y = 11$

20. $5x - 3y = -11$
$7x + 6y = -12$

21. $6x + 3y = -1$
$9x + 5y = 1$

22. $5x + 4y = -1$
$7x + 6y = -2$

23. $4x + 3y = 14$
$9x - 2y = 14$

24. $7x - 6y = 13$
$6x - 5y = 11$

25. $2x - 5y = 3$
$-4x + 10y = 3$

26. $3x - 2y = 1$
$-6x + 4y = -2$

27. $\frac{1}{4}x - \frac{1}{6}y = -2$
$-\frac{1}{6}x + \frac{1}{5}y = 4$

28. $-\frac{1}{3}x + \frac{1}{4}y = 0$
$\frac{1}{5}x - \frac{1}{10}y = 1$

29. $\frac{1}{2}x + \frac{1}{3}y = 13$
$\frac{2}{5}x + \frac{1}{4}y = 10$

30. $\frac{1}{2}x + \frac{1}{3}y = \frac{2}{3}$
$\frac{2}{3}x + \frac{2}{5}y = \frac{14}{15}$

C Solve each of the following systems by the substitution method. [Examples 7–8]

31. $7x - y = 24$
$x = 2y + 9$

32. $3x - y = -8$
$y = 6x + 3$

33. $6x - y = 10$
$y = -\frac{3}{4}x - 1$

34. $2x - y = 6$
$y = -\frac{4}{3}x + 1$

35. $3y + 4z = 23$
$6y + z = 32$

36. $2x - y = 650$
$3.5x - y = 1{,}400$

37. $y = 3x - 2$
$y = 4x - 4$

38. $y = 5x - 2$
$y = -2x + 5$

39. $2x - y = 5$
$4x - 2y = 10$

40. $-10x + 8y = -6$
$y = \frac{5}{4}x$

41. $\frac{1}{3}x - \frac{1}{2}y = 0$
$x = \frac{3}{2}y$

42. $\frac{2}{5}x - \frac{2}{3}y = 0$
$y = \frac{3}{5}x$

You may want to read Example 3 again before solving the systems that follow.

43. $4x - 7y = 3$
$5x + 2y = -3$

44. $3x - 4y = 7$
$6x - 3y = 5$

45. $9x - 8y = 4$
$2x + 3y = 6$

46. $4x - 7y = 10$
$-3x + 2y = -9$

47. $3x - 5y = 2$
$7x + 2y = 1$

48. $4x - 3y = -1$
$5x + 8y = 2$

Solve each of the following systems by using either the addition or substitution method. Choose the method that is most appropriate for the problem.

49. $x - 3y = 7$
$2x + y = -6$

50. $2x - y = 9$
$x + 2y = -11$

51. $y = \frac{1}{2}x + \frac{1}{3}$
$y = -\frac{1}{3}x + 2$

52. $y = \frac{3}{4}x - \frac{4}{5}$
$y = \frac{1}{2}x - \frac{1}{2}$

53. $3x - 4y = 12$
$x = \frac{2}{3}y - 4$

54. $-5x + 3y = -15$
$x = \frac{4}{5}y - 2$

55. $4x - 3y = -7$
$-8x + 6y = -11$

56. $3x - 4y = 8$
$y = \frac{3}{4}x - 2$

57. $3y + z = 17$
$5y + 20z = 65$

58. $x + y = 850$
$1.5x + y = 1{,}100$

59. $\frac{3}{4}x - \frac{1}{3}y = 1$
$y = \frac{1}{4}x$

60. $-\frac{2}{3}x - \frac{1}{2}y = -1$
$y = -\frac{1}{3}x$

61. $\frac{1}{4}x - \frac{1}{2}y = \frac{1}{3}$
$\frac{1}{3}x - \frac{1}{4}y = \frac{2}{3}$

62. $\frac{1}{5}x - \frac{1}{10}y = -\frac{1}{3}$
$\frac{2}{3}x - \frac{1}{2}y = -\frac{1}{6}$

63. $\frac{3}{4}x + \frac{1}{3}y = 2$
$\frac{1}{2}x - y = 0$

64. $\frac{5}{6}x - \frac{1}{2}y = 4$
$x - \frac{1}{3}y = 2$

65. $\frac{3}{5}x - \frac{1}{2}y = \frac{7}{10}$
$\frac{1}{6}x - \frac{2}{3}y = -\frac{1}{2}$

66. $\frac{5}{3}x - \frac{1}{2}y = -\frac{4}{3}$
$\frac{1}{2}x + \frac{3}{4}y = \frac{1}{4}$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.

67. Work each problem according to the instructions given.

a. Simplify: $(3x - 4y) - 3(x - y)$
b. Find y when x is 0 in $3x - 4y = 8$.
c. Find the y-intercept: $3x - 4y = 8$
d. Graph: $3x - 4y = 8$
e. Find the point where the graphs of $3x - 4y = 8$ and $x - y = 2$ cross.

68. Work each problem according to the instructions given.

a. Solve: $4x - 5 = 20$
b. Solve for y: $4x - 5y = 20$
c. Solve for x: $x - y = 5$
d. Solve: $4x - 5y = 20$
$x - y = 5$

69. Multiply both sides of the second equation in the following system by 100, and then solve as usual.

$$x + y = 10{,}000$$
$$0.06x + 0.05y = 560$$

70. Multiply both sides of the second equation in the following system by 10, and then solve as usual.

$$x + y = 12$$
$$0.20x + 0.50y = 0.30(12)$$

71. What value of c will make the following system a dependent system (one in which the lines coincide)?

$$\begin{aligned} 6x - 9y &= 3 \\ 4x - 6y &= c \end{aligned}$$

72. What value of c will make the following system a dependent system?

$$\begin{aligned} 5x - 7y &= c \\ -15x + 21y &= 9 \end{aligned}$$

73. Where do the graphs of the lines $x + y = 4$ and $x - 2y = 4$ intersect?

74. Where do the graphs of the line $x = -1$ and $x - 2y = 4$ intersect?

Maintaining Your Skills

75. Find the slope of the line that contains $(-4, -1)$ and $(-2, 5)$.

76. A line has a slope of $\frac{2}{3}$. Find the slope of any line
   a. Parallel to it.
   b. Perpendicular to it.

77. Give the slope and y-intercept of the line $2x - 3y = 6$.

78. Give the equation of the line with slope -3 and y-intercept 5.

79. Find the equation of the line with slope $\frac{2}{3}$ that contains the point $(-6, 2)$.

80. Find the equation of the line through $(1, 3)$ and $(-1, -5)$.

81. Find the equation of the line with x-intercept 3 and y-intercept -2.

82. Find the equation of the line through $(-1, 4)$ whose graph is perpendicular to the graph of $y = 2x + 3$.

83. Find the equation of the line through $(-2, 3)$ whose graph is parallel to the graph of $4x - 3y = 6$.

84. Find the equation of the line through $(-3, 3)$ whose graph is perpendicular to the graph of the line through $(1,1)$ and $(-2, -8)$.

Getting Ready for the Next Section

Simplify.

85. $2 - 2(6)$

86. $2(1) - 2 + 3$

87. $(x + 3y) - 1(x - 2z)$

88. $(x + y + z) + (2x - y + z)$

Solve.

89. $-9y = -9$

90. $30x = 38$

91. $3(1) + 2z = 9$

92. $4\left(\frac{19}{15}\right) - 2y = 4$

Apply the distributive property, then simplify if possible.

93. $2(5x - z)$

94. $-1(x - 2z)$

95. $3(3x + y - 2z)$

96. $2(2x - y + z)$

Extending the Concepts

97. Find a and b so that the line $ax + by = 7$ passes through the points $(1, -2)$ and $(3, 1)$.

98. Find a and b so that the line $ax + by = 2$ passes through the points $(2, 2)$ and $(6, 7)$.

99. Find m and b so that the line $y = mx + b$ passes through the points $(-6, 0)$ and $(3, 6)$.

100. Find m and b so that the line $y = mx + b$ passes through the points $(4, 1)$ and $(-2, -8)$.

4.2 Systems of Linear Equations in Three Variables

OBJECTIVES

A Solve systems of linear equations in three variables.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. What is an ordered triple of numbers?
2. Explain what it means for (1, 2, 3) to be a solution to a system of linear equations in three variables.
3. Explain in a general way the procedure you would use to solve a system of three linear equations in three variables.
4. How would you know when a system of linear equations in three variables has no single ordered triple for a solution?

In the previous section we were working with systems of equations in two variables. However, imagine you are in charge of running a concession stand selling sodas, hot dogs, and chips for a fundraiser. The price for each soda is \$.50, each hot dog is \$2.00, and each bag of chips is \$1.00. You have not kept track of your inventory, but you know you have sold a total of 85 items for a total of \$100.00. You also know that you sold twce as many sodas as hot dogs. How many sodas, hot dogs, and chips did you sell? We will now begin to solve systems of equations with three unknowns like this one.

A Solve Systems in Three Variables

A solution to an equation in three variables such as

$$2x + y - 3z = 6$$

is an ordered triple of numbers (x, y, z). For example, the ordered triples $(0, 0, -2)$, $(2, 2, 0)$, and $(0, 9, 1)$ are solutions to the equation $2x + y - 3z = 6$ since they produce a true statement when their coordinates are substituted for x, y, and z in the equation.

Definition
The **solution set** for a system of three linear equations in three variables is the set of ordered triples that satisfy all three equations.

EXAMPLE 1 Solve the system.

$$
\begin{aligned}
x + y + z &= 6 \qquad (1)\\
2x - y + z &= 3 \qquad (2)\\
x + 2y - 3z &= -4 \qquad (3)
\end{aligned}
$$

SOLUTION We want to find the ordered triple (x, y, z) that satisfies all three equations. We have numbered the equations so it will be easier to keep track of where they are and what we are doing.

There are many ways to proceed. The main idea is to take two different pairs of equations and eliminate the same variable from each pair. We begin by adding equations (1) and (2) to eliminate the y-variable. The resulting equation is numbered (4):

$$
\begin{aligned}
x + y + z &= 6 \qquad (1)\\
2x - y + z &= 3 \qquad (2)\\
\hline
3x + 2z &= 9 \qquad (4)
\end{aligned}
$$

Adding twice equation (2) to equation (3) will also eliminate the variable y. The resulting equation is numbered (5):

$$
\begin{aligned}
4x - 2y + 2z &= 6 \qquad \text{Twice (2)}\\
x + 2y - 3z &= -4 \qquad (3)\\
\hline
5x - z &= 2 \qquad (5)
\end{aligned}
$$

Equations (4) and (5) form a linear system in two variables. By multiplying equation (5) by 2 and adding the result to equation (4), we succeed in eliminating the variable z from the new pair of equations:

$$
\begin{aligned}
3x + 2z &= 9 \qquad (4)\\
10x - 2z &= 4 \qquad \text{Twice (5)}\\
\hline
13x &= 13\\
x &= 1
\end{aligned}
$$

Substituting $x = 1$ into equation (4), we have

$$
\begin{aligned}
3(1) + 2z &= 9\\
2z &= 6\\
z &= 3
\end{aligned}
$$

Using $x = 1$ and $z = 3$ in equation (1) gives us

$$
\begin{aligned}
1 + y + 3 &= 6\\
y + 4 &= 6\\
y &= 2
\end{aligned}
$$

The solution is the ordered triple $(1, 2, 3)$.

NOTE
Solving a system in three variables can be overwhelming at first. Don't get discouraged. Remember to read the solution's explanation slowly, and you may need to read it more than once to fully grasp each step.

EXAMPLE 2 Solve the system.

$$
\begin{aligned}
2x + y - z &= 3 \qquad (1)\\
3x + 4y + z &= 6 \qquad (2)\\
2x - 3y + z &= 1 \qquad (3)
\end{aligned}
$$

SOLUTION It is easiest to eliminate z from the equations using the addition method. The equation produced by adding (1) and (2) is

$$5x + 5y = 9 \qquad (4)$$

The equation that results from adding (1) and (3) is

$$4x - 2y = 4 \qquad (5)$$

Equations (4) and (5) form a linear system in two variables. We can eliminate the variable y from this system as follows:

$$\begin{array}{rcl} 5x + 5y = 9 & \xrightarrow{\text{Multiply by 2}} & 10x + 10y = 18 \\ 4x - 2y = 4 & \xrightarrow[\text{Multiply by 5}]{} & \underline{20x - 10y = 20} \\ & & 30x \qquad = 38 \\ & & x \qquad = \frac{38}{30} \\ & & \qquad\quad = \frac{19}{15} \end{array}$$

Substituting $x = \frac{19}{15}$ into equation (5) or equation (4) and solving for y gives

$$y = \frac{8}{15}$$

Using $x = \frac{19}{15}$ and $y = \frac{8}{15}$ in equation (1), (2), or (3) and solving for z results in

$$z = \frac{1}{15}$$

The ordered triple that satisfies all three equations is $\left(\frac{19}{15}, \frac{8}{15}, \frac{1}{15}\right)$.

EXAMPLE 3 Solve the system.

$$\begin{aligned} 2x + 3y - z &= 5 \qquad (1) \\ 4x + 6y - 2z &= 10 \qquad (2) \\ x - 4y + 3z &= 5 \qquad (3) \end{aligned}$$

SOLUTION Multiplying equation (1) by -2 and adding the result to equation (2) looks like this:

$$\begin{aligned} -4x - 6y + 2z &= -10 \qquad -2 \text{ times } (1) \\ 4x + 6y - 2z &= 10 \qquad (2) \\ 0 &= 0 \end{aligned}$$

All three variables have been eliminated, and we are left with a true statement. As was the case in the previous section, this implies that the two equations are dependent. With a system of three equations in three variables, however, a system such as this one can have no solution or an infinite number of solutions. In either case, we have no unique solutions, meaning there is no single ordered triple that is the only solution to the system.

EXAMPLE 4 Solve the system.

$$\begin{aligned} x - 5y + 4z &= 8 \qquad (1) \\ 3x + y - 2z &= 7 \qquad (2) \\ -9x - 3y + 6z &= 5 \qquad (3) \end{aligned}$$

SOLUTION Multiplying equation (2) by 3 and adding the result to equation (3) produces

$$\begin{array}{rl} 9x + 3y - 6z = 21 & \text{3 times (2)} \\ \underline{-9x - 3y + 6z = 5} & \text{(3)} \\ 0 = 26 & \end{array}$$

In this case, all three variables have been eliminated, and we are left with a false statement. The system is inconsistent: there are no ordered triples that satisfy both equations. The solution set for the system is the empty set, $\varnothing$. If equations (2) and (3) have no ordered triples in common, then certainly (1), (2), and (3) do not either.

EXAMPLE 5 Solve the system.

$$\begin{array}{rl} x + 3y = 5 & (1) \\ 6y + z = 12 & (2) \\ x - 2z = -10 & (3) \end{array}$$

SOLUTION It may be helpful to rewrite the system as

$$\begin{array}{rl} x + 3y = 5 & (1) \\ 6y + z = 12 & (2) \\ x - 2z = -10 & (3) \end{array}$$

Equation (2) does not contain the variable x. If we multiply equation (3) by -1 and add the result to equation (1), we will be left with another equation that does not contain the variable x.

$$\begin{array}{rl} x + 3y = 5 & (1) \\ \underline{-x + 2z = 10} & -1 \text{ times (3)} \\ 3y + 2z = 15 & (4) \end{array}$$

Equations (2) and (4) form a linear system in two variables. Multiplying equation (2) by -2 and adding the result to equation (4) eliminates the variable z.

$$\begin{array}{rcr} 6y + z = 12 & \xrightarrow{\text{Multiply by } -2} & -12y - 2z = -24 \\ 3y + 2z = 15 & \xrightarrow[\text{No change}]{} & \underline{3y + 2z = 15} \\ & & -9y = -9 \\ & & y = 1 \end{array}$$

Using $y = 1$ in equation (4) and solving for z, we have

$$z = 6$$

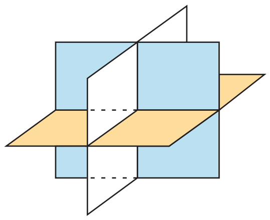
Substituting $y = 1$ into equation (1) gives

$$x = 2$$

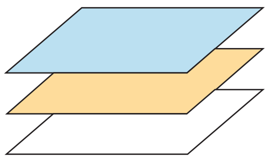
The ordered triple that satisfies all three equations is (2, 1, 6).

The Geometry Behind Equations in Three Variables

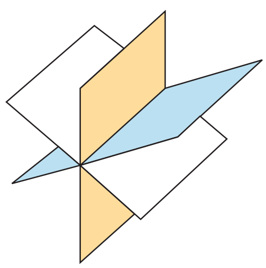
We can graph an ordered triple on a coordinate system with three axes. The graph will be a point in space. The coordinate system is drawn in perspective; you have to imagine that the x-axis comes out of the paper and is perpendicular to both the y-axis and the z-axis. To graph the point (3, 4, 5), we move 3 units in the x-direction, 4 units in the y-direction, and then 5 units in the z-direction, as shown in Figure 1.



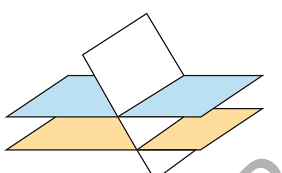
Case 1 The three planes have exactly one point in common. In this case we get one solution to our system, as in Examples 1, 2, and 5.



Case 2 The three planes have no points in common because they are all parallel to each other. The system they represent is an inconsistent system.



Case 3 The three planes intersect in a line. Any point on the line is a solution to the system.



Case 4 In this case the three planes have no points in common. There is no solution to the system; it is an inconsistent system.

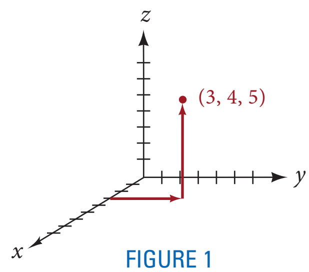


FIGURE 1

Although in actual practice, it is sometimes difficult to graph equations in three variables, if we have to graph a linear equation in three variables, we would find that the graph was a plane in space. A system of three equations in three variables is represented by three planes in space.

There are a number of possible ways in which these three planes can intersect, some of which are shown in the margin on this page. There are still other possibilities that are not among those shown in the margin.

In Example 3, we found that equations 1 and 2 were dependent equations. They represent the same plane; that is, they have all their points in common. But the system of equations that they came from has either no solution or an infinite number of solutions. It all depends on the third plane. If the third plane coincides with the first two, then the solution to the system is a plane. If the third plane is distinct from but parallel to the first two, then there is no solution to the system. And, finally, if the third plane intersects the first two, but does not coincide with them, then the solution to the system is that line of intersection.

In Example 4, we found that trying to eliminate a variable from the second and third equations resulted in a false statement. This means that the two planes represented by these equations are parallel. It makes no difference where the third plane is; there is no solution to the system in Example 4. (If we were to graph the three planes from Example 4, we would obtain a diagram similar to Case 4 in the margin.)

If, in the process of solving a system of linear equations in three variables, we eliminate all the variables from a pair of equations and are left with a false statement, we will say the system is inconsistent. If we eliminate all the variables and are left with a true statement, then we will say the system has no unique solution.

Problem Set 4.2

Moving Toward Success

"Even a mistake may turn out to be the one thing necessary to a worthwhile achievement."

—Henry Ford, 1863–1947, American industrialist and founder of Ford Motor Company

1. Why is making mistakes important to the process of learning mathematics?
2. What will you do if you notice you are repeatedly making mistakes on certain types of problems?

A Solve the following systems. [Examples 1–5]

1. $\begin{aligned} x + y + z &= 4 \\ x - y + 2z &= 1 \\ x - y - 3z &= -4 \end{aligned}$

2. $\begin{aligned} x - y - 2z &= -1 \\ x + y + z &= 6 \\ x + y - z &= 4 \end{aligned}$

3. $\begin{aligned} x + y + z &= 6 \\ x - y + 2z &= 7 \\ 2x - y - 4z &= -9 \end{aligned}$

4. $\begin{aligned} x + y + z &= 0 \\ x + y - z &= 6 \\ x - y + 2z &= -7 \end{aligned}$

5. $\begin{aligned} x + 2y + z &= 3 \\ 2x - y + 2z &= 6 \\ 3x + y - z &= 5 \end{aligned}$

6. $\begin{aligned} 2x + y - 3z &= -14 \\ x - 3y + 4z &= 22 \\ 3x + 2y + z &= 0 \end{aligned}$

7. $\begin{aligned} 2x + 3y - 2z &= 4 \\ x + 3y - 3z &= 4 \\ 3x - 6y + z &= -3 \end{aligned}$

8. $\begin{aligned} 4x + y - 2z &= 0 \\ 2x - 3y + 3z &= 9 \\ -6x - 2y + z &= 0 \end{aligned}$

9. $\begin{aligned} -x + 4y - 3z &= 2 \\ 2x - 8y + 6z &= 1 \\ 3x - y + z &= 0 \end{aligned}$

10. $\begin{aligned} 4x + 6y - 8z &= 1 \\ -6x - 9y + 12z &= 0 \\ x - 2y - 2z &= 3 \end{aligned}$

11. $\begin{aligned} \frac{1}{2}x - y + z &= 0 \\ 2x + \frac{1}{3}y + z &= 2 \\ x + y + z &= -4 \end{aligned}$

12. $\begin{aligned} \frac{1}{3}x + \frac{1}{2}y + z &= -1 \\ x - y + \frac{1}{5}z &= 1 \\ x + y + z &= 5 \end{aligned}$

13. $\begin{aligned} 2x - y - 3z &= 1 \\ x + 2y + 4z &= 3 \\ 4x - 2y - 6z &= 2 \end{aligned}$

14. $\begin{aligned} 3x + 2y + z &= 3 \\ x - 3y + z &= 4 \\ -6x - 4y - 2z &= 1 \end{aligned}$

15. $\begin{aligned} 2x - y + 3z &= 4 \\ x + 2y - z &= -3 \\ 4x + 3y + 2z &= -5 \end{aligned}$

16. $\begin{aligned} 6x - 2y + z &= 5 \\ 3x + y + 3z &= 7 \\ x + 4y - z &= 4 \end{aligned}$

17. $\begin{aligned} x + y &= 9 \\ y + z &= 7 \\ x - z &= 2 \end{aligned}$

18. $\begin{aligned} x - y &= -3 \\ x + z &= 2 \\ y - z &= 7 \end{aligned}$

19. $\begin{aligned} 2x + y &= 2 \\ y + z &= 3 \\ 4x - z &= 0 \end{aligned}$

20. $\begin{aligned} 2x + y &= 6 \\ 3y - 2z &= -8 \\ x + z &= 5 \end{aligned}$

21. $\begin{aligned} 2x - 3y &= 0 \\ 6y - 4z &= 1 \\ x + 2z &= 1 \end{aligned}$

22. $\begin{aligned} 3x + 2y &= 3 \\ y + 2z &= 2 \\ 6x - 4z &= 1 \end{aligned}$

23. $\begin{aligned} x + y - z &= 2 \\ 2x + y + 3z &= 4 \\ x - 2y + 2z &= 6 \end{aligned}$

24. $\begin{aligned} x + 2y - 2z &= 4 \\ 3x + 4y - z &= -2 \\ 2x + 3y - 3z &= -5 \end{aligned}$

25. $\begin{aligned} 2x + 3y &= -\frac{1}{2} \\ 4x + 8z &= 2 \\ 3y + 2z &= -\frac{3}{4} \end{aligned}$

26. $\begin{aligned} 3x - 5y &= 2 \\ 4x + 6z &= \frac{1}{3} \\ 5y - 7z &= \frac{1}{6} \end{aligned}$

27. $\begin{aligned} \frac{1}{3}x + \frac{1}{2}y - \frac{1}{6}z &= 4 \\ \frac{1}{4}x - \frac{3}{4}y + \frac{1}{2}z &= \frac{3}{2} \\ \frac{1}{2}x - \frac{2}{3}y - \frac{1}{4}z &= -\frac{16}{3} \end{aligned}$

28. $\begin{aligned} -\frac{1}{4}x + \frac{3}{8}y + \frac{1}{2}z &= -1 \\ \frac{2}{3}x - \frac{1}{6}y - \frac{1}{2}z &= 2 \\ \frac{3}{4}x - \frac{1}{2}y - \frac{1}{8}z &= 1 \end{aligned}$

29. $\begin{aligned} x - \frac{1}{2}y - \frac{1}{3}z &= -\frac{4}{3} \\ \frac{1}{3}x + y - \frac{1}{2}z &= 5 \\ -\frac{1}{4}x + \frac{2}{3}y - z &= -\frac{3}{4} \end{aligned}$

30. $\begin{aligned} x + \frac{1}{3}y - \frac{1}{2}z &= -\frac{3}{2} \\ \frac{1}{2}x - y + \frac{1}{3}z &= 8 \\ \frac{1}{3}x - \frac{1}{4}y - z &= -\frac{5}{6} \end{aligned}$

31. $\begin{aligned} \frac{1}{2}x + \frac{2}{3}y &= \frac{5}{2} \\ \frac{1}{5}x - \frac{1}{2}z &= -\frac{3}{10} \\ \frac{1}{3}y - \frac{1}{4}z &= \frac{3}{4} \end{aligned}$

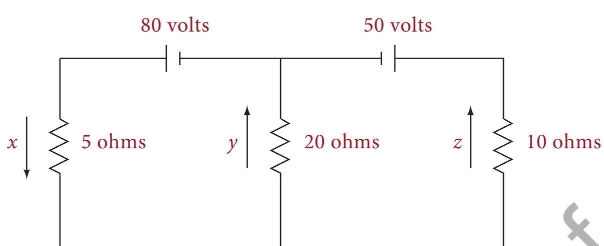
32. $\begin{aligned} \frac{1}{2}x - \frac{1}{3}y &= \frac{1}{6} \\ \frac{1}{3}y - \frac{1}{3}z &= 1 \\ \frac{1}{5}x - \frac{1}{2}z &= -\frac{4}{5} \end{aligned}$

33. $\begin{aligned} \frac{1}{2}x - \frac{1}{4}y + \frac{1}{2}z &= -2 \\ \frac{1}{4}x - \frac{1}{12}y - \frac{1}{3}z &= \frac{1}{4} \\ \frac{1}{6}x + \frac{1}{3}y - \frac{1}{2}z &= \frac{3}{2} \end{aligned}$

34. $\begin{aligned} \frac{1}{2}x + \frac{1}{2}y + z &= \frac{1}{2} \\ \frac{1}{2}x - \frac{1}{4}y - \frac{1}{4}z &= 0 \\ \frac{1}{4}x + \frac{1}{12}y + \frac{1}{6}z &= \frac{1}{6} \end{aligned}$

Applying the Concepts

35. Electric Current In the following diagram of an electrical circuit, x, y, and z represent the amount of current (in amperes) flowing across the 5-ohm, 20-ohm, and 10-ohm resistors, respectively. (In circuit diagrams resistors are represented by and potential differences by .)

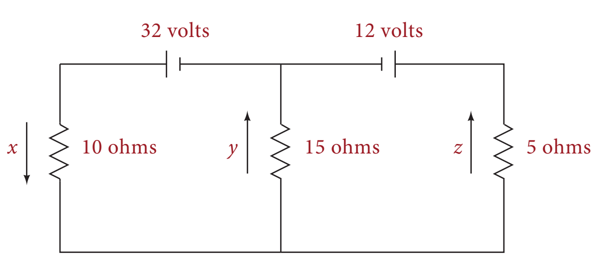


The system of equations used to find the three currents x, y, and z is

$$\begin{aligned} x - y - z &= 0 \\ 5x + 20y &= 80 \\ 20y - 10z &= 50 \end{aligned}$$

Solve the system for all variables.

36. Electric Current In the following diagram of an electrical circuit, x, y, and z represent the amount of current (in amperes) flowing across the 10-ohm, 15-ohm, and 5-ohm resistors, respectively.



The system of equations used to find the three currents x, y, and z is

$$\begin{aligned} x - y - z &= 0 \\ 10x + 15y &= 32 \\ 15y - 5z &= 12 \end{aligned}$$

Solve the system for all variables.

Getting Ready for the Next Section

Simplify.

37. $1(4) - 3(2)$

38. $3(7) - (-2)(5)$

39. $1(1) - 3(-2) + (-2)(-2)$

40. $-4(0)(-2) - (-1)(1)(1) - 1(2)(3)$

41. $-3(-1 - 1) + 4(-2 + 2) - 5[2 - (-2)]$

42. $12 + 4 - (-1) - 6$

Solve.

43. $-5x = 20$

44. $4x - 2x = 8$

Maintaining Your Skills

45. If y varies directly with the square of x, and y is 75 when x is 5, find y when x is 7.

46. Suppose y varies directly with the cube of x. If y is 16 when x is 2, find y when x is 3.

47. Suppose y varies inversely with x. If y is 10 when x is 25, find x when y is 5.

48. If y varies inversely with the cube of x, and y is 2 when x is 2, find y when x is 4.

49. Suppose z varies jointly with x and the square of y. If z is 40 when x is 5 and y is 2, find z when x is 2 and y is 5.

50. Suppose z varies jointly with x and the cube of y. If z is 48 when x is 3 and y is 2, find z when x is 4 and y is $\frac{1}{2}$.

Extending the Concepts

Solve each system for the solution (x, y, z, w).

51. $\begin{aligned} x + y + z + w &= 10 \\ x + 2y - z + w &= 6 \\ x - y - z + 2w &= 4 \\ x - 2y + z - 3w &= -12 \end{aligned}$

52. $\begin{aligned} x + y + z + w &= 16 \\ x - y + 2z - w &= 1 \\ x + 3y - z - w &= -2 \\ x - 3y - 2z + 2w &= -4 \end{aligned}$

4.3 Applications of Linear Systems

OBJECTIVES

A Solve application problems whose solutions are found through systems of linear equations.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. To apply the Blueprint for Problem Solving to the examples in this section, what is the first step?
2. When would you write a system of equations while working a problem in this section using the Blueprint for Problem Solving?
3. When working application problems involving boats moving in rivers, how does the current of the river affect the speed of the boat?
4. Write an application problem for which the solution depends on solving the system

$$x + y = 1{,}000$$
$$0.05x + 0.06y = 55$$

Image copyright ©Dmitry Kalinovsky, 2009. Used under license from Shutterstock.com

Suppose you want to be able to quantify your weekly driving habits. You put 25 gallons of gas in your car, then drive 425 miles in a week, using all the gas. You know your car gets 25 miles per gallon on the highway and 15 miles per gallon in the city. How many gallons were used in the highway and how many were used in the city? In this section, we will put our knowledge of systems of equations to use solving different kinds of applications.

Many times word problems involve more than one unknown quantity. If a problem is stated in terms of two unknowns and we represent each unknown quantity with a different variable, then we must write the relationships between the variables with two equations. The two equations written in terms of the two variables form a system of linear equations that we solve using the methods developed in this chapter. If we find a problem that relates three unknown quantities, then we need three equations to form a linear system we can solve.

Here is our Blueprint for Problem Solving, modified to fit the application problems that you will find in this section.

A Application Problems

Strategy Problem Solving Using a System of Equations

Step 1: ***Read*** the problem, and then mentally ***list*** the items that are known and the items that are unknown.

Step 2: ***Assign variables*** to each of the unknown items; that is, let x = one of the unknown items and y = the other unknown item (and z = the third unknown item, if there is a third one). Then ***translate*** the other ***information*** in the problem to expressions involving the two (or three) variables.

Step 3: ***Reread*** the problem, and then ***write a system of equations,*** using the items and variables listed in steps 1 and 2, that describes the situation.

Step 4: ***Solve the system*** found in step 3.

Step 5: ***Write your answers*** using complete sentences.

Step 6: ***Reread*** the problem, and ***check*** your solution with the original words in the problem.

EXAMPLE 1 One number is 2 more than 3 times another. Their sum is 26. Find the two numbers.

SOLUTION Applying the steps from our Blueprint, we have

Step 1: Read and list.
We know that we have two numbers, whose sum is 26. One of them is 2 more than 3 times the other. The unknown quantities are the two numbers.

Step 2: Assign variables and translate information.
Let x = one of the numbers and y = the other number.

Step 3: Write a system of equations.
The first sentence in the problem translates into $y = 3x + 2$. The second sentence gives us a second equation: $x + y = 26$. Together, these two equations give us the following system of equations:

$$\begin{aligned} x + y &= 26 \\ y &= 3x + 2 \end{aligned}$$

Step 4: Solve the system.
Substituting the expression for y from the second equation into the first and solving for x yields

$$\begin{aligned} x + (3x + 2) &= 26 \\ 4x + 2 &= 26 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

Using $x = 6$ in $y = 3x + 2$ gives the second number.

$$\begin{aligned} y &= 3(6) + 2 \\ y &= 20 \end{aligned}$$

Step 5: ***Write answers.***
The two numbers are 6 and 20.

Step 6: ***Reread and check.***
The sum of 6 and 20 is 26, and 20 is 2 more than 3 times 6.

EXAMPLE 2

Suppose 850 tickets were sold for a game for a total of $1,100. If adult tickets cost $1.50 and children's tickets cost $1.00, how many of each kind of ticket were sold?

SOLUTION

Step 1: ***Read and list.***
The total number of tickets sold is 850. The total income from tickets is $1,100. Adult tickets are $1.50 each. Children's tickets are $1.00 each. We don't know how many of each type of ticket have been sold.

Step 2: ***Assign variables and translate information.***
We let x = the number of adult tickets and y = the number of children's tickets.

Step 3: ***Write a system of equations.***
The total number of tickets sold is 850, giving us our first equation.

$$x + y = 850$$

Because each adult ticket costs $1.50 and each children's ticket costs $1.00 and the total amount of money paid for tickets was $1,100, a second equation is

$$1.50x + 1.00y = 1{,}100$$

The same information can also be obtained by summarizing the problem with a table. One such table follows. Notice that the two equations we obtained previously are given by the two rows of the table.

	Adult Tickets	Children's Tickets	Total
Number	x	y	850
Value	$1.50x$	$1.00y$	1,100

Whether we use a table to summarize the information in the problem or just talk our way through the problem, the system of equations that describes the situation is

$$x + y = 850$$
$$1.50x + 1.00y = 1{,}100$$

Step 4: ***Solve the system.***
If we multiply the second equation by 10 to clear it of decimals, we have the system

$$x + y = 850$$
$$15x + 10y = 11{,}000$$

Multiplying the first equation by -10 and adding the result to the second equation eliminates the variable y from the system.

$$\begin{array}{rcr} -10x - 10y & = & -8{,}500 \\ 15x + 10y & = & 11{,}000 \\ \hline 5x & = & 2{,}500 \\ x & = & 500 \end{array}$$

The number of adult tickets sold was 500. To find the number of children's tickets, we substitute $x = 500$ into $x + y = 850$ to get

$$500 + y = 850$$
$$y = 350$$

Step 5: ***Write answers.***
The number of children's tickets is 350, and the number of adult tickets is 500.

Step 6: ***Reread and check.***
The total number of tickets is $350 + 500 = 850$. The amount of money from selling the two types of tickets is

350 children's tickets at \$1.00 each is $350(1.00) = \$350$
500 adult tickets at \$1.50 each is $500(1.50) = \$750$
The total income from ticket sales is \$1,100

EXAMPLE 3 Suppose a person invests a total of \$10,000 in two accounts. One account earns 8% annually, and the other earns 9% annually. If the total interest earned from both accounts in a year is \$860, how much was invested in each account?

SOLUTION ***Step 1:*** ***Read and list.***
The total investment is \$10,000 split between two accounts. One account earns 8% annually, and the other earns 9% annually. The interest from both accounts is \$860 in 1 year. We don't know how much is in each account.

Step 2: ***Assign variables and translate information.***
We let x equal the amount invested at 9% and y be the amount invested at 8%.

Step 3: ***Write a system of equations.***
Because the total investment is \$10,000, one relationship between x and y can be written as

$$x + y = 10{,}000$$

The total interest earned from both accounts is \$860. The amount of interest earned on x dollars at 9% is $0.09x$, whereas the amount of interest earned on y dollars at 8% is $0.08y$. This relationship is represented by the equation

$$0.09x + 0.08y = 860$$

The two equations we have just written can also be found by first summarizing the information from the problem in a table. Again, the two rows of the table yield the two equations just written. Here is the table:

	Dollars at 9%	Dollars at 8%	Total
Number	x	y	10,000
Interest	$0.09x$	$0.08y$	860

The system of equations that describes this situation is given by

$$x + y = 10{,}000$$
$$0.09x + 0.08y = 860$$

Step 4: ***Solve the system.***
Multiplying the second equation by 100 will clear it of decimals. The system that results after doing so is

$$x + y = 10{,}000$$
$$9x + 8y = 86{,}000$$

We can eliminate y from this system by multiplying the first equation by -8 and adding the result to the second equation.

$$\begin{array}{r} -8x - 8y = -80{,}000 \\ 9x + 8y = \quad 86{,}000 \\ \hline x \qquad = \quad 6{,}000 \end{array}$$

The amount of money invested at 9% is \$6,000. Because the total investment was \$10,000, the amount invested at 8% must be \$4,000.

Step 5: ***Write answers.***
The amount invested at 8% is \$4,000, and the amount invested at 9% is \$6,000.

Step 6: ***Reread and check.***
The total investment is \$4,000 + \$6,000 = \$10,000. The amount of interest earned from the two accounts is

In 1 year, \$4,000 invested at 8% earns 0.08(4,000) = \$320
In 1 year, \$6,000 invested at 9% earns 0.09(6,000) = \$540
The total interest from the two accounts is \$860

EXAMPLE 4 How much 20% alcohol solution and 50% alcohol solution must be mixed to get 12 gallons of 30% alcohol solution?

SOLUTION To solve this problem, we must first understand that a 20% alcohol solution is 20% alcohol and 80% water.

Step 1: ***Read and list.***
We will mix two solutions to obtain 12 gallons of solution that is 30% alcohol. One of the solutions is 20% alcohol and the other is 50% alcohol. We don't know how much of each solution we need.

Step 2: ***Assign variables and translate information.***
Let x = the number of gallons of 20% alcohol solution needed and y = the number of gallons of 50% alcohol solution needed.

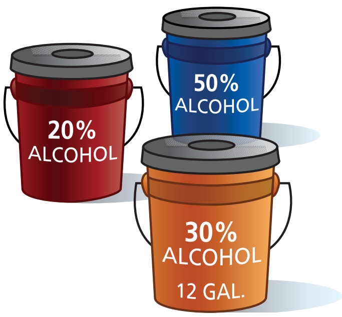
Step 3: ***Write a system of equations.***
Because we must end up with a total of 12 gallons of solution, one equation for the system is

$$x + y = 12$$

The amount of alcohol in the x gallons of 20% solution is $0.20x$, whereas the amount of alcohol in the y gallons of 50% solution is $0.50y$. Because the total amount of alcohol in the 20% and 50% solutions must add up to the amount of alcohol in the 12 gallons of 30% solution, the second equation in our system can be written as

$$0.20x + 0.50y = 0.30(12)$$

Again, let's make a table that summarizes the information we have to this point in the problem.



	20% Solution	50% Solution	Final Solution
Total number of gallons	x	y	12
Gallons of alcohol	$0.20x$	$0.50y$	$0.30(12)$

Our system of equations is

$$\begin{aligned} x + \quad y &= 12 \\ 0.20x + 0.50y &= 0.30(12) = 3.6 \end{aligned}$$

Step 4: Solve the system.

Multiplying the second equation by 10 gives us an equivalent system.

$$\begin{aligned} x + y &= 12 \\ 2x + 5y &= 36 \end{aligned}$$

Multiplying the top equation by -2 to eliminate the x-variable, we have

$$\begin{aligned} -2x - 2y &= -24 \\ 2x + 5y &= 36 \\ \hline 3y &= 12 \\ y &= 4 \end{aligned}$$

Substituting $y = 4$ into $x + y = 12$, we solve for x.

$$x + 4 = 12$$

$$x = 8$$

Step 5: Write answers.

It takes 8 gallons of 20% alcohol solution and 4 gallons of 50% alcohol solution to produce 12 gallons of 30% alcohol solution.

Step 6: Reread and check.

If we mix 8 gallons of 20% solution and 4 gallons of 50% solution, we end up with a total of 12 gallons of solution. To check the percentages, we look for the total amount of alcohol in the two initial solutions and in the final solution.

In the initial solutions

The amount of alcohol in 8 gallons of 20% solution is $0.20(8) = 1.6$ gallons
The amount of alcohol in 4 gallons of 50% solution is $0.50(4) = 2.0$ gallons

The total amount of alcohol in the initial solutions is 3.6 gallons

In the final solution

The amount of alcohol in 12 gallons of 30% solution is $0.30(12) = 3.6$ gallons.

EXAMPLE 5

It takes 2 hours for a boat to travel 28 miles downstream (with the current). The same boat can travel 18 miles upstream (against the current) in 3 hours. What is the speed of the boat in still water, and what is the speed of the current of the river?

SOLUTION

Step 1: Read and list.

A boat travels 18 miles upstream and 28 miles downstream. The trip upstream takes 3 hours. The trip downstream takes 2 hours. We don't know the speed of the boat or the speed of the current.

Step 2: Assign variables and translate information.

Let x = the speed of the boat in still water and let y = the speed of the current. The average speed (rate) of the boat upstream is $x - y$ because it is traveling against the current. The rate of the boat downstream is $x + y$ because the boat is traveling with the current.

Step 3: Write a system of equations.

Putting the information into a table, we have

	d (distance, miles)	r (rate, mph)	t (time, h)
Upstream	18	$x - y$	3
Downstream	28	$x + y$	2

The formula for the relationship between distance d, rate r, and time t is $d = rt$ (the rate equation). Because $d = r \cdot t$, the system we need to solve the problem is

$$18 = (x - y) \cdot 3$$
$$28 = (x + y) \cdot 2$$

which is equivalent to

$$6 = x - y$$
$$14 = x + y$$

Step 4: Solve the system.

Adding the two equations, we have

$$20 = 2x$$
$$x = 10$$

Substituting $x = 10$ into $14 = x + y$, we see that

$$y = 4$$

Step 5: Write answers.

The speed of the boat in still water is 10 miles per hour; the speed of the current is 4 miles per hour.

Step 6: ***Reread and check.***

The boat travels at $10 + 4 = 14$ miles per hour downstream, so in 2 hours it will travel $14 \cdot 2 = 28$ miles. The boat travels at $10 - 4 = 6$ miles per hour upstream, so in 3 hours it will travel $6 \cdot 3 = 18$ miles.

EXAMPLE 6 A coin collection consists of 14 coins with a total value of \$1.35. If the coins are nickels, dimes, and quarters, and the number of nickels is 3 less than twice the number of dimes, how many of each coin is there in the collection?

SOLUTION This problem will require three variables and three equations.

Step 1: ***Read and list.***

We have 14 coins with a total value of \$1.35. The coins are nickels, dimes, and quarters. The number of nickels is 3 less than twice the number of dimes. We do not know how many of each coin we have.

Step 2: ***Assign variables and translate information.***

Because we have three types of coins, we will have to use three variables. Let's let x = the number of nickels, y = the number of dimes, and z = the number of quarters.

Step 3: ***Write a system of equations.***

Because the total number of coins is 14, our first equation is

$$x + y + z = 14$$

Because the number of nickels is 3 less than twice the number of dimes, a second equation is

$$x = 2y - 3 \qquad \text{which is equivalent to} \qquad x - 2y = -3$$

Our last equation is obtained by considering the value of each coin and the total value of the collection. Let's write the equation in terms of cents, so we won't have to clear it of decimals later.

$$5x + 10y + 25z = 135$$

Here is our system, with the equations numbered for reference:

$$\begin{aligned} x + y + z &= 14 && (1) \\ x - 2y &= -3 && (2) \\ 5x + 10y + 25z &= 135 && (3) \end{aligned}$$

Step 4: ***Solve the system.***

Let's begin by eliminating x from the first and second equations and the first and third equations. Adding -1 times the second equation to the first equation gives us an equation in only y and z. We call this equation (4).

$$3y + z = 17 \qquad (4)$$

Adding -5 times equation (1) to equation (3) gives us

$$5y + 20z = 65 \qquad (5)$$

We can eliminate z from equations (4) and (5) by adding -20 times (4) to (5). Here is the result:

$$-55y = -275$$
$$y = 5$$

Substituting $y = 5$ into equation (4) gives us $z = 2$. Substituting $y = 5$ and $z = 2$ into equation (1) gives us $x = 7$.

Step 5: ***Write answers.***

The collection consists of 7 nickels, 5 dimes, and 2 quarters.

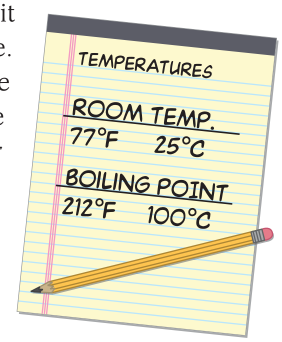
Step 6: ***Reread and check.***

The total number of coins is $7 + 5 + 2 = 14$. The number of nickels, 7, is 3 less than twice the number of dimes, 5. To find the total value of the collection, we have

The value of the 7 nickels is $7(0.05) = \$0.35$
The value of the 5 dimes is $5(0.10) = \$0.50$
The value of the 2 quarters is $2(0.25) = \$0.50$
The total value of the collection is $1.35

If you go on to take a chemistry class, you may see the next example (or one much like it).

EXAMPLE 7 In a chemistry lab, students record the temperature of water at room temperature and find that it is 77° on the Fahrenheit temperature scale and 25° on the Celsius temperature scale. The water is then heated until it boils. The temperature of the boiling water is 212°F and 100°C. Assume that the relationship between the two temperature scales is a linear one, and then use the preceding data to find the formula that gives the Celsius temperature C in terms of the Fahrenheit temperature F.



SOLUTION The data is summarized in Table 1.

TABLE 1
CORRESPONDING TEMPERATURES

In Degrees Fahrenheit	In Degrees Celsius
77	25
212	100

If we assume the relationship is linear, then the formula that relates the two temperature scales can be written in slope-intercept form as

$$C = mF + b$$

Substituting $C = 25$ and $F = 77$ into this formula gives us

$$25 = 77m + b$$

Substituting $C = 100$ and $F = 212$ into the formula yields

$$100 = 212m + b$$

Together, the two equations form a system of equations, which we can solve using the addition method.

$$25 = 77m + b \xrightarrow{\text{Multiply by } -1} -25 = -77m - b$$
$$100 = 212m + b \xrightarrow[\text{No change}]{} \underline{100 = 212m + b}$$
$$75 = 135m$$

$$m = \frac{75}{135} = \frac{5}{9}$$

To find the value of b, we substitute $m = \frac{5}{9}$ into $25 = 77m + b$ and solve for b.

$$25 = 77\left(\frac{5}{9}\right) + b$$
$$25 = \frac{385}{9} + b$$
$$b = 25 - \frac{385}{9} = \frac{225}{9} - \frac{385}{9} = -\frac{160}{9}$$

The equation that gives C in terms of F is

$$C = \frac{5}{9}F - \frac{160}{9}$$

Problem Set 4.3

Moving Toward Success

"Quality means doing it right when no one is looking."

—Henry Ford, 1863–1947, American industrialist and founder of Ford Motor Company

1. Why is it important to show all your work when you solve problems?
2. How do you make sure your answers are correct and that you fully understand how you calculated them?

A Number Problems [Example 1]

1. One number is 3 more than twice another. The sum of the numbers is 18. Find the two numbers.
2. The sum of two numbers is 32. One of the numbers is 4 less than 5 times the other. Find the two numbers.
3. The difference of two numbers is 6. Twice the smaller is 4 more than the larger. Find the two numbers.
4. The larger of two numbers is 5 more than twice the smaller. If the smaller is subtracted from the larger, the result is 12. Find the two numbers.
5. The sum of three numbers is 8. Twice the smallest is 2 less than the largest, and the sum of the largest and smallest is 5. Use a linear system in three variables to find the three numbers.
6. The sum of three numbers is 14. The largest is 4 times the smallest, the sum of the smallest and twice the largest is 18. Use a linear system in three variables to find the three numbers.

Ticket and Interest Problems [Examples 2–3]

7. A total of 925 tickets were sold for a game for a total of $1,150. If adult tickets sold for $2.00 and children's tickets sold for $1.00, how many of each kind of ticket were sold?
8. If tickets for a show cost $2.00 for adults and $1.50 for children, how many of each kind of ticket were sold if a total of 300 tickets were sold for $525?
9. Mr. Jones has $20,000 to invest. He invests part at 6% and the rest at 7%. If he earns $1,280 in interest after 1 year, how much did he invest at each rate?
10. A man invests $17,000 in two accounts. One account earns 5% interest per year and the other earns 6.5%. If his total interest after one year is $970, how much did he invest at each rate?
11. Susan invests twice as much money at 7.5% as she does at 6%. If her total interest after a year is $840, how much does she have invested at each rate?

12. A woman earns $1,350 in interest from two accounts in a year. If she has three times as much invested at 7% as she does at 6%, how much does she have in each account?

13. A man invests $2,200 in three accounts that pay 6%, 8%, and 9% in annual interest, respectively. He has three times as much invested at 9% as he does at 6%. If his total interest for the year is $178, how much is invested at each rate?

14. A student has money in three accounts that pay 5%, 7%, and 8% in annual interest. She has three times as much invested at 8% as she does at 5%. If the total amount she has invested is $1,600 and her interest for the year comes to $115, how much money does she have in each account?

15. Martin invests money in three accounts that pay 6%, 8%, and 12%. He has twice as much mony invested at 8% than as 6%. If the total amount he has invested is $1,300 and his interest for the year comes to $114, how much money does he have in each account?

16. Albert invests money in three accounts that pay 4%, 10%, an 15%. He has three times as much money invested at 10% than he does at 15%. If the total amount he has invested is $5,550 and his interest for the year comes to $570, how much money does he have in each account?

Mixture Problems [Example 4]

17. How many gallons of 20% alcohol solution and 50% alcohol solution must be mixed to get 9 gallons of 30% alcohol solution?

18. How many ounces of 30% hydrochloric acid solution and 80% hydrochloric acid solution must be mixed to get 10 ounces of 50% hydrochloric acid solution?

19. A mixture of 16% disinfectant solution is to be made from 20% and 14% disinfectant solutions. How much of each solution should be used if 15 gallons of the 16% solution are needed?

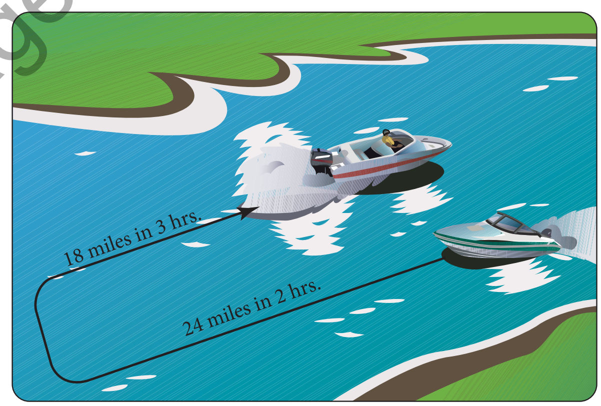
20. Paul mixes nuts worth $1.55 per pound with oats worth $1.35 per pound to get 25 pounds of trail mix worth $1.45 per pound. How many pounds of nuts and how many pounds of oats did he use?

21. Metal Alloys Metal workers solve systems of equations when forming metal alloys. If a certain metal alloy is 40% copper and another alloy is 60% copper, then a system of equations may be written to determine the amount of each alloy necessary to make 50 pounds of a metal alloy that is 55% copper. Write the system and determine this amount.

22. A chemist has three different acid solutions. The first acid solution contains 20% acid, the second contains 40%, and the third contains 60%. He wants to use all three solutions to obtain a mixture of 60 liters containing 50% acid, using twice as much of the 60% solution as the 40% solution. How many liters of each solution should be used?

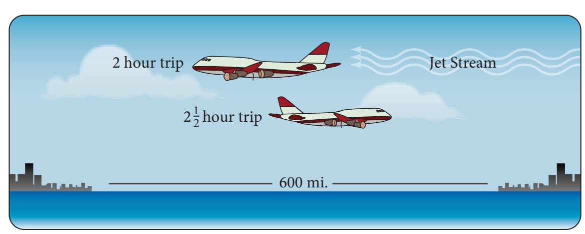
Rate Problems [Example 5]

23. It takes about 2 hours to travel 24 miles downstream and 3 hours to travel 18 miles upstream. What is the speed of the boat in still water? What is the speed of the current of the river?



24. A boat on a river travels 20 miles downstream in only 2 hours. It takes the same boat 6 hours to travel 12 miles upstream. What are the speed of the boat and the speed of the current?

25. An airplane flying with the wind can cover a certain distance in 2 hours. The return trip against the wind takes $2\frac{1}{2}$ hours. How fast is the plane and what is the speed of the wind, if the one-way distance is 600 miles?



26. An airplane covers a distance of 1,500 miles in 3 hours when it flies with the wind and $3\frac{1}{3}$ hours when it flies against the wind. What is the speed of the plane in still air?

Coin Problems [Example 6]

27. Bob has 20 coins totaling \$1.40. If he has only dimes and nickels, how many of each coin does he have?

28. If Amy has 15 coins totaling \$2.70, and the coins are quarters and dimes, how many of each coin does she have?

29. A collection of nickels, dimes, and quarters consists of 9 coins with a total value of \$1.20. If the number of dimes is equal to the number of nickels, find the number of each type of coin.

30. A coin collection consists of 12 coins with a total value of \$1.20. If the collection consists only of nickels, dimes, and quarters, and the number of dimes is two more than twice the number of nickels, how many of each type of coin are in the collection?

31. A collection of nickels, dimes, and quarters amounts to \$10.00. If there are 140 coins in all and there are twice as many dimes as there are quarters, find the number of nickels.

32. A cash register contains a total of 95 coins consisting of pennies, nickels, dimes, and quarters. There are only 5 pennies and the total value of the coins is \$12.05. Also, there are 5 more quarters than dimes. How many of each coin is in the cash register?

33. Justin has \$5.25 in quarters, dimes, and nickels. If he has a total of 31 coins and he has twice as many dimes as nickels, how many of each coin does he have?

34. Russ has \$46.00 in 1, 5, and 10 dollar bills. If he has 11 bills in his wallet and he has twice as many \$1 bills as \$10 bills, how many of each bill does he have?

Additional Problems

35. Price and Demand A manufacturing company finds that they can sell 300 items if the price per item is \$2.00, and 400 items if the price is \$1.50 per item. If the relationship between the number of items sold x and the price per item p is a linear one, find a formula that gives x in terms of p. Then use the formula to find the number of items they will sell if the price per item is \$3.00.

36. Price and Demand A company manufactures and sells bracelets. They have found from experience that they can sell 300 bracelets each week if the price per bracelet is \$2.00, but only 150 bracelets are sold if the price is \$2.50 per bracelet. If the relationship between the number of bracelets sold x and the price per bracelet p is a linear one, find a formula that gives x in terms of p. Then use the formula to find the number of bracelets they will sell at \$3.00 each.

37. Height of a Ball A ball is tossed into the air so that the height after 1, 3, and 5 seconds is as given in the following table. If the relationship between the height of the ball h and the time t is quadratic, then the relationship can be written as

$$h = at^2 + bt + c$$

Use the information in the table to write a system of three equations in three variables a, b, and c. Solve the system to find the exact relationship between h and t.

t (sec)	h (ft)
1	128
3	128
5	0

38. Height of a Ball A ball is tossed into the air and its height above the ground after 1, 3, and 4 seconds is recorded as shown in the following table. The relationship between the height of the ball h and the time t is quadratic and can be written as

$$h = at^2 + bt + c$$

Use the information in the table to write a system of three equations in three variables a, b, and c. Solve the system to find the exact relationship between the variables h and t.

t (sec)	h (ft)
1	96
3	64
4	0

Getting Ready for the Next Section

39. Does the graph of $x + y < 4$ include the boundary line?

40. Does the graph of $-x + y \le 3$ include the boundary line?

41. Where do the graphs of the lines $x + y = 4$ and $x - 2y = 4$ intersect?

42. Where do the graphs of the line $x = -1$ and $x - 2y = 4$ intersect?

Solve.

43. $20x + 9{,}300 > 18{,}000$

44. $20x + 4{,}800 > 18{,}000$

Maintaining Your Skills

Graph each inequality.

45. $2x + 3y < 6$

46. $2x + y < -5$

47. $y \ge -3x - 4$

48. $y \ge 2x - 1$

49. $x \ge 3$

50. $y > -5$

Extending the Concepts

High School Dropout Rate The high school dropout rates for males and females over the years 1965 to 2007 are shown in the following table.

Year	Female Dropout Rate (%)	Male Dropout Rate (%)
1965	18	16
1970	16	14
1975	15	13
1980	13	16
1985	12	15
1990	13	14
1995	12	12
2000	10	12
2005	8	11
2007	8	10

51. Plotting the years along the horizontal axis and the dropout rates along the vertical axis, draw a line graph for these data. Draw the female line dashed and the male line solid for easier reading and comparison.

52. Refer to the data in the preceding exercise to answer the following questions.

- **a.** Using the slope, determine the time interval when the decline in the female dropout rate was the steepest.
- **b.** Using the slope, determine the time interval when the increase in the male dropout rate was the steepest.
- **c.** What appears unusual about the time period from 1975 to 1980?
- **d.** Are the dropout rates for males and females generally increasing or generally decreasing?

53. Refer to the data about high school dropout rates for males and females.

- **a.** Write a linear equation for the dropout rate of males, M, based on the year, x, for the years 1975 and 1980.
- **b.** Write a linear equation for the dropout rate of females, F, based on the year, x, for the years 1975 and 1980.
- **c.** Using your results from parts a and b, determine when the dropout rates for males and females were the same.

4.4 Matrix Solutions to Linear Systems

OBJECTIVES

A Solve a system of linear equations using an augmented matrix.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. What is an augmented matrix?
2. What does the vertical line in a matrix represent?
3. Explain the three row operations for an augmented matrix?
4. Briefly explain how you would transform an augmented matrix into a matrix that has 1s down the diagonal of the coefficient matrix, and 0's below it.

In mathematics, a matrix is a rectangular array of elements considered as a whole. We can use matrices to represent systems of linear equations. The Chinese, between 200 BC and 100 BC, in a text called *Nine Chapters on the Mathematical Art*, written during the Han Dynasty, gives the first known example of the use of matrices. The following problem appears in that text:

> *There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?*

The author creates linear equations in three variables using that information and sets up the coefficients of the three unknowns as a table on a 'counting board'.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 26 \\ 2 & 3 & 1 & 24 \\ 3 & 2 & 1 & 39 \end{array}\right]$$

The counting board shown above is an example of a matrix, the focus of this section. We will begin our work with matrices by writing the coefficients of the variables and the constant terms in the same position in the matrix as they occur in the system of equations. To show where the coefficients end and the constant terms begin, we use vertical lines instead of equal signs. For example, the system

$$\begin{aligned} 2x + 5y &= -4 \\ x - 3y &= 9 \end{aligned}$$

can be represented by the matrix

$$\left[\begin{array}{cc|c} 2 & 5 & -4 \\ 1 & -3 & 9 \end{array}\right]$$

which is called an **augmented matrix** because it includes both the coefficients of the variables and the constant terms.

To solve a system of linear equations by using the augmented matrix for that system, we need the following row operations as the tools of that solution process. The row operations tell us what we can do to an augmented matrix that may change the numbers in the matrix, but will always produce a matrix that represents a system of equations with the same solution as that of our original system.

A Row Operations

1. We can interchange any two rows of a matrix.
2. We can multiply any row by a nonzero constant.
3. We can add to any row a constant multiple of another row.

The three row operations are simply a list of the properties we use to solve systems of linear equations, translated to fit an augmented matrix. For instance, the second operation in our list is actually just another way to state the multiplication property of equality.

We solve a system of linear equations by first transforming the augmented matrix into a matrix that has 1's down the diagonal of the coefficient matrix, and 0's below it. For instance, we will solve the system

$$\begin{aligned} 2x + 5y &= -4 \\ x - 3y &= 9 \end{aligned}$$

by transforming the matrix

$$\left[\begin{array}{cc|c} 2 & 5 & -4 \\ 1 & -3 & 9 \end{array}\right]$$

using the row operations listed earlier to get a matrix of the form

$$\left[\begin{array}{cc|c} 1 & \_ & \_ \\ 0 & 1 & \_ \end{array}\right]$$

To accomplish this, we begin with the first column and try to produce a 1 in the first position and a 0 below it. Interchanging rows 1 and 2 gives us a 1 in the top position of the first column:

$$\left[\begin{array}{cc|c} 1 & -3 & 9 \\ 2 & 5 & -4 \end{array}\right] \quad \text{Interchange rows 1 and 2}$$

Multiplying row 1 by -2 and adding the result to row 2 gives us a 0 where we want it.

$$\left[\begin{array}{cc|c} 1 & -3 & 9 \\ 0 & 11 & -22 \end{array}\right] \quad \text{Multiply row 1 by } -2 \text{ and add the result to row 2}$$

Continue to produce 1's down the diagonal by multiplying row 2 by $\frac{1}{11}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 9 \\ 0 & 1 & -2 \end{array}\right] \quad \text{Multiply row 2 by } \tfrac{1}{11}$$

Taking this last matrix and writing the system of equations it represents, we have

$$\begin{aligned} x - 3y &= 9 \\ y &= -2 \end{aligned}$$

Substituting -2 for y in the top equation gives us

$$x = 3$$

The solution to our system is $(3, -2)$.

EXAMPLE 1 Solve the following system using an augmented matrix:

$$\begin{aligned} x + y - z &= 2 \\ 2x + 3y - z &= 7 \\ 3x - 2y + z &= 9 \end{aligned}$$

SOLUTION We begin by writing the system in terms of an augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 2 & 3 & -1 & 7 \\ 3 & -2 & 1 & 9 \end{array}\right]$$

Next, we want to produce 0's in the second two postions of column 1:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 3 & -5 & 1 & 9 \end{array}\right] \quad \text{Multiply row 1 by } -2 \text{ and add the result to row 2}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -5 & 4 & 3 \end{array}\right] \quad \text{Multiply row 1 by } -3 \text{ and add the result to row 3}$$

Note that we could have done these two steps in one single step. As you become more familiar with this method of solving systems of equations, you will do just that.

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 9 & 18 \end{array}\right] \quad \text{Multiply row 2 by 5 and add the result to row 3}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{Multiply row 3 by } \tfrac{1}{9}$$

Converting back to a system of equations, we have

$$\begin{aligned} x + y - z &= 2 \\ y + z &= 3 \\ z &= 2 \end{aligned}$$

This system is equivalent to our first one, but much easier to solve.

Substituting $z = 2$ into the second equation, we have

$$y = 1$$

Substituting $z = 2$ and $y = 1$ into the first equation, we have

$$x = 3$$

The solution to our original system is $(3, 1, 2)$. It satisfies each of our original equations. You can check this, if you want.

Problem Set 4.4

Moving Toward Success

"One's first step in wisdom is to question everything— and one's last is to come to terms with everything."

—Georg Christoph Lichtenberg, 1742–1799, German scientist and philosopher

1. Why is it important to recognize the different types of instructions for a problem?
2. Should you still add to your list of difficult problems this far into the class? Why or why not?

A Solve the following systems of equations by using matrices.

1. $x + y = 5$, $3x - y = 3$

2. $x + y = -2$, $2x - y = -10$

3. $3x - 5y = 7$, $-x + y = -1$

4. $2x - y = 4$, $x + 3y = 9$

5. $2x - 8y = 6$, $3x - 8y = 13$

6. $3x - 6y = 3$, $-2x + 3y = -4$

7. $2x - y = -10$, $4x + 3y = 0$

8. $3x - 7y = 36$, $5x - 4y = 14$

9. $5x - 3y = 27$, $6x + 2y = -18$

10. $3x + 4y = 2$, $5x + 3y = 29$

11. $5x + 2y = -14$, $y = 2x + 11$

12. $3x + 5y = 3$, $x = 4y + 1$

13. $x + y + z = 4$, $x - y + 2z = 1$, $x - y - z = -2$

14. $x - y - 2z = -1$, $x + y + z = 6$, $x + y - z = 4$

15. $x + 2y + z = 3$, $2x - y + 2z = 6$, $3x + y - z = 5$

16. $x - 3y + 4z = -4$, $2x + y - 3z = 14$, $3x + 2y + z = 10$

17. $x - 2y + z = -4$, $2x + y - 3z = 7$, $5x - 3y + z = -5$

18. $3x - 2y + 3z = -3$, $x + y + z = 4$, $x - 4y + 2z = -9$

19. $5x - 3y + z = 10$, $x - 2y - z = 0$, $3x - y + 2z = 10$

20. $2x - y - z = 1$, $x + 3y + 2z = 13$, $4x + y - z = 7$

21. $2x - 5y + 3z = 2$, $3x - 7y + z = 0$, $x + y + 2z = 5$

22. $3x - 4y + 2z = -2$, $2x + y + 3z = 13$, $x - 3y + 2z = -3$

23. $x + 2y = 3$, $y + z = 3$, $4x - z = 2$

24. $x + y = 2$, $3y - 2z = -8$, $x + z = 5$

25. $x + 3y = 7$, $3x - 4z = -8$, $5y - 2z = -5$

26. $x + 4y = 13$, $2x - 5z = -3$, $4y - 3z = 9$

27. $x + 2y = 13$, $x - 3z = 11$, $3y + 4z = 4$

28. $x - 2y = 5$, $4x + 3z = 11$, $5y + 4z = -12$

Solve each system using matrices. Remember, multiplying a row by a nonzero constant will not change the solution to a system.

29. $\frac{1}{3}x + \frac{1}{5}y = 2$, $\frac{1}{3}x - \frac{1}{2}y = -\frac{1}{3}$

30. $\frac{1}{2}x + \frac{1}{3}y = 13$, $\frac{1}{5}x + \frac{1}{8}y = 5$

The systems that follow are inconsistent systems. In both cases, the lines are parallel. Try solving each system using matrices and see what happens.

31. $2x - 3y = 4$, $4x + 6y = 4$

32. $10x - 15y = 5$, $-4x + 6y = -4$

The systems that follow are dependent systems. In each case, the lines coincide. Try solving each system using matrices and see what happens

33. $-6x + 4y = 8$, $-3x + 2y = 4$

34. $x + 2y = 5$, $-x - 2y = -5$

Getting Ready for the Next Section

Graph the following equations.

35. $x + y = 3$

36. $x - y = 5$

37. $y = -\frac{2}{3}x + 4$

38. $y = 3x - 5$

39. $y = -\frac{3}{4}x$

40. $x = -3$

4.5 Systems of Linear Inequalities

OBJECTIVES

A Graph the solution to a system of linear inequalities in two variables.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. What is the solution set to a system of inequalities?
2. For the boundary lines of a system of linear inequalities, when would you use a dotted line rather than a solid line?
3. Once you have graphed the solution set for each inequality in a system, how would you determine the region to shade for the solution to the system of inequalities?
4. How would you find the solution set by graphing a system that contained three linear inequalities?

Imagine you work at a local coffee house and your boss wants you to create a new house blend using the three most popular coffees. The dark roast coffee you are to use costs \$20.00 per pound, the medium roast costs \$14.50 per pound, and the flavored coffee costs \$16.00 per pound. Your boss wants the cost of the new blend to be at most \$17.00 per pound. She will want to see possible combinations of the coffee to keep the price at most \$17.00. As discussed in Chapter 1, words like "at most" imply the use of inequalities. Working through this section will help us visualize solutions to inequalities in two and three variables.

A Graphing Solutions to Linear Inequalities

In the previous chapter, we graphed linear inequalities in two variables. To review, we graph the boundary line using a solid line if the boundary is part of the solution set and a broken line if the boundary is not part of the solution set. Then we test any point that is not on the boundary line in the original inequality. A true statement tells us that the point lies in the solution set; a false statement tells us the solution set is the other region.

Figure 1 shows the graph of the inequality $x + y < 4$. Note that the boundary is not included in the solution set and is therefore drawn with a broken line. Figure 2 shows the graph of $-x + y \le 3$. Note that the boundary is drawn with a solid line because it is part of the solution set.

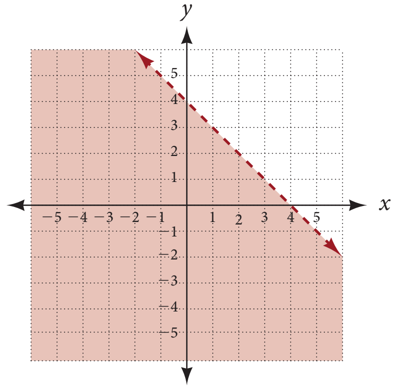


FIGURE 1

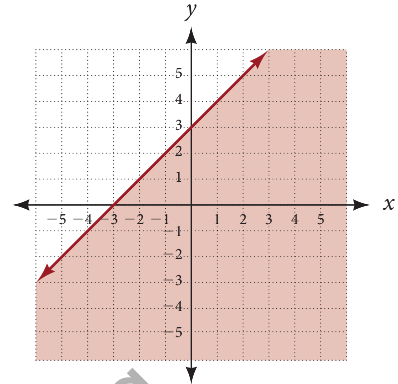


FIGURE 2

If we form a system of inequalities with the two inequalities, the solution set will be all the points common to both solution sets shown in the two figures above; it is the intersection of the two solution sets. Therefore, the solution set for the system of inequalities

$$x + y < 4$$
$$-x + y \le 3$$

is all the ordered pairs that satisfy both inequalities. It is the set of points that are below the line $x + y = 4$ and also below (and including) the line $-x + y = 3$. The graph of the solution set to this system is shown in Figure 3. We have written the system in Figure 3 with the word *and* just to remind you that the solution set to a system of equations or inequalities is all the points that satisfy both equations or inequalities.

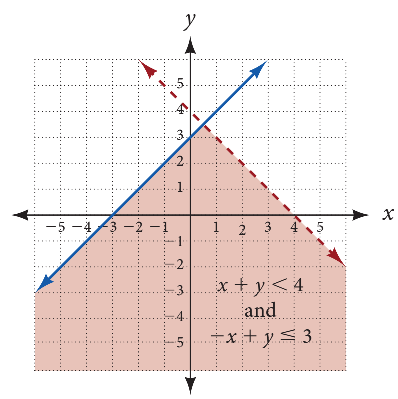


FIGURE 3

EXAMPLE 1 Graph the solution to the system of linear inequalities.

$$y < \frac{1}{2}x + 3$$
$$y \ge \frac{1}{2}x - 2$$

SOLUTION Figures 4 and 5 show the solution set for each of the inequalities separately.

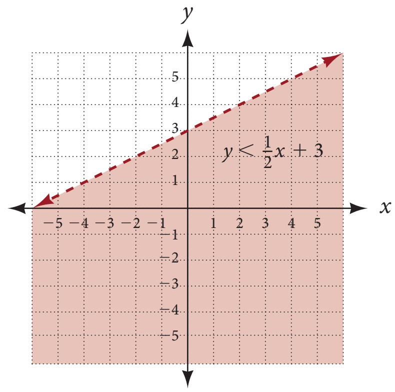


FIGURE 4

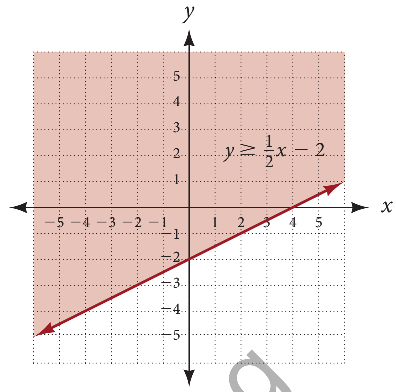


FIGURE 5

Figure 6 is the solution set to the system of inequalities. It is the region consisting of points whose coordinates satisfy both inequalities

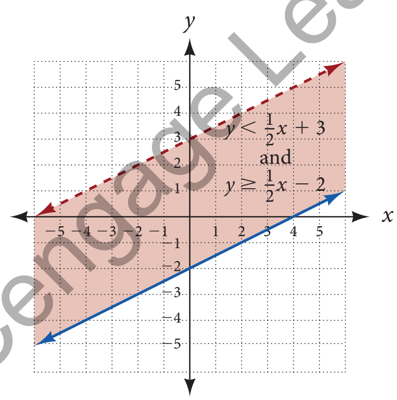


FIGURE 6

EXAMPLE 2 Graph the solution to the system of linear inequalities.

$$x + y < 4$$
$$x \geq 0$$
$$y \geq 0$$

SOLUTION We graphed the first inequality, $x + y < 4$, in Figure 1 at the beginning of this section. The solution set to the inequality $x \geq 0$, shown in Figure 7, is all the points to the right of the y-axis; that is, all the points with x-coordinates that are greater than or equal to 0. Figure 8 shows the graph of $y \geq 0$. It consists of all points with y-coordinates greater than or equal to 0; that is, all points from the x-axis up.

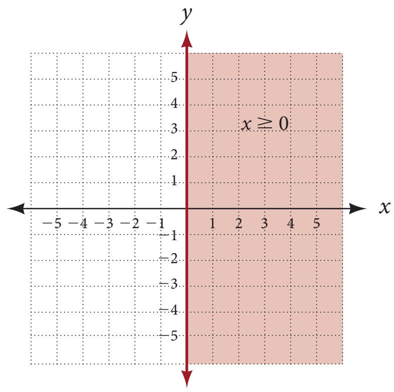


FIGURE 7

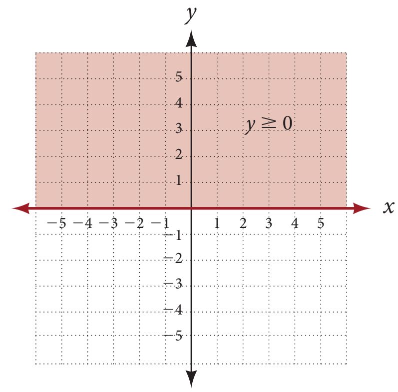


FIGURE 8

The regions shown in Figures 7 and 8 overlap in the first quadrant. Therefore, putting all three regions together we have the points in the first quadrant that are below the line $x + y = 4$. This region is shown in Figure 9, and it is the solution to our system of inequalities.

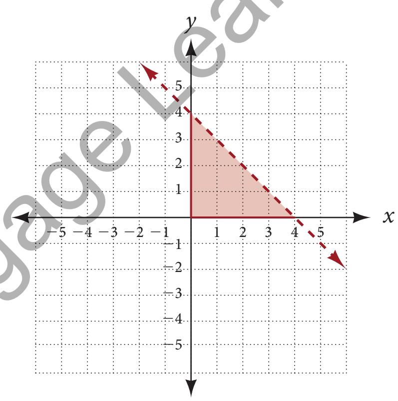


FIGURE 9

Extending the discussion in Example 2, we can name the points in each of the four quadrants using systems of inequalities.

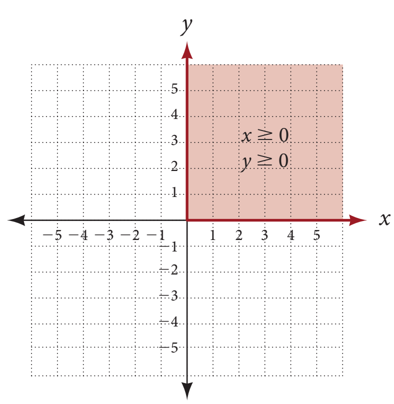


FIGURE 10

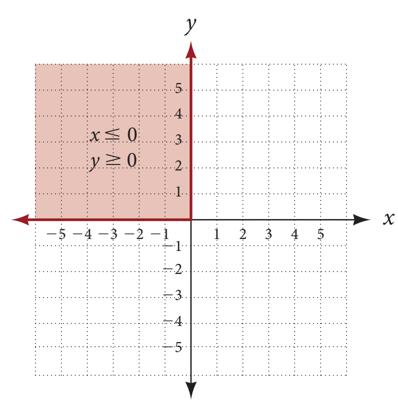


FIGURE 11

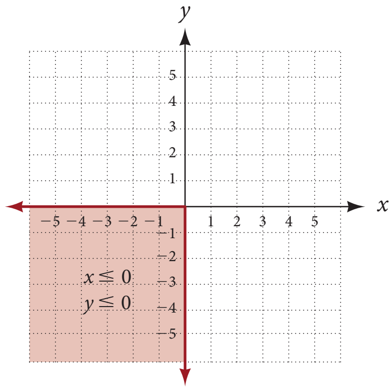


FIGURE 12

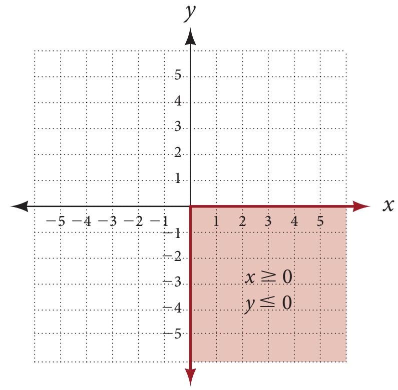


FIGURE 13

EXAMPLE 3 Graph the solution to the system of linear inequalities.

$$x \le 4$$
$$y \ge -3$$

SOLUTION The solution to this system will consist of all points to the left of and including the vertical line $x = 4$ that intersect with all points above and including the horizontal line $y = -3$. The solution set is shown in Figure 14.

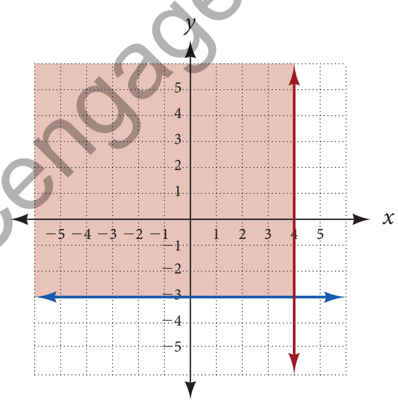


FIGURE 14

EXAMPLE 4 Graph the solution set for the following system.

$$x - 2y \le 4$$
$$x + y \le 4$$
$$x \ge -1$$

SOLUTION We have three linear inequalities, representing three sections of the coordinate plane. The graph of the solution set for this system will be the intersection of these three sections. The graph of $x - 2y \le 4$ is the section above and including the boundary $x - 2y = 4$. The graph of $x + y \le 4$ is the section below and including the boundary line $x + y = 4$. The graph of $x \ge -1$ is all the points to the right of, and including, the vertical line $x = -1$. The intersection of these three graphs is shown in Figure 15.

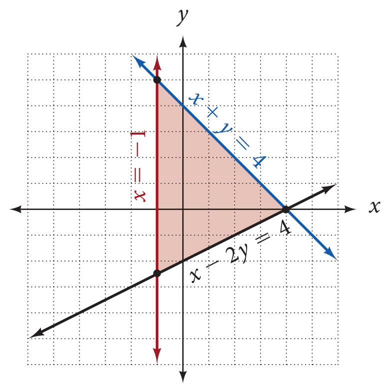


FIGURE 15

EXAMPLE 5 A college basketball arena plans on charging \$20 for certain seats and \$15 for others. They want to bring in more than \$18,000 from all ticket sales and they have reserved at least 500 tickets at the \$15 rate. Find a system of inequalities describing all possibilities and sketch the graph. If 620 tickets are sold for \$15, at least how many tickets are sold for \$20?

© Reuters/CORBIS

SOLUTION Let x = the number of \$20 tickets and y = the number of \$15 tickets. We need to write a list of inequalities that describe this situation. That list will form our system of inequalities. First of all, we note that we cannot use negative numbers for either x or y. So, we have our first inequalities:

$$x \geq 0$$
$$y \geq 0$$

Next, we note that they are selling at least 500 tickets for \$15, so we can replace our second inequality with $y \geq 500$. Now our system is

$$x \geq 0$$
$$y \geq 500$$

Now, the amount of money brought in by selling \$20 tickets is $20x$, and the amount of money brought in by selling \$15 tickets is $15y$. If the total income from ticket sales is to be more than \$18,000, then $20x + 15y$ must be greater than 18,000. This gives us our last inequality and completes our system.

$$20x + 15y > 18{,}000$$
$$x \geq 0$$
$$y \geq 500$$

We have used all the information in the problem to arrive at this system of inequalities. The solution set contains all the values of x and y that satisfy all the conditions given in the problem. Here is the graph of the solution set.

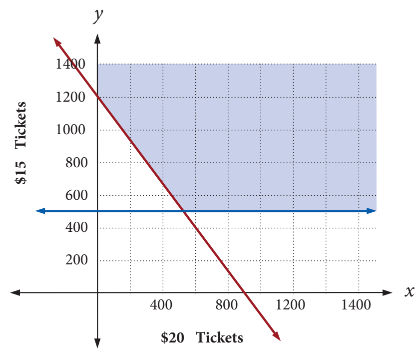


FIGURE 16

If 620 tickets are sold for \$15, then we substitute 620 for y in our first inequality to obtain

$$20x + 15(620) > 18{,}000 \qquad \text{Substitute 620 for } y$$
$$20x + 9{,}300 > 18{,}000 \qquad \text{Multiply}$$
$$20x > 8{,}700 \qquad \text{Add } -9{,}300 \text{ to each side}$$
$$x > 435 \qquad \text{Divide each side by 20}$$

If they sell 620 tickets for \$15 each, then they need to sell more than 435 tickets at \$20 each to bring in more than \$18,000.

Problem Set 4.5

Moving Toward Success

"There is no comparison between that which is lost by not succeeding and that which is lost by not trying."

—Francis Bacon, 1561–1626, English philosopher and statesman

1. Do you take notes in class? Why or why not?
2. Research some note taking techniques online. What are some tips that you should employ when taking notes for this class?

A Graph the solution set for each system of linear inequalities. [Examples 1–4]

1. $x + y < 5$
$2x - y > 4$

2. $x + y < 5$
$2x - y < 4$

3. $y < \frac{1}{3}x + 4$
$y \geq \frac{1}{3}x - 3$

4. $y < 2x + 4$
$y \geq 2x - 3$

5. $x \geq -3$
$y < 2$

6. $x \leq 4$
$y \geq -2$

7. $1 \leq x \leq 3$
$2 \leq y \leq 4$

8. $-4 \leq x \leq -2$
$1 \leq y \leq 3$

9. $x + y \leq 4$
$x \geq 0$
$y \geq 0$

10. $x - y \leq 2$
$x \geq 0$
$y \leq 0$

11. $x + y \leq 3$
$x - 3y \leq 3$
$x \geq -2$

12. $x - y \leq 4$
$x + 2y \leq 4$
$x \geq -1$

13. $x + y \leq 2$
$-x + y \leq 2$
$y \geq -2$

14. $x - y \leq 3$
$-x - y \leq 3$
$y \leq -1$

15. $x + y < 5$
$y > x$
$y \geq 0$

16. $x + y < 5$
$y > x$
$x \geq 0$

17. $2x + 3y \leq 6$
$x \geq 0$
$y \geq 0$

18. $x + 2y \leq 10$
$3x + y \leq 12$
$x \geq 0$
$y \geq 0$

For each figure below, find a system of inequalities that describes the shaded region.

19.

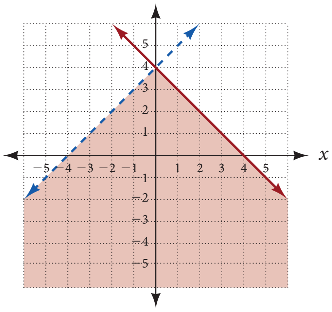


FIGURE 17

20.

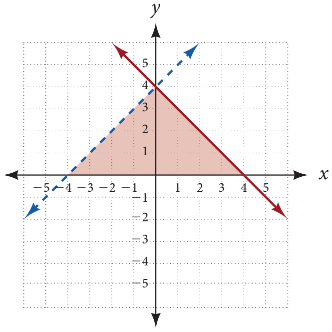


FIGURE 18

21.

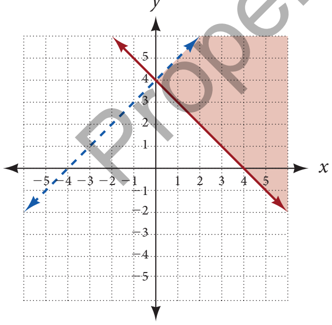


FIGURE 19

22.

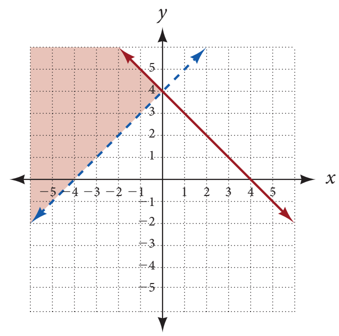


FIGURE 20

Applying the Concepts [Example 5]

23. Office Supplies An office worker wants to purchase some \$0.55 postage stamps and also some \$0.65 postage stamps totaling no more than \$40. He also wants to have at least twice as many \$0.55 stamps and more than 15 of the \$0.55 stamps.

a. Find a system of inequalities describing all the possibilities and sketch the graph.

b. If he purchases 20 of the \$0.55 stamps, what is the maximum number of \$0.65 stamps he can purchase?

24. Inventory A store sells two brands of DVD players. Customer demand indicates that it is necessary to stock at least twice as many DVD players of brand A as of brand B. At least 30 of brand A and 15 of brand B must be on hand. There is room for not more than 100 DVD players in the store.

a. Find a system of inequalities describing all possibilities, then sketch the graph.

b. If there are 35 DVD players of brand A, what is the maximum number of brand B DVD players on hand?

Maintaining Your Skills

For each of the following straight lines, identify the x-intercept, y-intercept, and slope, and sketch the graph.

25. $2x + y = 6$

26. $y = \frac{3}{2}x + 4$

27. $x = -2$

Find the equation for each line.

28. Give the equation of the line through $(-1, 3)$ that has slope $m = 2$.

29. Give the equation of the line through $(-3, 2)$ and $(4, -1)$.

30. Line l contains the point $(5, -3)$ and has a graph parallel to the graph of $2x - 5y = 10$. Find the equation for l.

31. Give the equation of the vertical line through $(4, -7)$.

State the domain and range for the following relations, and indicate which relations are also functions.

32. $\{(-2, 0), (-3, 0), (-2, 1)\}$

33. $y = x^2 - 9$

Let $f(x) = x - 2$, $g(x) = 3x + 4$ and $h(x) = 3x^2 - 2x - 8$, and find the following.

34. $f(3) + g(2)$

35. $h(0) + g(0)$

36. $f(g(2))$

37. $g(f(2))$

Solve the following variation problems.

38. Direct Variation Quantity y varies directly with the square of x. If y is 50 when x is 5, find y when x is 3.

39. Joint Variation Quantity z varies jointly with x and the cube of y. If z is 15 when x is 5 and y is 2, find z when x is 2 and y is 3.

Chapter 4 Summary

EXAMPLES

Systems of Linear Equations [4.1, 4.2]

1. The solution to the system
$$x + 2y = 4$$
$$x - y = 1$$
is the ordered pair (2, 1). It is the only ordered pair that satisfies both equations.

A system of linear equations consists of two or more linear equations considered simultaneously. The solution set to a linear system in two variables is the set of ordered pairs that satisfy both equations. The solution set to a linear system in three variables consists of all the ordered triples that satisfy each equation in the system.

To Solve a System by the Addition Method [4.1]

2. We can eliminate the y-variable from the system in Example 1 by multiplying both sides of the second equation by 2 and adding the result to the first equation.

$$\begin{aligned} x + 2y &= 4 \xrightarrow{\text{No change}} & x + 2y &= 4 \\ x - y &= 1 \xrightarrow[\text{Multiply by 2}]{} & 2x - 2y &= 2 \\ & & 3x &= 6 \\ & & x &= 2 \end{aligned}$$

Substituting $x = 2$ into either of the original two equations gives $y = 1$. The solution is (2, 1).

Step 1: Look the system over to decide which variable will be easier to eliminate.

Step 2: Use the multiplication property of equality on each equation separately, if necessary, to ensure that the coefficients of the variable to be eliminated are opposites.

Step 3: Add the left and right sides of the system produced in step 2, and solve the resulting equation.

Step 4: Substitute the solution from step 3 back into any equation with both x- and y-variables, and solve.

Step 5: Check your solution in both equations if necessary.

To Solve a System by the Substitution Method [4.1]

3. We can apply the substitution method to the system in Example 1 by first solving the second equation for x to get
$$x = y + 1$$
Substituting this expression for x into the first equation we have
$$\begin{aligned} (y + 1) + 2y &= 4 \\ 3y + 1 &= 4 \\ 3y &= 3 \\ y &= 1 \end{aligned}$$
Using $y = 1$ in either of the original equations gives $x = 2$.

Step 1: Solve either of the equations for one of the variables (this step is not necessary if one of the equations has the correct form already).

Step 2: Substitute the results of step 1 into the other equation, and solve.

Step 3: Substitute the results of step 2 into an equation with both x- and y-variables, and solve. (The equation produced in step 1 is usually a good one to use.)

Step 4: Check your solution if necessary.

Inconsistent and Dependent Equations [4.1, 4.2]

4. If the two lines are parallel, then the system will be inconsistent and the solution is $\varnothing$. If the two lines coincide, then the equations are dependent.

A system of two linear equations that have no solutions in common is said to be an *inconsistent* system, whereas two linear equations that have all their solutions in common are said to be *dependent* equations.

Applications of Linear Systems [4.3]

Strategy Problem Solving Using a System of Equations

Step 1: ***Read*** the problem, and then mentally ***list*** the items that are known and the items that are unknown.

Step 2: ***Assign variables*** to each of the unknown items; that is, let x = one of the unknown items and y = the other unknown item (and z = the third unknown item, if there is a third one). Then ***translate*** the other ***information*** in the problem to expressions involving the two (or three) variables.

Step 3: ***Reread*** the problem, and then ***write a system of equations,*** using the items and variables listed in steps 1 and 2, that describes the situation.

Step 4: ***Solve the system*** found in step 3.

Step 5: ***Write your answers*** using complete sentences.

Step 6: ***Reread*** the problem, and ***check*** your solution with the original words in the problem.

5. One number is 2 more than 3 times another. Their sum is 26. Find the numbers.

Step 1: Read and List.
Known items: two numbers, whose sum is 26. One is 2 more than 3 times the other. Unknown items are the two numbers.

Step 2: Assign variables.
Let x = one of the numbers. Then y = the other number.

Step 3: Write a system of equations.

$$x + y = 26$$
$$y = 3x + 2$$

Step 4: Solve the system.
Substituting the expression for y from the second equation into the first and solving for x yields

$$\begin{aligned} x + (3x + 2) &= 26 \\ 4x + 2 &= 26 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

Using $x = 6$ in $y = 3x + 2$ gives the second number: $y = 20$.

Step 5: Write answers.
The two numbers are 6 and 20.

Step 6: Reread and check.
The sum of 6 and 20 is 26, and 20 is 2 more than 3 times 6.

Matrix Solutions [4.4]

A matrix is a rectangular array of elements considered as a whole. We can use matrices to represent systems of linear equations. An augmented matrix includes both the coefficients of the variables and the constant terms of the linear system.

To solve a system of linear equations using an augmented matrix, we must follow row operations.

1. We can interchange any two rows of a matrix.
2. We can multiply any row by a nonzero constant.
3. We can add to any row a constant multiple of another row.

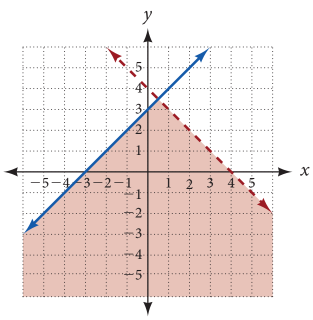
Systems of Linear Inequalities [4.5]

A system of linear inequalities is two or more linear inequalities considered at the same time. To find the solution set to the system, we graph each of the inequalities on the same coordinate system. The solution set is the region that is common to all the regions graphed.

6. The solution set for the system

$$\begin{aligned} x + y &< 4 \\ -x + y &\le 3 \end{aligned}$$

is shown below.



Chapter 4 Review

Solve each system using the addition method. [4.1]

1. $\begin{aligned} x + y &= 4 \\ 2x - y &= 14 \end{aligned}$

2. $\begin{aligned} 3x + y &= 2 \\ 2x + y &= 0 \end{aligned}$

3. $\begin{aligned} 2x - 4y &= 5 \\ -x + 2y &= 3 \end{aligned}$

4. $\begin{aligned} 5x - 2y &= 7 \\ 3x + y &= 2 \end{aligned}$

5. $\begin{aligned} 6x - 5y &= -5 \\ 3x + y &= 1 \end{aligned}$

6. $\begin{aligned} 6x + 4y &= 8 \\ 9x + 6y &= 12 \end{aligned}$

7. $\begin{aligned} 3x - 7y &= 2 \\ -4x + 6y &= -6 \end{aligned}$

8. $\begin{aligned} 6x + 5y &= 9 \\ 4x + 3y &= 6 \end{aligned}$

9. $\begin{aligned} -7x + 4y &= -1 \\ 5x - 3y &= 0 \end{aligned}$

10. $\begin{aligned} \frac{1}{2}x - \frac{3}{4}y &= -4 \\ \frac{1}{4}x + \frac{3}{2}y &= 13 \end{aligned}$

11. $\begin{aligned} \frac{2}{3}x - \frac{1}{6}y &= 0 \\ \frac{4}{3}x + \frac{5}{6}y &= 14 \end{aligned}$

12. $\begin{aligned} -\frac{1}{2}x + \frac{1}{3}y &= -\frac{13}{6} \\ \frac{4}{5}x + \frac{3}{4}y &= \frac{9}{10} \end{aligned}$

Solve each system by the substitution method. [4.1]

13. $\begin{aligned} x + y &= 2 \\ y &= x - 1 \end{aligned}$

14. $\begin{aligned} 2x - 3y &= 5 \\ y &= 2x - 7 \end{aligned}$

15. $\begin{aligned} x + y &= 4 \\ 2x + 5y &= 2 \end{aligned}$

16. $\begin{aligned} x + y &= 3 \\ 2x + 5y &= -6 \end{aligned}$

17. $\begin{aligned} 3x + 7y &= 6 \\ x &= -3y + 4 \end{aligned}$

18. $\begin{aligned} 5x - y &= 4 \\ y &= 5x - 3 \end{aligned}$

Solve each system. [4.2]

19. $\begin{aligned} x + y + z &= 6 \\ x - y - 3z &= -8 \\ x + y - 2z &= -6 \end{aligned}$

20. $\begin{aligned} 3x + 2y + z &= 4 \\ 2x - 4y + z &= -1 \\ x + 6y + 3z &= -4 \end{aligned}$

21. $\begin{aligned} 5x + 8y - 4z &= -7 \\ 7x + 4y + 2z &= -2 \\ 3x - 2y + 8z &= 8 \end{aligned}$

22. $\begin{aligned} 5x - 3y - 6z &= 5 \\ 4x - 6y - 3z &= 4 \\ -x + 9y + 9z &= 7 \end{aligned}$

23. $\begin{aligned} 5x - 2y + z &= 6 \\ -3x + 4y - z &= 2 \\ 6x - 8y + 2z &= -4 \end{aligned}$

24. $\begin{aligned} 4x - 6y + 8z &= 4 \\ 5x + y - 2z &= 4 \\ 6x - 9y + 12z &= 6 \end{aligned}$

25. $\begin{aligned} 2x - y &= 5 \\ 3x - 2z &= -2 \\ 5y + z &= -1 \end{aligned}$

26. $\begin{aligned} x - y &= 2 \\ y - z &= -3 \\ x - z &= -1 \end{aligned}$

Use systems of equations to solve each application problem. In each case, be sure to show the system used. [4.3]

27. Ticket Prices Tickets for the show cost \$2.00 for adults and \$1.50 for children. How many adult tickets and how many children's tickets were sold if a total of 127 tickets were sold for \$214?

28. Coin Collection John has 20 coins totaling \$3.20. If he has only dimes and quarters, how many of each coin does he have?

29. Investments Ms. Jones invests money in two accounts, one of which pays 12% per year, and the other pays 15% per year. If her total investment is \$12,000 and the interest after 1 year is \$1,650, how much is invested in each account?

30. Speed It takes a boat on a river 2 hours to travel 28 miles downstream and 3 hours to travel 30 miles upstream. What is the speed of the boat and the current of the river?

Solve the following systems of equations by using matrices. [4.4]

31. $5x - 3y = 13$
$2x + 6y = -2$

32. $2x + 3y = 6$
$x - 2y = -11$

33. $5x + 3y + z = 6$
$x - 4y - 3z = 4$
$3x + y - 5z = -6$

34. $4x - 3y + 4z = -2$
$x - y + 2z = 1$
$2x + y + 3z = -5$

Graph the solution set for each system of linear inequalities. [4.5]

35. $3x + 4y < 12$
$-3x + 2y \le 6$

36. $3x + 4y < 12$
$-3x + 2y \le 6$
$y \ge 0$

37. $3x + 4y < 12$
$x \ge 0$
$y \ge 0$

38. $x > -3$
$y > -2$

Simplify each of the following.

1. $4^2 - 8^2$

2. $(4 - 8)^2$

3. $15 - 12 \div 4 - 3 \cdot 2$

4. $6(11 - 13)^3 - 5(8 - 11)^2$

5. $8\left(\dfrac{3}{2}\right) - 24\left(\dfrac{3}{2}\right)^2$

6. $16\left(\dfrac{3}{8}\right) - 32\left(\dfrac{3}{4}\right)^2$

Find the value of each expression when x is -3.

7. $x^2 + 12x - 36$

8. $(x - 6)(x + 6)$

Simplify each of the following expressions.

9. $4(3x - 2) + 3(2x + 5)$

10. $5 - 3[2x - 4(x - 2)]$

11. $-4\left(\dfrac{3}{4}x - \dfrac{3}{2}y\right)$

12. $-6\left(\dfrac{5}{3}x + \dfrac{3}{6}y\right)$

Solve.

13. $-4y - 2 = 6y + 8$

14. $-6 + 2(2x + 3) = 0$

15. $|x + 5| + 3 = 2$

16. $|2x - 3| + 7 = 1$

Solve the following equations for y.

17. $y - 3 = -5(x - 2)$

18. $y - 3 = \dfrac{2}{3}(x - 6)$

Solve the following equations for x.

19. $ax - 5 = bx - 3$

20. $ax + 7 = cx - 2$

Solve each inequality and write your answers using interval notation.

21. $-3t \geq 12$

22. $|2x - 1| \geq 5$

Let $A = \{0, 3, 6, 9\}$ and $B = \{2, 4, 6\}$. Find the following.

23. $A \cup B$

24. $A \cap B$

For the set $\{-1, 0, \sqrt{2}, 2.35, \sqrt{3}, 4\}$ list all the elements in the following sets.

25. Rational numbers

26. Integers

Solve each of the following systems.

27.
$$\begin{aligned} 2x - 5y &= -7 \\ -3x + 4y &= 0 \end{aligned}$$

28.
$$\begin{aligned} 8x + 6y &= 4 \\ 12x + 9y &= 8 \end{aligned}$$

29.
$$\begin{aligned} 2x + y &= 3 \\ y &= -2x + 3 \end{aligned}$$

30.
$$\begin{aligned} 2x + 2y &= -8z \\ 4y - 2z &= -9 \\ 3x - 2z &= -6 \end{aligned}$$

Graph on a rectangular coordinate system.

31. $0.03x + 0.04y = 0.04$

32. $3x - y < -2$

33. Graph the solution set to the system.
$$\begin{aligned} x - y &< -3 \\ y &< 5 \end{aligned}$$

34. Find the slope of the line through $\left(\frac{2}{3}, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, -\frac{5}{6}\right)$.

35. Find the slope and y-intercept of $3x - 5y = 15$.

36. Give the equation of a line with slope $-\frac{2}{3}$ and y-intercept $= -3$.

37. Find the equation of the line that is perpendicular to $2x - 5y = 10$ and contains the point $(0, 1)$.

38. Find the equation of the line through $(1, 4)$ and $(-1, -2)$.

Let $f(x) = 2x + 3$ and $g(x) = x^2 - 5$. Find the following.

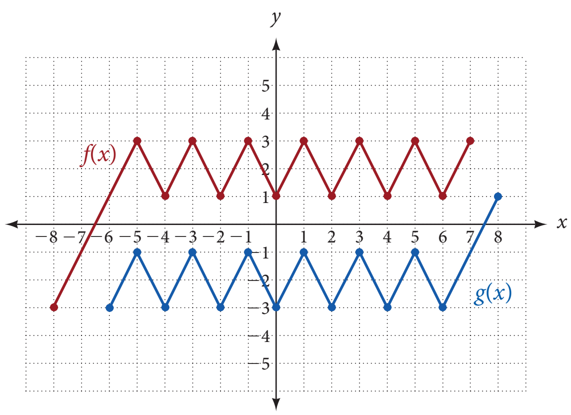
39. $f(-2)$

40. $g(-2) + 6$

41. $(f - g)(x)$

42. $(g \circ f)(x)$

Use the graph below to work Problems 43–46.



43. Find $f(3)$.

44. Find $(f + g)(-1)$.

45. Find $(g \circ f)(6)$.

46. Find x if $f(x) = -1$.

Chapter 4 Test

Solve the following systems by the addition method. [4.1]

1. $2x - 5y = -8$
$3x + y = 5$

2. $\frac{1}{3}x - \frac{1}{6}y = 3$
$-\frac{1}{5}x + \frac{1}{4}y = 0$

Solve the following systems by the substitution method. [4.1]

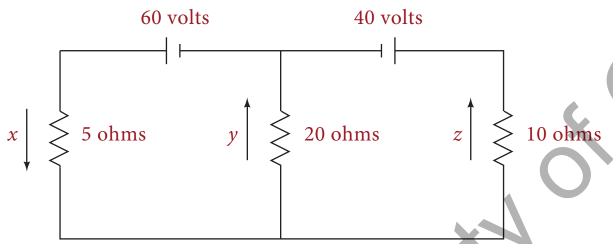
3. $2x - 5y = 14$
$y = 3x + 8$

4. $6x - 3y = 0$
$x + 2y = 5$

Solve the system. [4.2]

5. $2x - y + z = 9$
$x + y - 3z = -2$
$3x + y - z = 6$

6. **Electric Current** In the following diagram of an electrical circuit, x, y, and z represent the amount of current (in amperes) flowing across the 5-ohm, 20-ohm, and 10-ohm resistors, respectively. (In circuit diagrams, resistors are represented by and potential differences by .)



The system of equations used to find the three currents x, y, and z is

$$x - y - z = 0$$
$$5x + 20y = 60$$
$$20y - 10z = 40$$

Solve the system for all variables.

Solve each word problem. [4.3]

7. **Number Problem** A number is 1 less than twice another. Their sum is 14. Find the two numbers.

8. **Investing** John invests twice as much money at 6% as he does at 5%. If his investments earn a total of \$680 in 1 year, how much does he have invested at each rate?

9. **Ticket Cost** There were 750 tickets sold for a basketball game for a total of \$1,090. If adult tickets costs \$2.00 and children's tickets cost \$1.00, how many of each kind were sold?

10. **Speed of a Boat** A boat can travel 20 miles downstream in 2 hours. The same boat can travel 18 miles upstream in 3 hours. What is the speed of the boat in still water, and what is the speed of the current?

11. **Coin Problem** A collection of nickels, dimes, and quarters consists of 15 coins with a total value of \$1.10. If the number of nickels is one less than 4 times the number of dimes, how many of each coin are contained in the collection?

Sovle the following systems of equations by using matrices. [4.4]

12. $2x - y = -13$
$y = 4x + 23$

13. $3x - 4y - 3z = -3$
$x + 3y + 4z = 19$
$2y - 5z = -20$

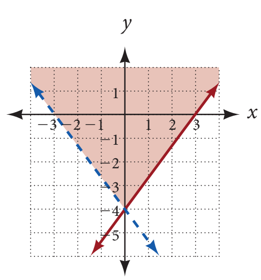
Graph the solution set for each system of linear inequalities. [4.5]

14. $x + 4y \le 4$
$-3x + 2y > -12$

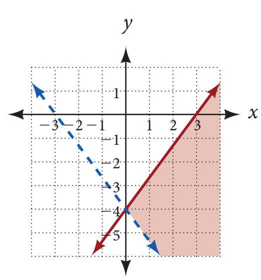
15. $y < -\frac{1}{2}x + 4$
$x \ge 0$
$y \ge 0$

Find a system of inequalities that describes the shaded region. [4.5]

16.



17.



Chapter 4 Projects

SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

GROUP PROJECT

Break-Even Point

Number of People 2 or 3

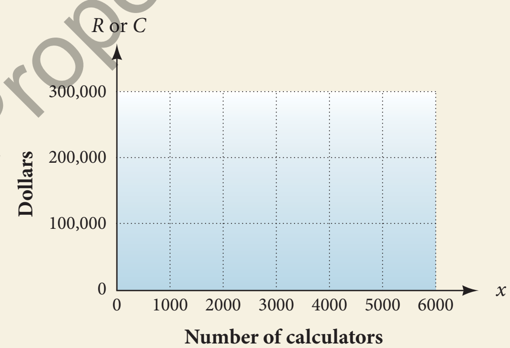
Time Needed 10–15 minutes

Equipment Pencil and paper

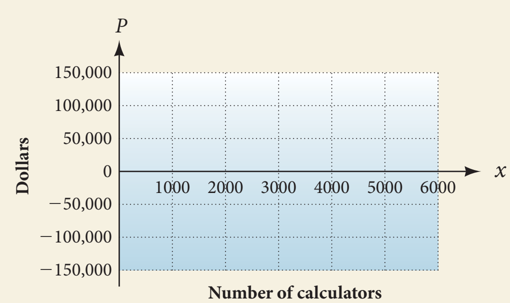
Background The break-even point for a company occurs when the revenue from sales of a product equals the cost of producing the product. This group project is designed to give you more insight into revenue, cost, and the break-even point.

Procedure A company is planning to open a factory to manufacture calculators.

1. It costs them \$120,000 to open the factory, and it will cost \$10 for each calculator they make. What is the expression for $C(x)$, the cost of making x calculators?
2. They can sell the calculators for \$50 each. What is the expression for $R(x)$, their revenue from selling x calculators? Remember that $R = px$, where p is the price per calculator.
3. Graph both the cost equation $C(x)$ and the revenue equation $R(x)$ on a coordinate system like the one below.



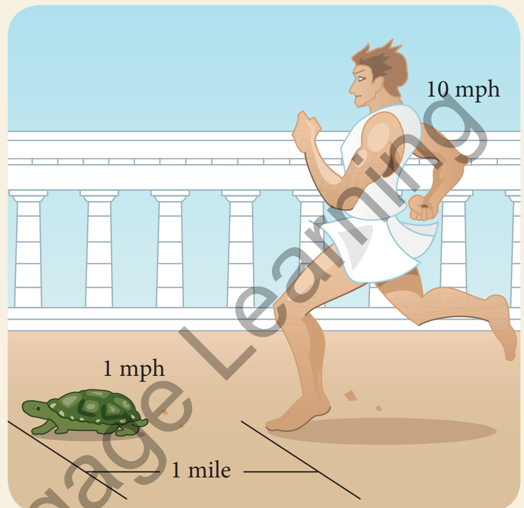
4. The break-even point is the value of x (the number of calculators) for which the revenue is equal to the cost. Where is the break-even point on the graph you produced in Part 3? Estimate the break-even point from the graph.
5. Set the cost equal to the revenue and solve for x to find the exact value of the break-even point. How many calculators do they need to make and sell to exactly break even? What will be their revenue and their cost for that many calculators?
6. Write an inequality that gives the values of x that will produce a profit for the company. (A profit occurs when the revenue is larger than the cost.)
7. Write an inequality that gives the values of x that will produce a loss for the company. (A loss occurs when the cost is larger than the revenue.)
8. Profit is the difference between revenue and cost, or $P(x) = R(x) - C(x)$. Write the equation for profit and then graph it on a coordinate system like the one below.



9. How do you recognize the break-even point and the regions of loss and profit on the graph you produced above?

RESEARCH PROJECT

Zeno's Paradoxes



Zeno of Elea was born at about the same time that Pythagoras (of the Pythagorean theorem) died. He is responsible for three paradoxes that have come to be known as Zeno's paradoxes. One of the three has to do with a race between Achilles and a tortoise. Achilles is much faster than the tortoise, but the tortoise has a head start. According to Zeno's method of reasoning, Achilles can never pass the tortoise because each time he reaches the place where the tortoise was, the tortoise is gone. Research Zeno's paradox concerning Achilles and the tortoise. Put your findings into essay form that begins with a definition for the word "paradox." Then use Zeno's method of reasoning to describe a race between Achilles and the tortoise — if Achilles runs at 10 miles per hour, the tortoise runs at 1 mile per hour, and the tortoise has a 1-mile head start. Next, use the methods shown in this chapter to find the distance and the time at which Achilles reaches the tortoise. Conclude your essay by summarizing what you have done and showing how the two results you have obtained form a paradox.

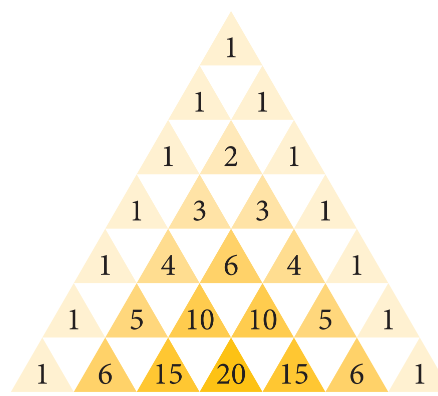
5 Exponents and Polynomials

Chapter Outline

Introduction

The French mathematician and philosopher, Blaise Pascal, was born in France in 1623. Both a scientist and a mathematician, he is credited for defining the scientific method. In the image below, Pascal carries a barometer to the top of the bell tower at the church of Saint-Jacques-de-la-Boucherie, overlooking Paris, to test a scientific theory.

Mary Evans Picture Library/Alamy



The triangular array of numbers to the right of the painting is called Pascal's triangle. Pascal's triangle is connected to the work we will do in this chapter when we find increasing powers of binomials.

Getting Ready for Chapter 5

Simplify the following.

1. $-6 - (-9)$ **2.** $-5 - (-7)$ **3.** $2 - 18$ **4.** $3 - 10$

5. $-8(4)$ **6.** $-3(2)$ **7.** $\frac{6}{12}$ **8.** $\frac{18}{36}$

9. 3^4 **10.** 2^3 **11.** 10^3 **12.** 10^4

13. $(-2)^3$ **14.** $(-3)^3$ **15.** $\frac{1}{5^2}$ **16.** $\frac{1}{3^2}$

17. $\frac{1}{(-2)^3}$ **18.** $\frac{1}{(-5)^3}$ **19.** $\frac{1}{\left(\frac{2}{3}\right)^2}$ **20.** $\frac{1}{\left(\frac{3}{4}\right)^2}$

21. $4.52 \times 1{,}000$ **22.** $3.91 \times 10{,}000$ **23.** $376{,}000 \div 100{,}000$ **24.** $27{,}400 \div 10{,}000$

Chapter Outline

5.1 Properties of Exponents

A Simplify expressions using the properties of exponents.

B Convert back and forth between scientific notation and expanded form.

C Multiply and divide expressions written in scientific notation.

5.2 Polynomials, Sums, and Differences

A Give the degree of a polynomial.

B Add and subtract polynomials.

C Evaluate a polynomial for a given value of its variable.

5.3 Multiplication of Polynomials

A Multiply polynomials.

B Multiply binomials using the FOIL method.

C Find the square of a binomial.

D Multiply binomials to find the difference of two squares.

5.4 The Greatest Common Factor and Factoring by Grouping

A Factor by factoring out the greatest common factor.

B Factor by grouping.

5.5 Factoring Trinomials

A Factor trinomials in which the leading coefficient is 1.

B Factor trinomials in which the leading coefficient is a number other than 1.

5.6 Special Factoring

A Factor perfect square trinomials.

B Factor the difference of two squares.

C Factor the sum or difference of two cubes.

5.7 Factoring: A General Review

A Factor a variety of polynomials.

5.8 Solving Equations by Factoring

A Solve equations by factoring.

B Apply the Blueprint for Problem Solving to solve application problems whose solutions involve quadratic equations.

C Solve problems that contain formulas that are quadratic.

5.1 Properties of Exponents

OBJECTIVES

A Simplify expressions using the properties of exponents.

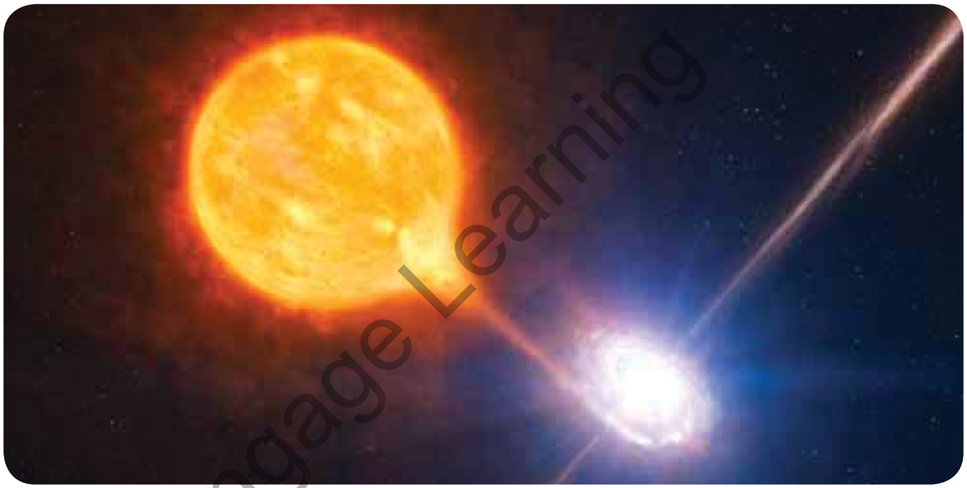
B Convert back and forth between scientific notation and expanded form.

C Multiply and divide expressions written in scientific notation.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

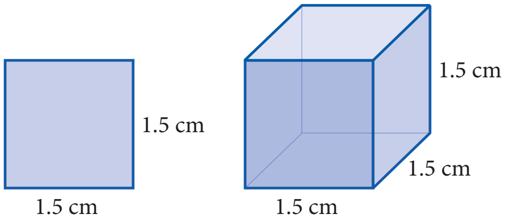
1. Using symbols, explain the first property for exponents.
2. Compare and contrast the second and third property for exponents.
3. How does a negative integer exponent affect the base number?
4. What is scientific notation?



ESO/L.Calçada/M.Kornmesser

A black hole is a region in space that is so dense not even light can escape its gravitational pull. Astronomers have discovered a large bubble in space, shown above, fueled by a black hole. This bubble is making its way through a nearby galaxy called NGC 7793, which is 12 million light years away. In this section, we will begin our work with exponents, allowing us to put very large numbers, like 12 million, into a form that is easier for us to work with.

The figure shows a square and a cube, each with a side of length 1.5 centimeters. To find the area of the square, we raise 1.5 to the second power: 1.5^2. To find the volume of the cube, we raise 1.5 to the third power: 1.5^3.



Because the area of the square is 1.5^2, we say second powers are *squares;* that is, x^2 is read "x squared." Likewise, since the volume of the cube is 1.5^3, we say third powers are *cubes,* that is, x^3 is read "x cubed." Exponents and the vocabulary associated with them are topics we will study in this section.

A Properties of Exponents

In this section, we will be concerned with the simplification of expressions that involve exponents. We begin by making some generalizations about exponents, based on specific examples.

EXAMPLE 1 Write the product $x^3 \cdot x^4$ with a single exponent.

SOLUTION
$$\begin{aligned} x^3 \cdot x^4 &= (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x) \\ &= (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x) \\ &= x^7 \qquad \text{Note: } 3 + 4 = 7 \end{aligned}$$

We can generalize this result into the first property of exponents.

Property 1 for Exponents
If a is a real number and r and s are integers, then
$$a^r \cdot a^s = a^{r+s}$$

EXAMPLE 2 Write $(5^3)^2$ with a single exponent.

SOLUTION
$$\begin{aligned} (5^3)^2 &= 5^3 \cdot 5^3 \\ &= 5^6 \qquad \text{Note: } 3 \cdot 2 = 6 \end{aligned}$$

Generalizing this result, we have a second property of exponents.

Property 2 for Exponents
If a is a real number and r and s are integers, then
$$(a^r)^s = a^{r \cdot s}$$

A third property of exponents arises when we have the product of two or more numbers raised to an integer power.

EXAMPLE 3 Expand $(3x)^4$ and then multiply.

SOLUTION
$$\begin{aligned} (3x)^4 &= (3x)(3x)(3x)(3x) \\ &= (3 \cdot 3 \cdot 3 \cdot 3)(x \cdot x \cdot x \cdot x) \\ &= 3^4 \cdot x^4 \qquad \text{Note: The exponent 4 distributes over the product } 3x \\ &= 81x^4 \end{aligned}$$

Generalizing Example 3, we have property 3 for exponents.

Property 3 for Exponents
If a and b are any two real numbers and r is an integer, then
$$(ab)^r = a^r \cdot b^r$$

Here are some examples that use combinations of the first three properties of exponents to simplify expressions involving exponents.

EXAMPLE 4 Simplify $(-3x^2)(5x^4)$ using the properties of exponents.

SOLUTION

$$\begin{aligned}(-3x^2)(5x^4) &= -3(5)(x^2 \cdot x^4) && \text{Commutative and associative}\\ &= -15x^6 && \text{Property 1 for exponents}\end{aligned}$$

EXAMPLE 5 Simplify $(-2x^2)^3(4x^5)$ using the properties of exponents.

SOLUTION

$$\begin{aligned}(-2x^2)^3(4x^5) &= (-2)^3(x^2)^3(4x^5) && \text{Property 3}\\ &= -8x^6 \cdot (4x^5) && \text{Property 2}\\ &= (-8 \cdot 4)(x^6 \cdot x^5) && \text{Commutative and associative}\\ &= -32x^{11} && \text{Property 1}\end{aligned}$$

EXAMPLE 6 Simplify $(x^2)^4(x^2y^3)^2(y^4)^3$ using the properties of exponents.

SOLUTION

$$\begin{aligned}(x^2)^4(x^2y^3)^2(y^4)^3 &= x^8 \cdot x^4 \cdot y^6 \cdot y^{12} && \text{Properties 2 and 3}\\ &= x^{12}y^{18} && \text{Property 1}\end{aligned}$$

The next property of exponents deals with negative integer exponents.

NOTE
This property is actually a definition; that is, we are defining negative-integer exponents as indicating reciprocals. Doing so gives us a way to write an expression with a negative exponent as an equivalent expression with a positive exponent.

Property 4 for Exponents

If a is any nonzero real number and r is a positive integer, then

$$a^{-r} = \frac{1}{a^r}$$

EXAMPLE 7 Write 5^{-2} with positive exponents, then simplify.

SOLUTION $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$

EXAMPLE 8 Write $(-2)^{-3}$ with positive exponents, then simplify.

SOLUTION $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$

EXAMPLE 9 Write $\left(\dfrac{3}{4}\right)^{-2}$ with positive exponents, then simplify.

SOLUTION $\left(\dfrac{3}{4}\right)^{-2} = \dfrac{1}{\left(\frac{3}{4}\right)^2} = \dfrac{1}{\frac{9}{16}} = \dfrac{16}{9}$

If we generalize the result in Example 9, we have the following extension of property 4,

$$\left(\frac{a}{b}\right)^{-r} = \left(\frac{b}{a}\right)^{r}$$

which indicates that raising a fraction to a negative power is equivalent to raising the reciprocal of the fraction to the positive power.

Property 3 indicated that exponents distribute over products. Since division is defined in terms of multiplication, we can expect that exponents will distribute over quotients as well. Property 5 is the formal statement of this fact.

Property 5 for Exponents

If a and b are any two real numbers with $b \neq 0$, and r is an integer, then

$$\left(\frac{a}{b}\right)^{r} = \frac{a^r}{b^r}$$

Proof of Property 5

$$\left(\frac{a}{b}\right)^r = \underbrace{\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\cdots\left(\frac{a}{b}\right)}_{r \text{ factors}}$$

$$= \frac{a \cdot a \cdot a \cdots a}{b \cdot b \cdot b \cdots b} \begin{array}{l} \leftarrow r \text{ factors} \\ \leftarrow r \text{ factors} \end{array}$$

$$= \frac{a^r}{b^r}$$

Since multiplication with the same base resulted in addition of exponents, it seems reasonable to expect division with the same base to result in subtraction of exponents.

Property 6 for Exponents

If a is any nonzero real number, and r and s are any two integers, then

$$\frac{a^r}{a^s} = a^{r-s}$$

Notice again that we have specified r and s to be any integers. Our definition of negative exponents is such that the properties of exponents hold for all integer exponents, whether positive or negative integers. Here is proof of property 6.

Proof of Property 6

Our proof is centered on the fact that division by a number is equivalent to multiplication by the reciprocal of the number.

$$\frac{a^r}{a^s} = a^r \cdot \frac{1}{a^s} \quad \text{Dividing by } a^s \text{ is equivalent to multiplying by } \frac{1}{a^s}$$

$$= a^r a^{-s} \quad \text{Property 4}$$

$$= a^{r+(-s)} \quad \text{Property 1}$$

$$= a^{r-s} \quad \text{Definition of subtraction}$$

EXAMPLE 10 Apply property 6 to each expression, and then simplify the result. All answers that contain exponents should contain positive exponents only.

SOLUTION

a. $\dfrac{2^8}{2^3} = 2^{8-3} = 2^5 = 32$

b. $\dfrac{x^2}{x^{18}} = x^{2-18} = x^{-16} = \dfrac{1}{x^{16}}$

c. $\dfrac{a^6}{a^{-8}} = a^{6-(-8)} = a^{14}$

d. $\dfrac{m^{-5}}{m^{-7}} = m^{-5-(-7)} = m^2$

Let's complete our list of properties by looking at how the numbers 0 and 1 behave when used as exponents.

We can use the original definition for exponents when the number 1 is used as an exponent.

$$a^1 = \underbrace{a}_{1 \text{ factor}}$$

For 0 as an exponent, consider the expression $\frac{3^4}{3^4}$. Since $3^4 = 81$, we have

$$\frac{3^4}{3^4} = \frac{81}{81} = 1$$

However, because we have the quotient of two expressions with the same base, we can subtract exponents.

$$\frac{3^4}{3^4} = 3^{4-4} = 3^0$$

Hence, 3^0 must be the same as 1.

Summarizing these results, we have our last property for exponents.

Property 7 for Exponents

If a is any real number, then

$$a^1 = a$$

and

$$a^0 = 1 \qquad \text{(as long as } a \neq 0\text{)}$$

EXAMPLE 11 Simplify.

SOLUTION **a.** $(2x^2y^4)^0 = 1$

b. $(2x^2y^4)^1 = 2x^2y^4$

Here are some examples that use many of the properties of exponents. There are a number of ways to proceed on problems like these. You should use the method that works best for you.

EXAMPLE 12 Simplify $\dfrac{(x^3)^{-2}(x^4)^5}{(x^{-2})^7}$.

SOLUTION

$$\begin{aligned} \frac{(x^3)^{-2}(x^4)^5}{(x^{-2})^7} &= \frac{x^{-6}x^{20}}{x^{-14}} && \text{Property 2} \\ &= \frac{x^{14}}{x^{-14}} && \text{Property 1} \\ &= x^{28} && \text{Property 6: } x^{14-(-14)} = x^{28} \end{aligned}$$

EXAMPLE 13 Simplify $\dfrac{6a^5b^{-6}}{12a^3b^{-9}}$.

SOLUTION

$$\begin{aligned} \frac{6a^5b^{-6}}{12a^3b^{-9}} &= \frac{6}{12} \cdot \frac{a^5}{a^3} \cdot \frac{b^{-6}}{b^{-9}} && \text{Write as separate fractions} \\ &= \frac{1}{2}a^2b^3 && \text{Property 6} \end{aligned}$$

Note: This last answer also can be written as $\frac{a^2b^3}{2}$. Either answer is correct.

EXAMPLE 14 Simplify $\dfrac{(4x^{-5}y^3)^2}{(x^4y^{-6})^{-3}}$.

SOLUTION

$$\begin{aligned} \frac{(4x^{-5}y^3)^2}{(x^4y^{-6})^{-3}} &= \frac{16x^{-10}y^6}{x^{-12}y^{18}} && \text{Properties 2 and 3} \\ &= 16x^2y^{-12} && \text{Property 6} \\ &= 16x^2 \cdot \frac{1}{y^{12}} && \text{Property 4} \\ &= \frac{16x^2}{y^{12}} && \text{Multiplication} \end{aligned}$$

B Scientific Notation

Scientific notation is a way in which to write very large or very small numbers in a more manageable form. Here is the definition.

Definition

A number is written in **scientific notation** if it is written as the product of a number between 1 and 10 and an integer power of 10. A number written in scientific notation has the form

$$n \times 10^r$$

where $1 \le n < 10$ and $r =$ an integer.

EXAMPLE 15 Write 376,000 in scientific notation.

SOLUTION We must rewrite 376,000 as the product of a number between 1 and 10 and a power of 10. To do so, we move the decimal point five places to the left so that it appears between the 3 and the 7. Then we multiply this number by 10^5. The number that results has the same value as our original number and is written in scientific notation.

$$376{,}000 = 3.76 \times 10^5$$

Move five places

Decimal point originally here

Keep track of the five places we moved the decimal point

If a number written in expanded form is greater than or equal to 10, then when the number is written in scientific notation the exponent on 10 will be positive. A number that is less than 1 will have a negative exponent when written in scientific notation.

EXAMPLE 16 Write 4.52×10^3 in expanded form.

SOLUTION Since 10^3 is 1,000, we can think of this as simply a multiplication problem; that is,

$$4.52 \times 10^3 = 4.52 \times 1{,}000 = 4{,}520$$

However, we can think of the exponent 3 as indicating the number of places we need to move the decimal point to write our number in expanded form. Since our exponent is positive 3, we move the decimal point three places to the right.

$$4.52 \times 10^3 = 4{,}520$$

The following table lists some additional examples of numbers written in expanded form and in scientific notation. In each case, note the relationship between the number of places the decimal point is moved and the exponent on 10.

Number Written in Expanded Form		Number Written Again in Scientific Notation
376,000	=	3.76×10^5
49,500	=	4.95×10^4
3,200	=	3.2×10^3
591	=	5.91×10^2
46	=	4.6×10^1
8	=	8×10^0
0.47	=	4.7×10^{-1}
0.093	=	9.3×10^{-2}
0.00688	=	6.88×10^{-3}
0.0002	=	2×10^{-4}
0.000098	=	9.8×10^{-5}

Calculator Note Many calculators have a key that allows you to enter numbers in scientific notation. The key is labeled

EXP or **EE** or **SCI**

To enter the number 3.45×10^6, you first enter the decimal number, then press the scientific notation key, and finally enter the exponent.

3.45 **EXP** 6

We can use our properties of exponents to do arithmetic with numbers written in scientific notation. Here are some examples.

C Simplifying Expressions with Scientific Notation

EXAMPLE 17 Simplify each expression, and write all answers in scientific notation.

SOLUTION **a.** $(2 \times 10^8)(3 \times 10^{-3}) = (2)(3) \times (10^8)(10^{-3})$

$$= 6 \times 10^5$$

b. $\dfrac{4.8 \times 10^9}{2.4 \times 10^{-3}} = \dfrac{4.8}{2.4} \times \dfrac{10^9}{10^{-3}}$

$$= 2 \times 10^{9-(-3)}$$

$$= 2 \times 10^{12}$$

c. $\dfrac{(6.8 \times 10^5)(3.9 \times 10^{-7})}{7.8 \times 10^{-4}} = \dfrac{(6.8)(3.9)}{7.8} \times \dfrac{(10^5)(10^{-7})}{10^{-4}}$

$$= 3.4 \times 10^2$$

NOTE
Remember, on some calculators the scientific notation key may be labeled EE or SCI.

Calculator Note On a scientific calculator with a scientific notation key, you would use the following sequence of keys to do Example 17 (b):

4.8 **EXP** 9 ÷ 2.4 **EXP** 3 +/− =

Problem Set 5.1

Moving Toward Success

"Vision without action is a daydream. Action without vision is a nightmare."

—Japanese proverb

1. Why do you think it is important to create pictures in your head as you learn mathematics?
2. Why can it be helpful to sometimes read this book or your notes out loud as you study?

Evaluate each of the following.

1. 4^2

2. $(-4)^2$

3. -4^2

4. $-(-4)^2$

5. -0.3^3

6. $(-0.3)^3$

7. 2^5

8. 2^4

9. $\left(\frac{1}{2}\right)^3$

10. $\left(\frac{3}{4}\right)^2$

11. $-\left(\frac{5}{6}\right)^2$

12. $-\left(\frac{7}{8}\right)^2$

A Use the properties of exponents to simplify each of the following as much as possible. [Examples 1–6]

13. $x^5 \cdot x^4$

14. $x^6 \cdot x^3$

15. $(2^3)^2$

16. $(3^2)^2$

17. $\left(-\frac{2}{3}x^2\right)^3$

18. $\left(-\frac{3}{5}x^4\right)^3$

19. $-3a^2(2a^4)$

20. $5a^7(-4a^6)$

A Write each of the following with positive exponents. Then simplify as much as possible. [Examples 7–9]

21. 3^{-2}

22. $(-5)^{-2}$

23. $(-2)^{-5}$

24. 2^{-5}

25. $\left(\frac{3}{4}\right)^{-2}$

26. $\left(\frac{3}{5}\right)^{-2}$

27. $\left(\frac{1}{3}\right)^{-2} + \left(\frac{1}{2}\right)^{-3}$

28. $\left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{3}\right)^{-3}$

A Simplify each expression. Write all answers with positive exponents only. (Assume all variables are nonzero.)

29. $x^{-4}x^7$

30. $x^{-3}x^8$

31. $(a^2b^{-5})^3$

32. $(a^4b^{-3})^3$

33. $(5y^4)^{-3}(2y^{-2})^3$

34. $(3y^5)^{-2}(2y^{-4})^3$

35. $\left(\frac{1}{2}x^3\right)\left(\frac{2}{3}x^4\right)\left(\frac{3}{5}x^{-7}\right)$

36. $\left(\frac{1}{7}x^{-3}\right)\left(\frac{7}{8}x^{-5}\right)\left(\frac{8}{9}x^8\right)$

37. $(4a^5b^2)(2b^{-5}c^2)(3a^7c^4)$

38. $(3a^{-2}c^3)(5b^{-6}c^5)(4a^6b^{-2})$

39. $(2x^2y^{-5})^3(3x^{-4}y^2)^{-4}$

40. $(4x^{-4}y^9)^{-2}(5x^4y^{-3})^2$

A Use the properties of exponents to simplify each expression. Write all answers with positive exponents only. (Assume all variables are nonzero.) [Examples 10–14]

41. $\frac{x^{-1}}{x^9}$

42. $\frac{x^{-3}}{x^5}$

43. $\frac{a^4}{a^{-6}}$

44. $\frac{a^5}{a^{-2}}$

45. $\frac{t^{-10}}{t^{-4}}$

46. $\frac{t^{-8}}{t^{-5}}$

47. $\left(\frac{x^5}{x^3}\right)^6$

48. $\left(\frac{x^7}{x^4}\right)^5$

49. $\frac{(x^5)^6}{(x^3)^4}$

50. $\frac{(x^7)^3}{(x^4)^5}$

51. $\frac{(x^{-2})^3(x^3)^{-2}}{x^{10}}$

52. $\frac{(x^{-4})^3(x^3)^{-4}}{x^{10}}$

53. $\frac{5a^8b^3}{20a^5b^{-4}}$

54. $\frac{7a^6b^{-2}}{21a^2b^{-5}}$

55. $\frac{(3x^{-2}y^8)^4}{(9x^4y^{-3})^2}$

56. $\frac{(6x^{-3}y^{-5})^2}{(3x^{-4}y^{-3})^4}$

57. $\left(\frac{8x^2y}{4x^4y^{-3}}\right)^4$

58. $\left(\frac{5x^4y^5}{10xy^{-2}}\right)^3$

59. $\left(\frac{x^{-5}y^2}{x^{-3}y^5}\right)^{-2}$

60. $\left(\frac{x^{-8}y^{-3}}{x^{-5}y^6}\right)^{-1}$

Write each expression as a perfect square.

61. $x^4y^2 = (\quad)^2$

62. $x^8y^6 = (\quad)^2$

63. $9a^2b^4 = (\quad)^2$

64. $225x^6y^{12} = (\quad)^2$

Write each expression as a perfect cube.

65. $8a^3 = (\quad)^3$

66. $27b^3 = (\quad)^3$

67. $64x^3y^{12} = (\quad)^3$

68. $216x^{15}y^{21} = (\quad)^3$

69. Let $x = 2$ in each of the following expressions and simplify.

a. x^3x^2

b. $(x^3)^2$

c. x^5

d. x^6

70. Let $x = -1$ in each of the following expressions and simplify.

a. x^3x^4

b. $(x^3)^4$

c. x^7

d. x^{12}

71. Let $x = 2$ in each of the following expressions and simplify.

a. $\frac{x^5}{x^2}$

b. x^3

c. $\frac{x^2}{x^6}$

d. x^{-4}

72. Let $x = -1$ in each of the following expressions and simplify.

a. $\frac{x^{14}}{x^9}$

b. x^5

c. $\frac{x^{13}}{x^9}$

d. x^4

73. Write each expression as a perfect square.

a. $\frac{1}{49} = \left(\quad\right)^2$

b. $\frac{1}{121} = \left(\quad\right)^2$

c. $\frac{1}{4x^2} = \left(\quad\right)^2$

d. $\frac{1}{64x^4} = \left(\quad\right)^2$

74. Write each expression as a perfect cube.

a. $\frac{1}{125x^3} = \left(\quad\right)^3$

b. $\frac{1}{64y^{12}} = \left(\quad\right)^3$

c. $\frac{x^6}{216y^9} = \left(\quad\right)^3$

d. $\frac{8a^9}{27b^{15}} = \left(\quad\right)^3$

Simplify.

75. $2 \cdot 2^{n-1}$

76. $3 \cdot 3^{n-1}$

77. $\frac{ar^6}{ar^3}$

78. $\frac{ar^7}{ar^4}$

B Write each number in scientific notation. [Example 15]

79. 378,000

80. 3,780,000

81. 4,900

82. 490

83. 0.00037

84. 0.000037

85. 0.00495

86. 0.0495

B Write each number in expanded form. [Example 16]

87. 5.34×10^3

88. 5.34×10^2

89. 7.8×10^6

90. 7.8×10^4

91. 3.44×10^{-3}

92. 3.44×10^{-5}

93. 4.9×10^{-1}

94. 4.9×10^{-2}

C Use the properties of exponents to simplify each of the following expressions. Write all answers in scientific notation. [Example 17]

95. $(4 \times 10^{10})(2 \times 10^{-6})$

96. $(3 \times 10^{-12})(3 \times 10^4)$

97. $\frac{8 \times 10^{14}}{4 \times 10^5}$

98. $\frac{6 \times 10^8}{2 \times 10^3}$

99. $\frac{(5 \times 10^6)(4 \times 10^{-8})}{8 \times 10^4}$

100. $\frac{(6 \times 10^{-7})(3 \times 10^9)}{5 \times 10^6}$

Problems 101–110 are problems you will see later in the book.

Multiply.

101. $8x^3 \cdot 10y^6$

102. $5y^2 \cdot 4x^2$

103. $8x^3 \cdot 9y^3$

104. $4y^3 \cdot 3x^2$

105. $3x \cdot 5y$

106. $3xy \cdot 5z$

107. $4x^6y^6 \cdot 3x$

108. $16x^4y^4 \cdot 3y$

109. $27a^6c^3 \cdot 2b^2c$

110. $8a^3b^3 \cdot 5a^2b$

Divide.

111. $\dfrac{10x^5}{5x^2}$

112. $\dfrac{-15x^4}{5x^2}$

113. $\dfrac{20x^3}{5x^2}$

114. $\dfrac{25x^7}{-5x^2}$

115. $\dfrac{8x^3y^5}{-2x^2y}$

116. $\dfrac{-16x^2y^2}{-2x^2y}$

117. $\dfrac{4x^4y^3}{-2x^2y}$

118. $\dfrac{10a^4b^2}{4a^2b^2}$

119. $\dfrac{7x^4y^3}{21xy^4}$

120. $\dfrac{12a^6b^4c}{16a^4b^6c^4}$

121. $\dfrac{25x^5y^3}{10x^4y^2}$

122. $\dfrac{27a^6b^5}{6a^3b^7}$

Applying the Concepts

123. Google Earth This Google Earth image is of the Luxor Hotel in Las Vegas, Nevada. The casino has a square base with sides of 525 feet. What is the area of the casino floor?

124. Google Earth This is a three dimensional model created by Google Earth of the Louvre Museum in Paris, France. The pyramid that dominates the Napoleon Courtyard has a square base with sides of 35.50 meters. What is the area of the base of the pyramid?

125. Our Galaxy The galaxy the Earth resides in is called the Milky Way galaxy. It is a spiral galaxy that contains approximately 200,000,000,000 stars (our Sun is one of them). Write this number in words and in scientific notation.


NASA

126. Fingerprints The FBI has been collecting fingerprint cards since 1924. Their collection has grown to over 200 million cards. They are digitizing the fingerprints. Each fingerprint card turns into about 10 MB of data. (A megabyte [MB] is $2^{20} \approx$ one million bytes.)

a. How many bytes of storage will they need?

b. A compression routine called the WSQ method will compress the bytes by ratio of 12.9 to 1. Approximately how many bytes of storage will the FBI need for the compressed data? (Hint: Divide by 12.9.)

127. Light Year A light year, the distance light travels in 1-year, is approximately 5.9×10^{12} miles. The Andromeda galaxy is approximately 1.7×10^6 light years from our galaxy. Find the distance in miles between our galaxy and the Andromeda galaxy.

128. Distance to the Sun The distance from the Earth to the sun is approximately 9.3×10^7 miles. If light travels 1.2×10^7 miles in 1 minute, how many minutes does it take the light from the sun to reach the Earth?

129. Cone Nebula The photograph was taken by the Hubble telescope in April 2002. The object in the photograph is called the *Cone Nebula.* The distance across the photograph is about 2.5 light-years, which is 14,664,240,000,000 miles. Round this number to the nearest trillion and then write the result in scientific notation.


NASA

130. Computer Science We all use the language of computers to indicate how much memory our computers hold or how much information we can put on a storage device such as a flash drive. Scientific notation gives us a way to compare the actual numbers associated with the words we use to describe data storage in computers. The smallest amount of data that a computer can hold is measured in bits. A byte is the next largest unit and is equal to 8, or 2^3, bits. Fill in the following table.

NUMBER OF BYTES		
Unit	**Exponential Form**	**Scientific Notation**
Kilobyte	$2^{10} = 1{,}024$	
Megabyte	$2^{20} \approx 1{,}048{,}000$	
Gigabyte	$2^{30} \approx 1{,}074{,}000{,}000$	
Terabyte	$2^{40} \approx 1{,}099{,}500{,}000{,}000$	

Getting Ready for the Next Section

Simplify.

131. $-4x + 9x$

132. $-6x - 2x$

133. $5x^2 + 3x^2$

134. $7x^2 + 3x^2$

135. $-8x^3 + 10x^3$

136. $4x^3 - 7x^3$

137. $2x + 3 - 2x - 8$

138. $9x - 4 - 9x - 10$

139. $-1(2x - 3)$

140. $-1(-3x + 1)$

141. $-3(-3x - 2)$

142. $-4(-5x + 3)$

143. $-500 + 27(100) - 0.1(100)^2$

144. $-500 + 27(170) - 0.1(170)^2$

Maintaining Your Skills

Solve each system by the addition method.

145. $4x + 3y = 10$
$2x + y = 4$

146. $3x - 5y = -2$
$2x - 3y = 1$

147. $4x + 5y = 5$
$\frac{6}{5}x + y = 2$

148. $4x + 2y = -2$
$\frac{1}{2}x + y = 0$

Solve each system by the substitution method.

149. $x + y = 3$
$y = x + 3$

150. $x + y = 6$
$y = x - 4$

151. $2x - 3y = -6$
$y = 3x - 5$

152. $7x - y = 24$
$x = 2y + 9$

Solve each equation using matricies.

153. $4x - 3y = 11$
$x = 2y + 4$

154. $5x - 4y = 3$
$y = 3x - 6$

155. $x - y + z = 2$
$-x + 2y - z = 0$
$2x - y + 2z = 6$

156. $2x - 3y + z = 16$
$-3x + 4y + 2z = -9$
$x - 2y - 3z = -5$

Extending the Concepts

Assume all variable exponents represent positive integers and simplify each expression.

157. $x^{m+2} \cdot x^{-2m} \cdot x^{m-5}$

158. $x^{m-4}x^{m+9}x^{-2m}$

159. $(y^m)^2(y^{-3})^m(y^{m+3})$

160. $(y^m)^{-4}(y^3)^m(y^{m+6})$

161. $\dfrac{x^{n+2}}{x^{n-3}}$

162. $\dfrac{x^{n-3}}{x^{n-7}}$

5.2 Polynomials, Sums, and Differences

OBJECTIVES

A Give the degree of a polynomial.

B Add and subtract polynomials.

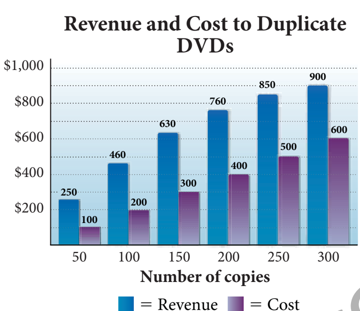
C Evaluate a polynomial for a given value of its variable.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. What is a monomial?
2. What is the degree of a polynomial?
3. Why do you have to consider similar terms when adding or subtracting polynomials?
4. Write a problem that contains a polynomial and then restate it using function notation.

Image copyright ©Taina Sohlman, 2009. Used under license from Shutterstock.com



The chart is from a company that duplicates DVDs. It shows the revenue and cost to duplicate a 30-minute DVD. From the chart you can see that 300 copies will bring in \$900 in revenue, with a cost of \$600. The profit is the difference between revenue and cost, or \$300.

The relationship between profit, revenue, and cost is one application of the polynomials we will study in this section. Let's begin with a definition that we will use to build polynomials.

Polynomials in General

Definition

A **term**, or **monomial**, is a constant or the product of a constant and one or more variables raised to whole-number exponents.

The following are monomials, or terms:

$$-16 \qquad 3x^2y \qquad -\frac{2}{5}a^3b^2c \qquad xy^2z$$

The numerical part of each monomial is called the *numerical coefficient,* or just *coefficient*. For the preceding terms, the coefficients are -16, 3, $-\frac{2}{5}$, and 1. Notice that the coefficient for xy^2z is understood to be 1.

Definition

A **polynomial** is any finite sum of terms. Because subtraction can be written in terms of addition, finite differences are also included in this definition.

The following are polynomials:

$$2x^2 - 6x + 3 \qquad -5x^2y + 2xy^2 \qquad 4a - 5b + 6c + 7d$$

Polynomials can be classified further according to the number of terms present. If a polynomial consists of two terms, it is said to be a *binomial.* If it has three terms, it is called a *trinomial.* And, as stated, a polynomial with only one term is said to be a *monomial.*

A Degree of a Polynomial

Definition

The **degree** of a polynomial with one variable is the highest power to which the variable is raised in any one term.

EXAMPLES

1. $6x^2 + 2x - 1$ — A trinomial of degree 2
2. $5x - 3$ — A binomial of degree 1
3. $7x^6 - 5x^3 + 2x - 4$ — A polynomial of degree 6
4. $-7x^4$ — A monomial of degree 4
5. 15 — A monomial of degree 0

Polynomials in one variable are usually written in decreasing powers of the variable. When this is the case, the coefficient of the first term is called the *leading coefficient.* In Example 1, the leading coefficient is 6. In Example 2, it is 5. The leading coefficient in Example 3 is 7.

Definition

Two or more terms that differ only in their numerical coefficients are called **similar**, or **like**, terms. Since similar terms can differ only in their coefficients, they have identical variable parts.

B Addition and Subtraction of Polynomials

To add two polynomials, we simply apply the commutative and associative properties to group similar terms together and then use the distributive property as we have in the following example.

NOTE
In practice it is not necessary to show all the steps shown in Example 6. It is important to understand that addition of polynomials is equivalent to combining similar terms.

EXAMPLE 6 Add $5x^2 - 4x + 2$ and $3x^2 + 9x - 6$.

SOLUTION $(5x^2 - 4x + 2) + (3x^2 + 9x - 6)$

$= (5x^2 + 3x^2) + (-4x + 9x) + (2 - 6)$ — Commutative and associative properties

$$= (5 + 3)x^2 + (-4 + 9)x + (2 - 6)$$ Distributive property

$$= 8x^2 + 5x + (-4)$$

$$= 8x^2 + 5x - 4$$

EXAMPLE 7 Find the sum of $-8x^3 + 7x^2 - 6x + 5$ and $10x^3 + 3x^2 - 2x - 6$.

SOLUTION We can add the two polynomials using the method in Example 6, or we can arrange similar terms in columns and add vertically. Using the column method, we have

$$\begin{array}{r} -8x^3 + 7x^2 - 6x + 5 \\ 10x^3 + 3x^2 - 2x - 6 \\ \hline 2x^3 + 10x^2 - 8x - 1 \end{array}$$

To find the difference of two polynomials, we need to use the fact that the opposite of a sum is the sum of the opposites; that is,

$$-(a + b) = -a + (-b)$$

One way to remember this is to observe that $-(a + b)$ is equivalent to $-1(a + b) = (-1)a + (-1)b = -a + (-b)$.

If a negative sign directly precedes the parentheses surrounding a polynomial, we may remove the parentheses and the preceding negative sign by changing the sign of each term within the parentheses. For example,

$$\begin{aligned} -(3x + 4) &= -3x + (-4) = -3x - 4 \\ -(5x^2 - 6x + 9) &= -5x^2 + 6x - 9 \\ -(-x^2 + 7x - 3) &= x^2 - 7x + 3 \end{aligned}$$

EXAMPLE 8 Subtract $(9x^2 - 3x + 5) - (4x^2 + 2x - 3)$.

SOLUTION We subtract by adding the opposite of each term in the polynomial that follows the subtraction sign.

$$(9x^2 - 3x + 5) - (4x^2 + 2x - 3)$$

$$= 9x^2 - 3x + 5 + (-4x^2) + (-2x) + 3$$ The opposite of a sum is the sum of the opposites

$$= (9x^2 - 4x^2) + (-3x - 2x) + (5 + 3)$$ Commutative and associative properties

$$= 5x^2 - 5x + 8$$ Combine similar terms

EXAMPLE 9 Subtract $4x^2 - 9x + 1$ from $-3x^2 + 5x - 2$.

SOLUTION Again, to subtract, we add the opposite.

$$\begin{aligned} &(-3x^2 + 5x - 2) - (4x^2 - 9x + 1) \\ &\quad = -3x^2 + 5x - 2 - 4x^2 + 9x - 1 \\ &\quad = (-3x^2 - 4x^2) + (5x + 9x) + (-2 - 1) \\ &\quad = -7x^2 + 14x - 3 \end{aligned}$$

EXAMPLE 10 Simplify $4x - 3[2 - (3x + 4)]$.

SOLUTION Removing the innermost parentheses first, we have

$$4x - 3[2 - (3x + 4)] = 4x - 3(2 - 3x - 4)$$

$$= 4x - 3(-3x - 2)$$
$$= 4x + 9x + 6$$
$$= 13x + 6$$

EXAMPLE 11 Simplify $(2x + 3) - [(3x + 1) - (x - 7)]$.

SOLUTION
$$\begin{aligned}(2x + 3) - [(3x + 1) - (x - 7)] &= (2x + 3) - (3x + 1 - x + 7)\\ &= (2x + 3) - (2x + 8)\\ &= 2x + 3 - 2x - 8\\ &= -5\end{aligned}$$

C Evaluating Polynomials

In the example that follows, we will find the value of a polynomial for a given value of the variable.

EXAMPLE 12 Find the value of $5x^3 - 3x^2 + 4x - 5$ when x is 2.

SOLUTION We begin by substituting 2 for x in the original polynomial:

$$\begin{aligned}&\text{When} \to & x &= 2\\ &\text{the polynomial} \to & &5x^3 - 3x^2 + 4x - 5\\ &\text{becomes} \to & &5 \cdot 2^3 - 3 \cdot 2^2 + 4 \cdot 2 - 5 = 5 \cdot 8 - 3 \cdot 4 + 4 \cdot 2 - 5\\ & & &= 40 - 12 + 8 - 5\\ & & &= 31\end{aligned}$$

Polynomials and Function Notation

Example 12 can be restated using function notation by calling the polynomial $P(x)$ and asking for $P(2)$. The solution would look like this:

$$\begin{aligned}\text{If} \quad P(x) &= 5x^3 - 3x^2 + 4x - 5\\ \text{then} \quad P(2) &= 5 \cdot 2^3 - 3 \cdot 2^2 + 4 \cdot 2 - 5\\ &= 31\end{aligned}$$

Our next example is stated in terms of function notation.

Three functions that occur very frequently in business and economics classes are profit, revenue, and cost functions. If a company manufactures and sells x items, then the revenue $R(x)$ is the total amount of money obtained by selling all x items. The cost $C(x)$ is the total amount of money it costs the company to manufacture the x items. The profit $P(x)$ obtained by selling all x items is the difference between the revenue and the cost and is given by the equation

$$P(x) = R(x) - C(x)$$

EXAMPLE 13 A company produces and sells copies of an accounting program for home computers. The total weekly cost (in dollars) to produce x copies of the program is $C(x) = 8x + 500$. Find its weekly profit if the total revenue obtained from selling all x programs is $R(x) = 35x - 0.1x^2$. How much profit will the company make if it produces and sells 100 programs a week? That is, find $P(100)$.

SOLUTION Using the equation $P(x) = R(x) - C(x)$ and the information given in the problem, we have

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 35x - 0.1x^2 - (8x + 500) \\ &= 35x - 0.1x^2 - 8x - 500 \\ &= -500 + 27x - 0.1x^2 \end{aligned}$$

If the company produces and sells 100 copies of the program, its weekly profit will be

$$\begin{aligned} P(100) &= -500 + 27(100) - 0.1(100)^2 \\ &= -500 + 27(100) - 0.1(10{,}000) \\ &= -500 + 2{,}700 - 1{,}000 \\ &= 1{,}200 \end{aligned}$$

The weekly profit is \$1,200.

Problem Set 5.2

Moving Toward Success

"To the man who only has a hammer in the toolkit, every problem looks like a nail."

—Abraham Maslow, 1908–1970, American psychologist

1. A mnemonic device is a mental tool that uses an acronym or a short verse to help a person remember. Do you think a mnemonic device is a helpful learning tool? Why?
2. Have you used a mnemonic device to help you study mathematics? If so, what was it? If not, create one for this section.

A Identify those of the following that are monomials, binomials, or trinomials. Give the degree of each, and name the leading coefficient. [Examples 1–5]

1. $5x^2 - 3x + 2$

2. $2x^2 + 4x - 1$

3. $3x - 5$

4. $5y + 3$

5. $8a^2 + 3a - 5$

6. $9a^2 - 8a - 4$

7. $4x^3 - 6x^2 + 5x - 3$

8. $9x^4 + 4x^3 - 2x^2 + x$

9. $-\frac{3}{4}$

10. -16

11. $4x - 5 + 6x^3$

12. $9x + 2 + 3x^3$

B Simplify each of the following by combining similar terms. [Examples 6–9]

13. $(4x + 2) + (3x - 1)$

14. $(8x - 5) + (-5x + 4)$

15. $2x^2 - 3x + 10x - 15$

16. $6x^2 - 4x - 15x + 10$

17. $12a^2 + 8ab - 15ab - 10b^2$

18. $28a^2 - 8ab + 7ab - 2b^2$

19. $(5x^2 - 6x + 1) - (4x^2 + 7x - 2)$

20. $(11x^2 - 8x) - (4x^2 - 2x - 7)$

21. $\left(\frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{6}\right) - \left(\frac{1}{4}x^2 + \frac{7}{12}x\right) + \left(\frac{1}{3}x - \frac{1}{12}\right)$

22. $\left(\frac{2}{3}x^2 - \frac{1}{2}x\right) - \left(\frac{1}{4}x^2 + \frac{1}{6}x\right) + \frac{1}{12} - \left(\frac{1}{2}x^2 + \frac{1}{4}\right)$

23. $(y^3 - 2y^2 - 3y + 4) - (2y^3 - y^2 + y - 3)$

24. $(8y^3 - 3y^2 + 7y + 2) - (-4y^3 + 6y^2 - 5y - 8)$

25. $(5x^3 - 4x^2) - (3x + 4) + (5x^2 - 7) - (3x^3 + 6)$

26. $(x^3 - x) - (x^2 + x) + (x^3 - 1) - (-3x + 2)$

27. $\left(\frac{4}{7}x^2 - \frac{1}{7}xy + \frac{1}{14}y^2\right) - \left(\frac{1}{2}x^2 - \frac{2}{7}xy - \frac{9}{14}y^2\right)$

28. $\left(\frac{1}{5}x^2 - \frac{1}{2}xy + \frac{1}{10}y^2\right) - \left(-\frac{3}{10}x^2 + \frac{2}{5}xy - \frac{1}{2}y^2\right)$

29. $(3a^3 + 2a^2b + ab^2 - b^3) - (6a^3 - 4a^2b + 6ab^2 - b^3)$

30. $(a^3 - 3a^2b + 3ab^2 - b^3) - (a^3 + 3a^2b + 3ab^2 + b^3)$

31. Subtract $2x^2 - 4x$ from $2x^2 - 7x$.

32. Subtract $-3x + 6$ from $-3x + 9$.

33. Find the sum of $x^2 - 6xy + y^2$ and $2x^2 - 6xy - y^2$.

34. Find the sum of $9x^3 - 6x^2 + 2$ and $3x^2 - 5x + 4$.

35. Subtract $-8x^5 - 4x^3 + 6$ from $9x^5 - 4x^3 - 6$.

36. Subtract $4x^4 - 3x^3 - 2x^2$ from $2x^4 + 3x^3 + 4x^2$.

37. Find the sum of $11a^2 + 3ab + 2b^2$, $9a^2 - 2ab + b^2$, and $-6a^2 - 3ab + 5b^2$.

38. Find the sum of $a^2 - ab - b^2$, $a^2 + ab - b^2$, and $a^2 + 2ab + b^2$.

B Simplify each of the following. Begin by working within the innermost parentheses. [Examples 10–11]

39. $-[2 - (4 - x)]$

40. $-[-3 - (x - 6)]$

41. $-5[-(x - 3) - (x + 2)]$

42. $-6[(2x - 5) - 3(8x - 2)]$

43. $4x - 5[3 - (x - 4)]$

44. $x - 7[3x - (2 - x)]$

45. $-(3x - 4y) - [(4x + 2y) - (3x + 7y)]$

46. $(8x - y) - [-(2x + y) - (-3x - 6y)]$

47. $4a - \{3a + 2[a - 5(a + 1) + 4]\}$

48. $6a - \{-2a - 6[2a + 3(a - 1) - 6]\}$

C [Examples 12–13]

49. Find the value of $2x^2 - 3x - 4$ when x is 2.

50. Find the value of $4x^2 + 3x - 2$ when x is -1.

51. If $P(x) = \frac{3}{2}x^2 - \frac{3}{4}x + 1$, find
   a. $P(12)$
   b. $P(-8)$

52. If $P(x) = \frac{2}{5}x^2 - \frac{1}{10}x + 2$, find
   a. $P(10)$
   b. $P(-10)$

53. If $Q(x) = x^3 - x^2 + x - 1$, find
   a. $Q(4)$
   b. $Q(-2)$

54. If $Q(x) = x^3 + x^2 + x - 1$, find
   a. $Q(5)$
   b. $Q(-2)$

55. If $R(x) = 11.5x - 0.05x^2$, find
   a. $R(10)$
   b. $R(-10)$

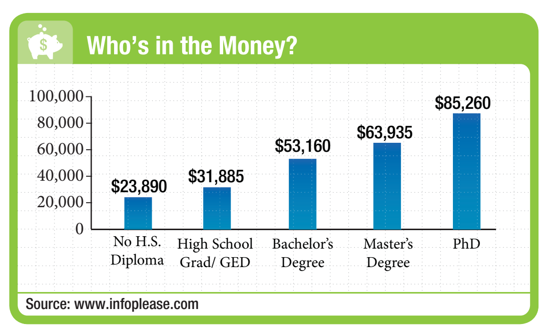
56. If $R(x) = 11.5x - 0.01x^2$, find
   a. $R(10)$
   b. $R(-10)$

57. If $P(x) = 600 + 1{,}000x - 100x^2$, find
   a. $P(-4)$
   b. $P(4)$

58. If $P(x) = 500 + 800x - 100x^2$, find
   a. $P(-6)$
   b. $P(8)$

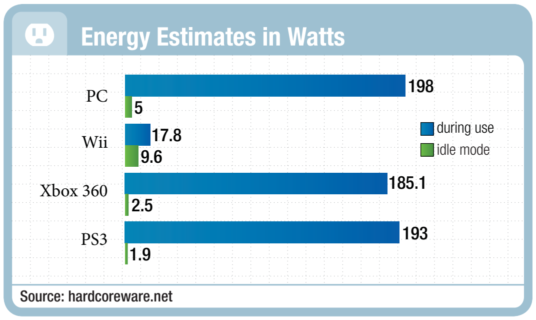
Applying the Concepts

59. **Education** The chart shows the average income for people with different levels of education. In a high school's graduating class, x students plan to get their Bachelor's Degree and y students plan to go on and get their Master's Degree. The next year, twice as many students plan to get their Bachelor's Degree than the year before but the number of students that plan to go on and get their Master's Degree stays the same as the year before. Write an expression that describes the total income of each year and then find the total income for both years in terms of x and y.



60. **Energy Usage** The chart shows how much energy is used by different gaming systems. If twice as many Wiis are being played as PS3s and one PC is being played, write an expression that describes the energy

played in this situation. Then find the value if 2 PS3s were played.



C Problems 61–66 may be solved using a graphing calculator. [Example 13]

61. Height of an Object If an object is thrown straight up into the air with a velocity of 128 feet/second, then its height $h(t)$ above the ground t seconds later is given by the formula

$$h(t) = -16t^2 + 128t$$

Find the height after 3 seconds and after 5 seconds. [Find $h(3)$ and $h(5)$.]

62. Height of an Object The formula for the height of an object that has been thrown straight up with a velocity of 64 feet/second is

$$h(t) = -16t^2 + 64t$$

Find the height after 1 second and after 3 seconds. [Find $h(1)$ and $h(3)$.]

63. Profits The total cost (in dollars) for a company to manufacture and sell x items per week is $C(x) = 60x + 300$. If the revenue brought in by selling all x items is $R(x) = 100x - 0.5x^2$, find the weekly profit. How much profit will be made by producing and selling 60 items each week?

64. Profits The total cost (in dollars) for a company to produce and sell x items per week is $C(x) = 200x + 1{,}600$. If the revenue brought in by selling all x items is $R(x) = 300x - 0.6x^2$, find the weekly profit. How much profit will be made by producing and selling 50 items each week?

65. Profits Suppose it costs a company selling patterns $C(x) = 800 + 6.5x$ dollars to produce and sell x patterns a month. If the revenue obtained by selling x patterns is $R(x) = 10x - 0.002x^2$, what is the profit equation? How much profit will be made if 1,000 patterns are produced and sold in May?

66. Profits Suppose a company manufactures and sells x picture frames each month with a total cost of $C(x) = 1{,}200 + 3.5x$ dollars. If the revenue obtained by selling x frames is $R(x) = 9x - 0.003x^2$, find the profit equation. How much profit will be made if 1,000 frames are manufactured and sold in June?

Getting Ready for the Next Section

Simplify.

67. $2x^2 - 3x + 10x - 15$

68. $12a^2 + 8ab - 15ab - 10b^2$

69. $(6x^3 - 2x^2y + 8xy^2) + (-9x^2y + 3xy^2 - 12y^3)$

70. $(3x^3 - 15x^2 + 18x) + (2x^2 - 10x + 12)$

71. $4x^3(-3x)$

72. $5x^2(-4x)$

73. $4x^3(5x^2)$

74. $5x^2(3x^2)$

75. $(a^3)^2$

76. $(a^4)^2$

77. $11.5(130) - 0.05(130)^2$

78. $-0.05(130)^2 + 9.5(130) - 200$

Maintaining Your Skills

Simplify each expression.

79. $-1(5 - x)$

80. $-1(a - b)$

81. $-1(7 - x)$

82. $-1(6 - y)$

83. $5\left(x - \frac{1}{5}\right)$

84. $7\left(x + \frac{1}{7}\right)$

85. $x\left(1 - \frac{1}{x}\right)$

86. $a\left(1 + \frac{1}{a}\right)$

87. $12\left(\frac{1}{4}x + \frac{2}{3}y\right)$

88. $20\left(\frac{2}{5}x + \frac{1}{4}y\right)$

Extending the Concepts

89. The graphs of two polynomial functions are given in Figures 1 and 2. Use the graphs to find the following.

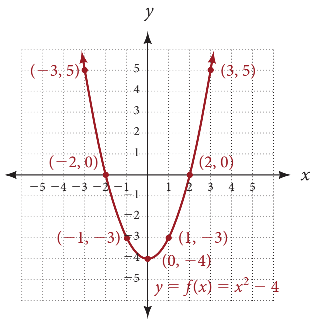


FIGURE 1

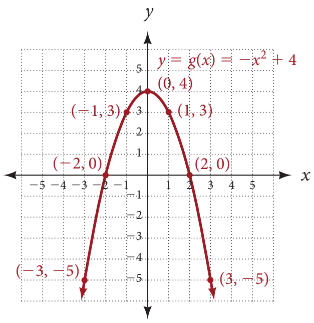


FIGURE 2

a. $f(-3)$ **b.** $f(0)$

c. $f(1)$ **d.** $g(-1)$

e. $g(0)$ **f.** $g(2)$

g. $f(g(2))$ **h.** $g(f(2))$

5.3 Multiplication of Polynomials

OBJECTIVES

A Multiply polynomials.

B Multiply binomials using the FOIL method.

C Find the square of a binomial.

D Multiply binomials to find the difference of two squares.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. In words, state the rule for multiplying polynomials.
2. How would you use the FOIL method to multiply two binomials?
3. Write a problem that demonstrates how to expand a binomial square.
4. Discuss the rule for a difference of two squares.

In the previous section, we found the relationship between profit, revenue, and cost to be

$$P(x) = R(x) - C(x)$$

Revenue itself can be broken down further by another formula common in the business world. The revenue obtained from selling all x items is the product of the number of items sold and the price per item; that is,

$$\text{Revenue} = (\text{number of items sold})(\text{price of each item})$$

$$R = xp$$

Many times, x and p are polynomials, which means that the expression xp is the product of two polynomials. In this section we learn how to multiply polynomials, and in so doing, increase our understanding of the equations and formulas that describe business applications.

A Multiplying Polynomials

EXAMPLE 1 Find the product of $4x^3$ and $5x^2 - 3x + 1$.

SOLUTION To multiply, we apply the distributive property:

$$\begin{aligned} &4x^3(5x^2 - 3x + 1) \\ &\quad = 4x^3(5x^2) + 4x^3(-3x) + 4x^3(1) && \text{Distributive property} \\ &\quad = 20x^5 - 12x^4 + 4x^3 \end{aligned}$$

Notice that we multiply coefficients and add exponents. ■

EXAMPLE 2 Multiply $2x - 3$ and $x + 5$.

SOLUTION Distributing the $2x - 3$ across the sum $x + 5$ gives us

$$\begin{aligned} &(2x - 3)(x + 5) \\ &\quad = (2x - 3)x + (2x - 3)5 && \text{Distributive property} \\ &\quad = 2x(x) + (-3)x + 2x(5) + (-3)5 && \text{Distributive property} \\ &\quad = 2x^2 - 3x + 10x - 15 \\ &\quad = 2x^2 + 7x - 15 && \text{Combine like terms} \end{aligned}$$

Notice the third line in this example. It consists of all possible products of terms in the first binomial and those of the second binomial. We can generalize this into a rule for multiplying two polynomials. ■

Rule

To multiply two polynomials, multiply each term in the first polynomial by each term in the second polynomial.

Multiplying polynomials can be accomplished by a method that looks very similar to long multiplication with whole numbers.

EXAMPLE 3 Multiply $(2x - 3y)$ and $(3x^2 - xy + 4y^2)$ vertically.

SOLUTION

$$\begin{array}{rrrrl} & 3x^2 - & xy + & 4y^2 & \\ & & 2x - & 3y & \\ \hline 6x^3 - & 2x^2y + & 8xy^2 & & \text{Multiply } (3x^2 - xy + 4y^2) \text{ by } 2x \\ - & 9x^2y + & 3xy^2 - & 12y^3 & \text{Multiply } (3x^2 - xy + 4y^2) \text{ by } -3y \\ \hline 6x^3 - & 11x^2y + & 11xy^2 - & 12y^3 & \text{Add similar terms} \end{array}$$

■

NOTE

The vertical method of multiplying polynomials does not directly show the use of the distributive property. It is, however, very useful since it always gives the correct result and is easy to remember.

B Multiplying Binomials—The FOIL Method

Consider the product of $(2x - 5)$ and $(3x - 2)$. Distributing $(3x - 2)$ over $2x$ and -5, we have

$$\begin{aligned}(2x - 5)(3x - 2) &= (2x)(3x - 2) + (-5)(3x - 2)\\ &= (2x)(3x) + (2x)(-2) + (-5)(3x) + (-5)(-2)\\ &= 6x^2 - 4x - 15x + 10\\ &= 6x^2 - 19x + 10\end{aligned}$$

> **NOTE**
> The FOIL method does not show the properties used in multiplying two binomials. It is simply a way of finding products of binomials quickly. Remember, the FOIL method applies only to products of two binomials. The vertical method applies to all products of polynomials with two or more terms.

Looking closely at the second and third lines, we notice the following:

1. $6x^2$ comes from multiplying the *first* terms in each binomial:

$$(2x - 5)(3x - 2) \qquad 2x(3x) = 6x^2 \qquad \textit{First terms}$$

2. $-4x$ comes from multiplying the *outside* terms in the product:

$$(2x - 5)(3x - 2) \qquad 2x(-2) = -4x \qquad \textit{Outside terms}$$

3. $-15x$ comes from multiplying the *inside* terms in the product:

$$(2x - 5)(3x - 2) \qquad -5(3x) = -15x \qquad \textit{Inside terms}$$

4. 10 comes from multiplying the *last* two terms in the product:

$$(2x - 5)(3x - 2) \qquad -5(-2) = 10 \qquad \textit{Last terms}$$

Once we know where the terms in the answer come from, we can reduce the number of steps used in finding the product:

$$(2x - 5)(3x - 2) = \underset{\text{First}}{6x^2} \underset{\text{Outside}}{- 4x} \underset{\text{Inside}}{- 15x} + \underset{\text{Last}}{10} = 6x^2 - 19x + 10$$

EXAMPLE 4 Multiply $(4a - 5b)(3a + 2b)$ using the FOIL method.

SOLUTION
$$\begin{aligned}(4a - 5b)(3a + 2b) &= \underset{\text{F}}{12a^2} + \underset{\text{O}}{8ab} - \underset{\text{I}}{15ab} - \underset{\text{L}}{10b^2}\\ &= 12a^2 - 7ab - 10b^2\end{aligned}$$

EXAMPLE 5 Multiply $(3 - 2t)(4 + 7t)$ using the FOIL method.

SOLUTION
$$\begin{aligned}(3 - 2t)(4 + 7t) &= \underset{\text{F}}{12} + \underset{\text{O}}{21t} - \underset{\text{I}}{8t} - \underset{\text{L}}{14t^2}\\ &= 12 + 13t - 14t^2\end{aligned}$$

EXAMPLE 6 Multiply $\left(2x + \frac{1}{2}\right)\left(4x - \frac{1}{2}\right)$ using the FOIL method.

SOLUTION
$$\left(2x + \frac{1}{2}\right)\left(4x - \frac{1}{2}\right) = \underset{\text{F}}{8x^2} \underset{\text{O}}{- x} + \underset{\text{I}}{2x} \underset{\text{L}}{- \frac{1}{4}} = 8x^2 + x - \frac{1}{4}$$

EXAMPLE 7 Multiply $(a^5 + 3)(a^5 - 7)$ using the FOIL method.

SOLUTION

$$
\begin{aligned}
(a^5 + 3)(a^5 - 7) &= \underset{\text{F}}{a^{10}} - \underset{\text{O}}{7a^5} + \underset{\text{I}}{3a^5} - \underset{\text{L}}{21} \\
&= a^{10} - 4a^5 - 21
\end{aligned}
$$

EXAMPLE 8 Multiply $(2x + 3)(5y - 4)$ using the FOIL method.

SOLUTION

$$
(2x + 3)(5y - 4) = \underset{\text{F}}{10xy} - \underset{\text{O}}{8x} + \underset{\text{I}}{15y} - \underset{\text{L}}{12}
$$

C The Square of a Binomial

EXAMPLE 9 Find $(4x - 6)^2$.

SOLUTION Applying the definition of exponents and then the FOIL method, we have

$$
\begin{aligned}
(4x - 6)^2 &= (4x - 6)(4x - 6) \\
&= \underset{\text{F}}{16x^2} - \underset{\text{O}}{24x} - \underset{\text{I}}{24x} + \underset{\text{L}}{36} \\
&= 16x^2 - 48x + 36
\end{aligned}
$$

This example is the square of a binomial. This type of product occurs frequently enough in algebra that we have special formulas for it. Here are the formulas for binomial squares:

$$
(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2
$$

$$
(a - b)^2 = (a - b)(a - b) = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2
$$

Observing the results in both cases, we have the following rule.

Rule

The square of a binomial is the sum of the square of the first term, twice the product of the two terms, and the square of the last term. Or:

$$
(a + b)^2 = \underset{\substack{\text{Square}\\\text{of}\\\text{first}\\\text{term}}}{a^2} + \underset{\substack{\text{Twice the}\\\text{product}\\\text{of the}\\\text{two terms}}}{2ab} + \underset{\substack{\text{Square}\\\text{of}\\\text{last}\\\text{term}}}{b^2}
$$

$$
(a - b)^2 = a^2 - 2ab + b^2
$$

EXAMPLES Use the preceding formulas to expand each binomial square.

10. $(x + 7)^2 = x^2 + 2(x)(7) + 7^2 = x^2 + 14x + 49$

11. $(3t - 5)^2 = (3t)^2 - 2(3t)(5) + 5^2 = 9t^2 - 30t + 25$

12. $(4x + 2y)^2 = (4x)^2 + 2(4x)(2y) + (2y)^2 = 16x^2 + 16xy + 4y^2$

13. $(5 - a^3)^2 = 5^2 - 2(5)(a^3) + (a^3)^2 = 25 - 10a^3 + a^6$

D Products Resulting in the Difference of Two Squares

Another frequently occurring kind of product is found when multiplying two binomials that differ only in the sign between their terms.

EXAMPLE 14 Multiply $(3x - 5)$ and $(3x + 5)$.

SOLUTION Applying the FOIL method, we have

$$(3x - 5)(3x + 5) = \underset{F}{9x^2} + \underset{O}{15x} - \underset{I}{15x} - \underset{L}{25} \qquad \text{Two middle terms add to 0}$$
$$= 9x^2 - 25$$

The outside and inside products in Example 14 are opposites and therefore add to 0. Here it is in general:

$$(a - b)(a + b) = a^2 + ab - ab - b^2 \qquad \text{Two middle terms add to 0}$$
$$= a^2 - b^2$$

Rule

To multiply two binomials that differ only in the sign between their two terms, simply subtract the square of the second term from the square of the first term:

$$(a + b)(a - b) = a^2 - b^2$$

The expression $a^2 - b^2$ is called the *difference of two squares.*

EXAMPLES Find the following products.

15. $(x - 5)(x + 5) = x^2 - 25$
16. $(2a - 3)(2a + 3) = 4a^2 - 9$
17. $(x^2 + 4)(x^2 - 4) = x^4 - 16$
18. $(x^3 - 2a)(x^3 + 2a) = x^6 - 4a^2$

More About Function Notation

From the introduction to this chapter, we know that the revenue obtained from selling x items at p dollars per item is

$$R = \text{Revenue} = xp \qquad \text{(The number of items} \times \text{price per item)}$$

For example, if a store sells 100 items at \$4.50 per item, the revenue is $100(4.50) = \$450$. If we have an equation that gives the relationship between x and p, then we can write the revenue in terms of x or in terms of p. With function notation, we would write the revenue as either $R(x)$ or $R(p)$, where

$R(x)$ is the revenue function that gives the revenue R in terms of the number of items x.

$R(p)$ is the revenue function that gives the revenue R in terms of the price per item p.

With function notation we can see exactly which variables we want our formulas written in terms of.

In the next two examples, we will use function notation to combine a number of problems we have worked previously.

EXAMPLE 19 A company manufactures and sells DVDs. They find that they can sell x DVDs each day at p dollars per disc, according to the equation $x = 230 - 20p$. Find $R(x)$ and $R(p)$.

SOLUTION The notation $R(p)$ tells us we are to write the revenue equation in terms of the variable p. To do so, we use the formula $R(p) = xp$ and substitute $230 - 20p$ for x to obtain

$$R(p) = xp = (230 - 20p)p = 230p - 20p^2$$

The notation $R(x)$ indicates that we are to write the revenue in terms of the variable x. We need to solve the equation $x = 230 - 20p$ for p. Let's begin by interchanging the two sides of the equation:

$$\begin{aligned} 230 - 20p &= x \\ -20p &= -230 + x && \text{Add } -230 \text{ to each side} \\ p &= \frac{-230 + x}{-20} && \text{Divide each side by } -20 \\ p &= 11.5 - 0.05x && \frac{230}{20} = 11.5 \text{ and } \frac{1}{20} = 0.05 \end{aligned}$$

Now we can find $R(x)$ by substituting $11.5 - 0.05x$ for p in the formula $R(x) = xp$:

$$R(x) = xp = x(11.5 - 0.05x) = 11.5x - 0.05x^2$$

Our two revenue functions are actually equivalent. To offer some justification for this, suppose that the company decides to sell each disc for \$5. The equation $x = 230 - 20p$ indicates that, at \$5 per disc, they will sell $x = 230 - 20(5) = 230 - 100 = 130$ discs per day. To find the revenue from selling the discs for \$5 each, we use $R(p)$ with $p = 5$:

$$\begin{aligned} \text{If} \quad p &= 5 \\ \text{then} \quad R(p) &= R(5) \\ &= 230(5) - 20(5)^2 \\ &= 1{,}150 - 500 \\ &= \$650 \end{aligned}$$

However, to find the revenue from selling 130 discs, we use $R(x)$ with $x = 130$:

$$\begin{aligned} \text{If} \quad x &= 130 \\ \text{then} \quad R(x) &= R(130) \\ &= 11.5(130) - 0.05(130)^2 \\ &= 1{,}495 - 845 \\ &= \$650 \end{aligned}$$

EXAMPLE 20 Suppose the daily cost function for the DVDs in Example 19 is $C(x) = 200 + 2x$. Find the profit function $P(x)$ and then find $P(130)$.

SOLUTION Since profit is equal to the difference of the revenue and the cost, we have

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 11.5x - 0.05x^2 - (200 + 2x) \\ &= -0.05x^2 + 9.5x - 200 \end{aligned}$$

Notice that we used the formula for $R(x)$ from Example 19 instead of the formula for $R(p)$. We did so because we were asked to find $P(x)$, meaning we want the profit P only in terms of the variable x.

Next, we use the formula we just obtained to find $P(130)$:

$$\begin{aligned} P(130) &= -0.05(130)^2 + 9.5(130) - 200 \\ &= -0.05(16{,}900) + 9.5(130) - 200 \\ &= -845 + 1{,}235 - 200 \\ &= \$190 \end{aligned}$$

Because $P(130) = \$190$, the company will make a profit of \$190 per day by selling 130 discs per day.

USING TECHNOLOGY

Graphing Calculators: More about Example 20

We can visualize the three functions given in Example 20 if we set up the functions list and graphing window on our calculator this way:

$Y_1 = 11.5X - 0.05X^2$ — This gives the graph of $R(x)$

$Y_2 = 200 + 2X$ — This gives the graph of $C(x)$

$Y_3 = Y_1 - Y_2$ — This gives the graph of $P(x)$

Window: X from 0 to 250, Y from 0 to 750

The graphs in Figure 1 are similar to what you will obtain using the functions list and window shown here. Next, find the value of $P(x)$ when $R(x)$ and $C(x)$ intersect.

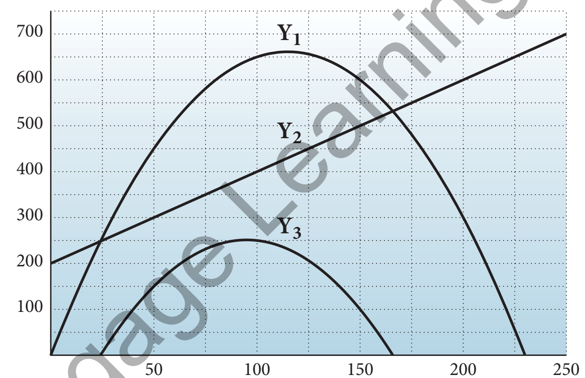


FIGURE 1

Problem Set 5.3

Moving Toward Success

"When what we are is what we want to be, that's happiness."

—Malcolm S. Forbes, 1917–1990, American publisher

1. What traits do you like in a study partner?
2. Have you possessed those traits when studying for this class? Why or why not?

A Multiply the following by applying the distributive property. [Examples 1–2]

1. $2x(6x^2 - 5x + 4)$
2. $-3x(5x^2 - 6x - 4)$
3. $-3a^2(a^3 - 6a^2 + 7)$
4. $4a^3(3a^2 - a + 1)$
5. $2a^2b(a^3 - ab + b^3)$
6. $5a^2b^2(8a^2 - 2ab + b^2)$

A Multiply the following vertically. [Example 3]

7. $(x - 5)(x + 3)$
8. $(x + 4)(x + 6)$
9. $(2x^2 - 3)(3x^2 - 5)$
10. $(3x^2 + 4)(2x^2 - 5)$
11. $(x + 3)(x^2 + 6x + 5)$
12. $(x - 2)(x^2 - 5x + 7)$
13. $(a - b)(a^2 + ab + b^2)$
14. $(a + b)(a^2 - ab + b^2)$
15. $(2x + y)(4x^2 - 2xy + y^2)$
16. $(x - 3y)(x^2 + 3xy + 9y^2)$
17. $(2a - 3b)(a^2 + ab + b^2)$
18. $(5a - 2b)(a^2 - ab - b^2)$

B Multiply the following using the FOIL method. [Examples 4–8]

19. $(x-2)(x+3)$

20. $(x+2)(x-3)$

21. $(2a+3)(3a+2)$

22. $(5a-4)(2a+1)$

23. $(5-3t)(4+2t)$

24. $(7-t)(6-3t)$

25. $(x^3+3)(x^3-5)$

26. $(x^3+4)(x^3-7)$

27. $(5x-6y)(4x+3y)$

28. $(6x-5y)(2x-3y)$

29. $\left(3t+\frac{1}{3}\right)\left(6t-\frac{2}{3}\right)$

30. $\left(5t-\frac{1}{5}\right)\left(10t+\frac{3}{5}\right)$

C **D** Find the following special products. [Examples 9–18]

31. $(5x+2y)^2$

32. $(3x-4y)^2$

33. $(5-3t^3)^2$

34. $(7-2t^4)^2$

35. $(2a+3b)(2a-3b)$

36. $(6a-1)(6a+1)$

37. $(3r^2+7s)(3r^2-7s)$

38. $(5r^2-2s)(5r^2+2s)$

39. $\left(y+\frac{3}{2}\right)^2$

40. $\left(y-\frac{7}{2}\right)^2$

41. $\left(a-\frac{1}{2}\right)^2$

42. $\left(a-\frac{5}{2}\right)^2$

43. $\left(x+\frac{1}{4}\right)^2$

44. $\left(x-\frac{3}{8}\right)^2$

45. $\left(t+\frac{1}{3}\right)^2$

46. $\left(t-\frac{2}{5}\right)^2$

47. $\left(\frac{1}{3}x-\frac{2}{5}\right)\left(\frac{1}{3}x+\frac{2}{5}\right)$

48. $\left(\frac{3}{4}x-\frac{1}{7}\right)\left(\frac{3}{4}x+\frac{1}{7}\right)$

Find the following products.

49. $(x-2)^3$

50. $(4x+1)^3$

51. $\left(x-\frac{1}{2}\right)^3$

52. $\left(x+\frac{1}{4}\right)^3$

53. $3(x-1)(x-2)(x-3)$

54. $2(x+1)(x+2)(x+3)$

55. $(b^2+8)(a^2+1)$

56. $(b^2+1)(a^4-5)$

57. $(x+1)^2+(x+2)^2+(x+3)^2$

58. $(x-1)^2+(x-2)^2+(x-3)^2$

59. $(2x+3)^2-(2x-3)^2$

60. $(x-3)^3-(x+3)^3$

Here are some problems you will see later in the book.

Simplify.

61. $(x+3)^2-2(x+3)-8$

62. $(x-2)^2-3(x-2)-10$

63. $(2a-3)^2-9(2a-3)+20$

64. $(3a-2)^2+2(3a-2)-3$

65. $2(4a+2)^2-3(4a+2)-20$

66. $6(2a+4)^2-(2a+4)-2$

67. Let $a=2$ and $b=3$, and evaluate each of the following expressions.

$a^4-b^4 \qquad (a-b)^4 \qquad (a^2+b^2)(a+b)(a-b)$

68. Let $a=2$ and $b=3$, and evaluate each of the following expressions.

$a^3+b^3 \qquad (a+b)^3 \qquad a^3+3a^2b+3ab^2+b^3$

Applying the Concepts [Examples 19–20]

Solar and Wind Energy The chart shows the cost to install either solar panels or a wind turbine. Use the chart to answer Problems 69 and 70.

Solar Versus Wind Energy Costs

Solar Energy Equipment Cost:		Wind Energy Equipment Cost:	
Modules	\$6200	Turbine	\$3300
Fixed Rack	\$1570	Tower	\$3000
Charge Controller	\$971	Cable	\$715
Cable	\$440		
TOTAL	**\$9181**	**TOTAL**	**\$7015**

Source: a Limited 2006

69. A homeowner is buying a certain number of solar panel modules. He is going to get a discount on each module that is equal to 25 dollars for each module he buys. Write an equation that describes this situation, then simplify and find the cost if he buys 3 modules.

70. A farmer is replacing several turbines in his field. He is going to get a discount on each turbine that is equal to 50 dollars for each turbine he buys. Write an expression that describes this situation, then simplify and find the cost if he replaces 5 turbines.

71. **Revenue** A store selling art supplies finds that it can sell x sketch pads per week at p dollars each, according to the formula $x = 900 - 300p$. Write formulas for $R(p)$ and $R(x)$. Then find the revenue obtained by selling the pads for \$1.60 each.

72. **Revenue** A company selling CDs finds that it can sell x CDs per day at p dollars per CD, according to the formula $x = 800 - 100p$. Write formulas for $R(p)$ and $R(x)$. Then find the revenue obtained by selling the CDs for \$3.80 each.

73. **Revenue** A company sells an inexpensive accounting program for home computers. If it can sell x programs per week at p dollars per program, according to the formula $x = 350 - 10p$, find formulas for $R(p)$ and $R(x)$. How much will the weekly revenue be if it sells 65 programs?

74. **Revenue** A company sells boxes of greeting cards through the mail. It finds that it can sell x boxes of cards each week at p dollars per box, according to the formula $x = 1{,}475 - 250p$. Write formulas for $R(p)$ and $R(x)$. What revenue will it bring in each week if it sells 200 boxes of cards?

75. **Profit** If the cost to produce the x programs in Problem 73 is $C(x) = 5x + 500$, find $P(x)$ and $P(60)$.

76. **Profit** If the cost to produce the x CDs in Problem 72 is $C(x) = 2x + 200$, find $P(x)$ and $P(40)$.

77. **Interest** If you deposit \$100 in an account with an interest rate r that is compounded annually, then the amount of money in that account at the end of 4 years is given by the formula $A = 100(1 + r)^4$. Expand the right side of this formula.

78. **Interest** If you deposit P dollars in an account with an annual interest rate r that is compounded twice a year, then at the end of a year the amount of money in that account is given by the formula

$$A = P\left(1 + \frac{r}{2}\right)^2$$

Expand the right side of this formula.

Getting Ready for the Next Section

79. $\dfrac{8a^3}{a}$

80. $\dfrac{-8a^2}{a}$

81. $\dfrac{-48a}{a}$

82. $\dfrac{-32a}{a}$

83. $\dfrac{16a^5b^4}{8a^2b^3}$

84. $\dfrac{12x^4y^5}{3x^3y^3}$

85. $\dfrac{-24a^5b^5}{8a^5b^3}$

86. $\dfrac{-15x^5y^3}{3x^3y^3}$

87. $\dfrac{x^3y^4}{-x^3}$

88. $\dfrac{x^2y^2}{-x^2}$

Maintaining Your Skills

Solve.

89. $\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ x + 2y - 3z &= -4 \end{aligned}$

90. $\begin{aligned} x + y + z &= 6 \\ x - y + 2z &= 7 \\ 2x - y - z &= 0 \end{aligned}$

91. $\begin{aligned} 3x + 4y &= 15 \\ 2x - 5z &= -3 \\ 4y - 3z &= 9 \end{aligned}$

92. $\begin{aligned} x + 3y &= 5 \\ 6y + z &= 12 \\ x - 2z &= -10 \end{aligned}$

Extending the Concepts

93. Multiply $(x + y - 4)(x + y + 5)$ by first writing it like this:

$$[(x + y) - 4][(x + y) + 5]$$

and then applying the FOIL method.

94. Multiply $(x - 5 - y)(x - 5 + y)$ by first writing it like this:

$$[(x - 5) - y][(x - 5) + y]$$

and then applying the FOIL method.

Assume n is a positive integer and multiply.

95. $(x^n - 2)(x^n - 3)$

96. $(x^{2n} + 3)(x^{2n} - 3)$

97. $(2x^n + 3)(5x^n - 1)$

98. $(4x^n - 3)(7x^n + 2)$

99. $(x^n + 5)^2$

100. $(x^n - 2)^2$

101. $(x^n + 1)(x^{2n} - x^n + 1)$

102. $(x^{3n} - 3)(x^{6n} + 3x^{3n} + 9)$

5.4 The Greatest Common Factor and Factoring by Grouping

OBJECTIVES

A Factor by factoring out the greatest common factor.

B Factor by grouping.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. What is the relationship between factoring and multiplication?
2. What is the greatest common factor for a polynomial?
3. After factoring a polynomial, how can you check your result?
4. When would you try to factor by grouping?

Picture a football player throwing a ball straight up in the air. The distance the ball is from the ground can be represented by the polynomial equations $d = -16t^2 + 80t$ where t is the time the ball is in the air. If we needed to solve this equation, our first step would be to factor out the greatest common factor.

In this section, we will begin our work factoring polynomials. In general, factoring is the reverse of multiplication. The diagram here illustrates the relationship between factoring and multiplication. Reading from left to right, we say the product of 3 and 7 is 21. Reading in the other direction, from right to left, we say 21 factors into 3 times 7. Or, 3 and 7 are factors of 21.

Multiplication

Factors $\rightarrow$ $3 \cdot 7 = 21$ $\leftarrow$ Product

Factoring

A Greatest Common Factor

Definition

The **greatest common factor** for a polynomial is the largest monomial that divides (is a factor of) each term of the polynomial.

NOTE
The term *largest monomial*, as used here, refers to the monomial with the largest integer exponents whose coefficient has the greatest absolute value.

The greatest common factor for the polynomial $25x^5 + 20x^4 - 30x^3$ is $5x^3$ since it is the largest monomial that is a factor of each term. We can apply the distributive property and write

$$\begin{aligned} 25x^5 + 20x^4 - 30x^3 &= 5x^3(5x^2) + 5x^3(4x) - 5x^3(6) \\ &= 5x^3(5x^2 + 4x - 6) \end{aligned}$$

The last line is written in factored form.

EXAMPLE 1 Factor the greatest common factor from $8a^3 - 8a^2 - 48a$.

SOLUTION The greatest common factor is $8a$. It is the largest monomial that divides each term of our polynomial. We can write each term in our polynomial as the product of $8a$ and another monomial. Then, we apply the distributive property to factor $8a$ from each term.

$$\begin{aligned} 8a^3 - 8a^2 - 48a &= 8a(a^2) - 8a(a) - 8a(6) \\ &= 8a(a^2 - a - 6) \end{aligned}$$

EXAMPLE 2 Factor the greatest common factor from $16a^5b^4 - 24a^2b^5 - 8a^3b^3$.

SOLUTION The largest monomial that divides each term is $8a^2b^3$. We write each term of the original polynomial in terms of $8a^2b^3$ and apply the distributive property to write the polynomial in factored form.

$$\begin{aligned} 16a^5b^4 - 24a^2b^5 - 8a^3b^3 &= 8a^2b^3(2a^3b) - 8a^2b^3(3b^2) - 8a^2b^3(a) \\ &= 8a^2b^3(2a^3b - 3b^2 - a) \end{aligned}$$

EXAMPLE 3 Factor the greatest common factor from $5x^2(a + b) - 6x(a + b) - 7(a + b)$.

SOLUTION The greatest common factor is $a + b$. Factoring it from each term, we have

$$5x^2(a + b) - 6x(a + b) - 7(a + b) = (a + b)(5x^2 - 6x - 7)$$

EXAMPLE 4 A company manufacturing DVDs finds that the total daily revenue for selling x DVDs is given by

$$R(x) = 11.5x - 0.05x^2$$

Factor x from each term on the right side of the equation to find the formula that gives the price p in terms of x.

SOLUTION We begin by factoring x from the right side of the equation.

If $\quad R(x) = 11.5x - 0.05x^2$
then $\quad R(x) = x(11.5 - 0.05x)$

Because R is always xp, the quantity in parentheses must be p. The price it should charge if it wants to sell x items per day is therefore

$$p = 11.5 - 0.05x$$

B Factoring by Grouping

The polynomial $5x + 5y + x^2 + xy$ can be factored by noticing that the first two terms have a 5 in common, whereas the last two have an x in common. Applying the distributive property, we have

$$5x + 5y + x^2 + xy = 5(x + y) + x(x + y)$$

This last expression can be thought of as having two terms, $5(x + y)$ and $x(x + y)$, each of which has a common factor $(x + y)$. We apply the distributive property again to factor $(x + y)$ from each term.

$$5(x + y) + x(x + y) = (x + y)(5 + x)$$

EXAMPLE 5 Factor $a^2b^2 + b^2 + 8a^2 + 8$.

SOLUTION The first two terms have b^2 in common; the last two have 8 in common.

$$\begin{aligned} a^2b^2 + b^2 + 8a^2 + 8 &= b^2(a^2 + 1) + 8(a^2 + 1) \\ &= (a^2 + 1)(b^2 + 8) \end{aligned}$$

EXAMPLE 6 Factor $15 - 5y^4 - 3x^3 + x^3y^4$.

SOLUTION Let's try factoring a 5 from the first two terms and an $-x^3$ from the last two terms.

$$\begin{aligned} 15 - 5y^4 - 3x^3 + x^3y^4 &= 5(3 - y^4) - x^3(3 - y^4) \\ &= (3 - y^4)(5 - x^3) \end{aligned}$$

EXAMPLE 7 Factor by grouping $x^3 + 2x^2 + 9x + 18$.

SOLUTION We begin by factoring x^2 from the first two terms and 9 from the second two terms.

$$\begin{aligned} x^3 + 2x^2 + 9x + 18 &= x^2(x + 2) + 9(x + 2) \\ &= (x + 2)(x^2 + 9) \end{aligned}$$

Problem Set 5.4

Moving Toward Success

"Kind words do not cost much. Yet they accomplish much."

—Blaise Pascal, 1623–1662, French mathematician

1. What would you do if your study partner is constantly complaining about the class or wanting to quit?
2. What are some other method you can use to prevent negative thoughts toward this class?

A Factor the greatest common factor from each of the following. [Examples 1–4]

1. $10x^3 - 15x^2$

2. $12x^5 + 18x^7$

3. $9y^6 + 18y^3$

4. $24y^4 - 8y^2$

5. $9a^2b - 6ab^2$

6. $30a^3b^4 + 20a^4b^3$

7. $21xy^4 + 7x^2y^2$

8. $14x^6y^3 - 6x^2y^4$

9. $3a^2 - 21a + 30$

10. $3a^2 - 3a - 6$

11. $4x^3 - 16x^2 - 20x$

12. $2x^3 - 14x^2 + 20x$

13. $10x^4y^2 + 20x^3y^3 - 30x^2y^4$

14. $6x^4y^2 + 18x^3y^3 - 24x^2y^4$

15. $-x^2y + xy^2 - x^2y^2$

16. $-x^3y^2 - x^2y^3 - x^2y^2$

17. $4x^3y^2z - 8x^2y^2z^2 + 6xy^2z^3$

18. $7x^4y^3z^2 - 21x^2y^2z^2 - 14x^2y^3z^4$

19. $20a^2b^2c^2 - 30ab^2c + 25a^2bc^2$

20. $8a^3bc^5 - 48a^2b^4c + 16ab^3c^5$
21. $5x(a - 2b) - 3y(a - 2b)$
22. $3a(x - y) - 7b(x - y)$
23. $3x^2(x + y)^2 - 6y^2(x + y)^2$
24. $10x^3(2x - 3y) - 15x^2(2x - 3y)$
25. $2x^2(x + 5) + 7x(x + 5) + 6(x + 5)$
26. $2x^2(x + 2) + 13x(x + 2) + 15(x + 2)$

B Factor each of the following by grouping. [Examples 5–7]

27. $3xy + 3y + 2ax + 2a$
28. $5xy^2 + 5y^2 + 3ax + 3a$
29. $x^2y + x + 3xy + 3$
30. $x^3y^3 + 2x^3 + 5x^2y^3 + 10x^2$
31. $3xy^2 - 6y^2 + 4x - 8$
32. $8x^2y - 4x^2 + 6y - 3$
33. $x^2 - ax - bx + ab$
34. $ax - x^2 - bx + ab$
35. $ab + 5a - b - 5$
36. $x^2 - xy - ax + ay$
37. $a^4b^2 + a^4 - 5b^2 - 5$
38. $2a^2 - a^2b - bc^2 + 2c^2$
39. $x^3 + 3x^2 - 4x - 12$
40. $x^3 + 5x^2 - 4x - 20$
41. $x^3 + 2x^2 - 25x - 50$
42. $x^3 + 4x^2 - 9x - 36$
43. $2x^3 + 3x^2 - 8x - 12$
44. $3x^3 + 2x^2 - 27x - 18$
45. $4x^3 + 12x^2 - 9x - 27$
46. $9x^3 + 18x^2 - 4x - 8$
47. The greatest common factor of the binomial $3x - 9$ is 3. The greatest common factor of the binomial $6x - 2$ is 2. What is the greatest common factor of their product, $(3x - 9)(6x - 2)$, when it has been multiplied out?
48. The greatest common factors of the binomials $5x - 10$ and $2x + 4$ are 5 and 2, respectively. What is the greatest common factor of their product, $(5x - 10)(2x + 4)$, when it has been multiplied out?

Applying the Concepts

49. **Investing** If P dollars are placed in a savings account in which the rate of interest r is compounded yearly, then at the end of one year the amount of money in the account can be written as $P + Pr$. At the end of two years the amount of money in the account is

$$P + Pr + (P + Pr)r$$

Use factoring by grouping to show that this last expression can be written as $P(1 + r)^2$.

50. **Investing** At the end of 3 years, the amount of money in the savings account in Problem 49 will be

$$P(1 + r)^2 + P(1 + r)^2r$$

Use factoring to show that this last expression can be written as $P(1 + r)^3$.

Use Example 4 as a guide in solving the next four problems.

51. **Price** A manufacturing company that produces prerecorded DVDs finds that the total daily revenue R for selling x items at p dollars per item is given by

$$R(x) = 11.5x - 0.05x^2$$

Factor x from each term on the right side of the equation to find the formula that gives the price p in terms of x. Then, use it to find the price they should charge if they want to sell 125 DVDs per day.

52. **Price** A company producing CDs for home computers finds that the total daily revenue for selling x items at p dollars per item is given by

$$R(x) = 8x - 0.01x^2$$

Use the fact that $R = xp$ and your knowledge of factoring to find a formula that gives the price p in terms of x. Then, use it to find the price they should charge if they want to sell 420 CDs per day.

53. Price The weekly revenue equation for a company selling an inexpensive accounting program for home computers is given by the equation

$$R(x) = 35x - 0.1x^2$$

where x is the number of programs they sell per week. What price p should they charge if they want to sell 65 programs per week?

54. Price The weekly revenue equation for a small mail-order company selling boxes of greeting cards is

$$R(x) = 5.9x - 0.004x^2$$

where x is the number of boxes they sell per week. What price p should they charge if they want to sell 200 boxes each week?

Getting Ready for the Next Section

Factor out the greatest common factor.

55. $3x^4 - 9x^3y - 18x^2y^2$

56. $5x^2 + 10x + 30$

57. $2x^2(x - 3) - 4x(x - 3) - 3(x - 3)$

58. $3x^2(x - 2) - 8x(x - 2) + 2(x - 2)$

Multiply.

59. $(x + 2)(3x - 1)$

60. $(x - 2)(3x + 1)$

61. $(x - 1)(3x - 2)$

62. $(x + 1)(3x + 2)$

63. $(x + 2)(x + 3)$

64. $(x - 2)(x - 3)$

65. $(2y + 5)(3y - 7)$

66. $(2y - 5)(3y + 7)$

67. $(4 - 3a)(5 - a)$

68. $(4 - 3a)(5 + a)$

Complete each table.

69.

Two Numbers a and b	Their Product ab	Their Sum $a + b$
$1, -24$		
$-1, 24$		
$2, -12$		
$-2, 12$		
$3, -8$		
$-3, 8$		
$4, -6$		
$-4, 6$		

70.

Two Numbers a and b	Their Product ab	Their Sum $a + b$
$1, -54$		
$-1, 54$		
$2, -27$		
$-2, 27$		
$3, -18$		
$-3, 18$		
$6, -9$		
$-6, 9$		

Maintaining Your Skills

Solve the following systems of equations.

71. $2x + 5y = 4$
$3x - 2y = -13$

72. $6x + 3y = -15$
$3x - 9y = 17$

73. $2x + 6y = -12$
$y = x - 14$

74. $4x - 6y = 2$
$x = 5y - 10$

5.5 Factoring Trinomials

OBJECTIVES

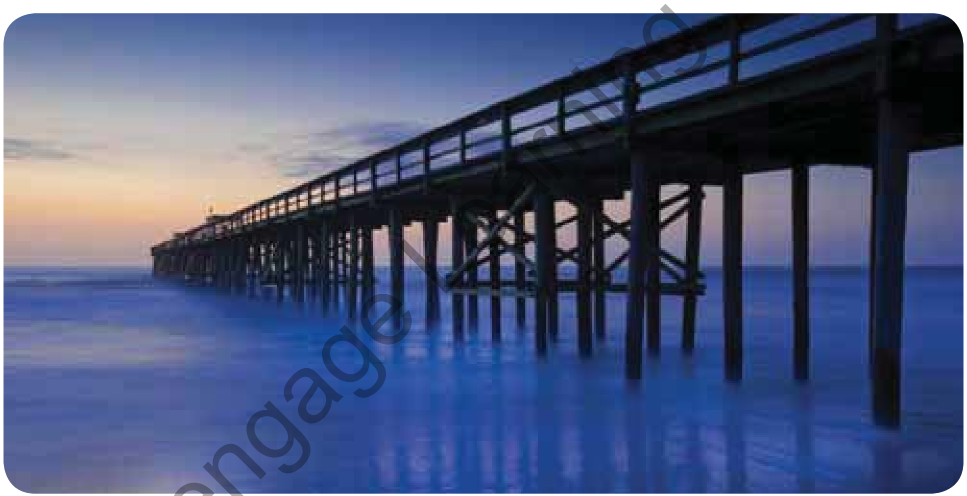
A Factor trinomials in which the leading coefficient is 1.

B Factor trinomials in which the leading coefficient is a number other than 1.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. What is a prime polynomial?
2. What is the first step in factoring a polynomial with a lead coefficient a number other than 1?
3. When would you factor a polynomial using trial and error?
4. Briefly explain the method of factoring a polynomial by grouping?



Suppose you drop a large shell off a pier into the ocean 32 feet below. The equation that gives the height of the shell at any time is $h = 32 + 16t - 16t^2$. How would you find the height of the shell at any given time? In this section, we will expand our work with factoring to look at how to factor several different kinds of trinomials.

A Factoring Trinomials with a Leading Coefficient of 1

Earlier in this chapter, we multiplied binomials.

$$(x - 2)(x + 3) = x^2 + x - 6$$

$$(x + 5)(x + 2) = x^2 + 7x + 10$$

In each case, the product of two binomials is a trinomial. The first term in the resulting trinomial is obtained by multiplying the first term in each binomial. The middle term comes from adding the product of the two inside terms with the product of the two outside terms. The last term is the product of the last terms in each binomial.

In general,

$$(x + a)(x + b) = x^2 + ax + bx + ab$$

$$= x^2 + (a + b)x + ab$$

Writing this as a factoring problem, we have

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

To factor a trinomial with a leading coefficient of 1, we simply find the two numbers a and b whose sum is the coefficient of the middle term and whose product is the constant term.

EXAMPLE 1 Factor $x^2 + 2x - 15$.

SOLUTION Since the leading coefficient is 1, we need two integers whose product is -15 and whose sum is 2. The integers are 5 and -3.

$$x^2 + 2x - 15 = (x + 5)(x - 3)$$

In the preceding example, we found factors of $x + 5$ and $x - 3$. These are the only two such factors for $x^2 + 2x - 15$. There is no other pair of binomials $x + a$ and $x + b$ whose product is $x^2 + 2x - 15$.

EXAMPLE 2 Factor $x^2 - xy - 12y^2$.

SOLUTION We need two expressions whose product is $-12y^2$ and whose sum is $-y$. The expressions are $-4y$ and $3y$.

$$x^2 - xy - 12y^2 = (x - 4y)(x + 3y)$$

Checking this result gives

$$\begin{aligned}(x - 4y)(x + 3y) &= x^2 + 3xy - 4xy - 12y^2\\ &= x^2 - xy - 12y^2\end{aligned}$$

EXAMPLE 3 Factor $x^2 - 8x + 6$.

SOLUTION Since there is no pair of integers whose product is 6 and whose sum is -8, the trinomial $x^2 - 8x + 6$ is not factorable. We say it is a *prime polynomial.*

B Factoring When the Lead Coefficient Is Not 1

EXAMPLE 4 Factor $3x^4 - 15x^3y - 18x^2y^2$.

NOTE
As a general rule, it is best to factor out the greatest common factor first.

SOLUTION The leading coefficient is not 1. Each term is divisible by $3x^2$, however. Factoring this out to begin with we have

$$3x^4 - 15x^3y - 18x^2y^2 = 3x^2(x^2 - 5xy - 6y^2)$$

Factoring the resulting trinomial as in the previous examples gives

$$3x^2(x^2 - 5xy - 6y^2) = 3x^2(x - 6y)(x + y)$$

Factoring Other Trinomials by Trial and Error

We want to turn our attention now to trinomials with leading coefficients other than 1 and with no greatest common factor other than 1.

Suppose we want to factor $3x^2 - x - 2$. The factors will be a pair of binomials. The product of the first terms will be $3x^2$, and the product of the last terms will be -2. We can list all the possible factors along with their products as follows.

Possible Factors	First Term	Middle Term	Last Term
$(x + 2)(3x - 1)$	$3x^2$	$+5x$	-2
$(x - 2)(3x + 1)$	$3x^2$	$-5x$	-2
$(x + 1)(3x - 2)$	$3x^2$	$+x$	-2
$(x - 1)(3x + 2)$	$3x^2$	$-x$	-2

From the last line we see that the factors of $3x^2 - x - 2$ are $(x - 1)(3x + 2)$. That is,

$$3x^2 - x - 2 = (x - 1)(3x + 2)$$

To factor trinomials with leading coefficients other than 1, when the greatest common factor is 1, we must use trial and error or list all the possible factors. In either case the idea is this: look only at pairs of binomials whose products give the correct first and last terms, then look for the combination that will give the correct middle term.

EXAMPLE 5 Factor $2x^2 + 13xy + 15y^2$.

SOLUTION Listing all possible factors the product of whose first terms is $2x^2$ and the product of whose last terms is $15y^2$ yields

Possible Factors	Middle Term of Product
$(2x - 5y)(x - 3y)$	$-11xy$
$(2x - 3y)(x - 5y)$	$-13xy$
$(2x + 5y)(x + 3y)$	$+11xy$
$(2x + 3y)(x + 5y)$	$+13xy$
$(2x + 15y)(x + y)$	$+17xy$
$(2x - 15y)(x - y)$	$-17xy$
$(2x + y)(x + 15y)$	$+31xy$
$(2x - y)(x - 15y)$	$-31xy$

The fourth line has the correct middle term.

$$2x^2 + 13xy + 15y^2 = (2x + 3y)(x + 5y)$$

Actually, we did not need to check the first two pairs of possible factors in the preceding list. Because all the signs in the trinomial $2x^2 + 13xy + 15y^2$ are positive, the binomial factors must be of the form $(ax + b)(cx + d)$, where a, b, c, and d are all positive.

There are other ways to reduce the number of possible factors to consider. For example, if we were to factor the trinomial $2x^2 - 11x + 12$, we would not have to consider the pair of possible factors $(2x - 4)(x - 3)$. If the original trinomial has no greatest common factor other than 1, then neither of its binomial factors will either. The trinomial $2x^2 - 11x + 12$ has a greatest common factor of 1, but the possible factor $2x - 4$ has a greatest common factor of 2: $2x - 4 = 2(x - 2)$. Therefore, we do not need to consider $2x - 4$ as a possible factor.

EXAMPLE 6 Factor $12x^4 + 17x^2 + 6$.

SOLUTION This is a trinomial in x^2:

$$12x^4 + 17x^2 + 6 = (4x^2 + 3)(3x^2 + 2)$$

EXAMPLE 7 Factor $2x^2(x - 3) - 5x(x - 3) - 3(x - 3)$.

SOLUTION We begin by factoring out the greatest common factor $(x - 3)$. Then we factor the trinomial that remains.

$$\begin{aligned} 2x^2(x - 3) - 5x(x - 3) - 3(x - 3) &= (x - 3)(2x^2 - 5x - 3) \\ &= (x - 3)(2x + 1)(x - 3) \\ &= (x - 3)^2(2x + 1) \end{aligned}$$

Another Method of Factoring Trinomials

As an alternative to the trial-and-error method of factoring trinomials, we present the following method. The new method does not require as much trial and error. To use this new method, we must rewrite our original trinomial in such a way that the factoring by grouping method can be applied.

Here are the steps we use to factor $ax^2 + bx + c$.

Step 1: Form the product ac.
Step 2: Find a pair of numbers whose product is ac and whose sum is b.
Step 3: Rewrite the polynomial to be factored so that the middle term bx is written as the sum of two terms whose coefficients are the two numbers found in step 2.
Step 4: Factor by grouping.

EXAMPLE 8 Factor $3x^2 - 10x - 8$ using these steps.

SOLUTION The trinomial $3x^2 - 10x - 8$ has the form $ax^2 + bx + c$, where $a = 3$, $b = -10$, and $c = -8$.

Step 1: The product ac is $3(-8) = -24$.

Step 2: We need to find two numbers whose product is -24 and whose sum is -10. Let's list all the pairs of numbers whose product is -24 to find the pair whose sum is -10.

Product	*Sum*
$1(-24) = -24$	$1 + (-24) = -23$
$-1(24) = -24$	$-1 + 24 = 23$
$2(-12) = -24$	$2 + (-12) = -10$
$-2(12) = -24$	$-2 + 12 = 10$
$3(-8) = -24$	$3 + (-8) = -5$
$-3(8) = -24$	$-3 + 8 = 5$
$4(-6) = -24$	$4 + (-6) = -2$
$-4(6) = -24$	$-4 + 6 = 2$

As you can see, of all the pairs of numbers whose product is -24, only 2 and -12 have a sum of -10.

Step 3: We now rewrite our original trinomial so the middle term $-10x$ is written as the sum of $-12x$ and $2x$:

$$3x^2 - 10x - 8 = 3x^2 - 12x + 2x - 8$$

Step 4: Factoring by grouping, we have

$$\begin{aligned} 3x^2 - 12x + 2x - 8 &= 3x(x - 4) + 2(x - 4) \\ &= (x - 4)(3x + 2) \end{aligned}$$

You can see that this method works by multiplying $x - 4$ and $3x + 2$ to get

$$3x^2 - 10x - 8$$

EXAMPLE 9 Factor $9x^2 + 15x + 4$.

SOLUTION In this case $a = 9$, $b = 15$, and $c = 4$. The product ac is $9 \cdot 4 = 36$. Listing all the pairs of numbers whose product is 36 with their corresponding sums, we have

Product	*Sum*
$1(36) = 36$	$1 + 36 = 37$
$2(18) = 36$	$2 + 18 = 20$
$3(12) = 36$	$3 + 12 = 15$
$4(9) = 36$	$4 + 9 = 13$
$6(6) = 36$	$6 + 6 = 12$

Notice we list only positive numbers since both the product and sum we are looking for are positive. The numbers 3 and 12 are the numbers we are looking for. Their product is 36, and their sum is 15. We now rewrite the original polynomial $9x^2 + 15x + 4$ with the middle term written as $3x + 12x$. We then factor by grouping:

$$\begin{aligned} 9x^2 + 15x + 4 &= 9x^2 + 3x + 12x + 4 \\ &= 3x(3x + 1) + 4(3x + 1) \\ &= (3x + 1)(3x + 4) \end{aligned}$$

The polynomial $9x^2 + 15x + 4$ factors into the product

$$(3x + 1)(3x + 4)$$

EXAMPLE 10 Factor $8x^2 - 2x - 15$.

SOLUTION The product ac is $8(-15) = -120$. There are many pairs of numbers whose product is -120. We are looking for the pair whose sum is also -2. The numbers are -12 and 10. Writing $-2x$ as $-12x + 10x$ and then factoring by grouping, we have

$$\begin{aligned} 8x^2 - 2x - 15 &= 8x^2 - 12x + 10x - 15 \\ &= 4x(2x - 3) + 5(2x - 3) \\ &= (2x - 3)(4x + 5) \end{aligned}$$

Problem Set 5.5

Moving Toward Success

"Human beings, by changing the inner attitudes of their minds, can change the outer aspects of their lives."

—William James, 1842–1910, American psychologist and author

1. Thoughts like "I can't do this problem" or "I'm going to get a bad grade so why try" are called negative self talk. How does this type of thinking hurt your success in this class?
2. What are some positive things you can say to yourself to combat any negative self talk? Why would these things help relieve stress in this class?

A Factor each of the following trinomials. [Examples 1–3]

1. $x^2 + 7x + 12$

2. $x^2 - 7x + 12$

3. $x^2 - x - 12$

4. $x^2 + x - 12$

5. $y^2 + y - 6$

6. $y^2 - y - 6$

7. $16 - 6x - x^2$

8. $3 + 2x - x^2$

9. $12 + 8x + x^2$

10. $15 - 2x - x^2$

B Factor completely by first factoring out the greatest common factor and then factoring the trinomial that remains. [Example 4]

11. $3a^2 - 21a + 30$

12. $3a^2 - 3a - 6$

13. $4x^3 - 16x^2 - 20x$

14. $2x^3 - 14x^2 + 20x$

A Factor. [Examples 1–3]

15. $x^2 + 3xy + 2y^2$

16. $x^2 - 5xy - 24y^2$

17. $a^2 + 3ab - 18b^2$

18. $a^2 - 8ab - 9b^2$

19. $x^2 - 2xa - 48a^2$

20. $x^2 + 14xa + 48a^2$

21. $x^2 - 12xb + 36b^2$

22. $x^2 + 10xb + 25b^2$

B Factor completely. Be sure to factor out the greatest common factor first if it is other than 1. [Examples 4–6]

23. $3x^2 - 6xy - 9y^2$

24. $5x^2 + 25xy + 20y^2$

25. $2a^5 + 4a^4b + 4a^3b^2$

26. $3a^4 - 18a^3b + 27a^2b^2$

27. $10x^4y^2 + 20x^3y^3 - 30x^2y^4$

28. $6x^4y^2 + 18x^3y^3 - 24x^2y^4$

29. $2x^2 + 7x - 15$

30. $2x^2 - 7x - 15$

31. $2x^2 + x - 15$

32. $2x^2 - x - 15$

33. $2x^2 - 13x + 15$

34. $2x^2 + 13x + 15$

35. $2x^2 - 11x + 15$

36. $2x^2 + 11x + 15$

37. $2x^2 + 7x + 15$

38. $2x^2 + x + 15$

39. $2 + 7a + 6a^2$

40. $2 - 7a + 6a^2$

41. $60y^2 - 15y - 45$

42. $72y^2 + 60y - 72$

43. $6x^4 - x^3 - 2x^2$

44. $3x^4 + 2x^3 - 5x^2$

45. $40r^3 - 120r^2 + 90r$

46. $40r^3 + 200r^2 + 250r$

47. $4x^2 - 11xy - 3y^2$

48. $3x^2 + 19xy - 14y^2$

49. $10x^2 - 3xa - 18a^2$

50. $9x^2 + 9xa - 10a^2$

51. $18a^2 + 3ab - 28b^2$

52. $6a^2 - 7ab - 5b^2$

53. $600 + 800t - 800t^2$

54. $200 - 600t - 350t^2$

55. $9y^4 + 9y^3 - 10y^2$

56. $4y^5 + 7y^4 - 2y^3$

57. $24a^2 - 2a^3 - 12a^4$

58. $60a^2 + 65a^3 - 20a^4$

59. $8x^4y^2 - 2x^3y^3 - 6x^2y^4$

60. $8x^4y^2 - 47x^3y^3 - 6x^2y^4$

61. $300x^4 + 1{,}000x^2 + 300$

62. $600x^4 - 100x^2 - 200$

63. $20a^4 + 37a^2 + 15$

64. $20a^4 + 13a^2 - 15$

65. $9 + 3r^2 - 12r^4$

66. $2 - 4r^2 - 30r^4$

B Factor each of the following by first factoring out the greatest common factor and then factoring the trinomial that remains. [Example 7]

67. $2x^2(x + 5) + 7x(x + 5) + 6(x + 5)$

68. $2x^2(x + 2) + 13x(x + 2) + 15(x + 2)$

69. $x^2(2x + 3) + 7x(2x + 3) + 10(2x + 3)$

70. $2x^2(x+1) + 7x(x+1) + 6(x+1)$

71. $3x^2(x-3) + 7x(x-3) - 20(x-3)$

72. $4x^2(x+6) + 23x(x+6) + 15(x+6)$

73. $6x^2(x-2) - 17x(x-2) + 12(x-2)$

74. $10x^2(x+4) - 33x(x+4) - 7(x+4)$

75. $12x^2(x+3) + 7x(x+3) - 45(x+3)$

76. $24x^2(x-6) + 38x(x-6) + 15(x-6)$

77. $6x^2(5x-2) - 11x(5x-2) - 10(5x-2)$

78. $14x^2(3x+4) - 39x(3x+4) + 10(3x+4)$

79. $20x^2(2x+3) + 47x(2x+3) + 21(2x+3)$

80. $15x^2(4x-5) - 2x(4x-5) - 24(4x-5)$

81. What polynomial, when factored, gives $(3x+5y)(3x-5y)$?

82. What polynomial, when factored, gives $(7x+2y)(7x-2y)$?

83. One factor of the trinomial $a^2 + 260a + 2{,}500$ is $a + 10$. What is the other factor?

84. One factor of the trinomial $a^2 - 75a - 2{,}500$ is $a + 25$. What is the other factor?

85. One factor of the trinomial $12x^2 - 107x + 210$ is $x - 6$. What is the other factor?

86. One factor of the trinomial $36x^2 + 134x - 40$ is $2x + 8$. What is the other factor?

87. One factor of the trinomial $54x^2 + 111x + 56$ is $6x + 7$. What is the other factor?

88. One factor of the trinomial $63x^2 + 110x + 48$ is $7x + 6$. What is the other factor?

89. One factor of the trinomial $35x^2 + 19x - 24$ is $5x - 3$. What is the other factor?

90. One factor of the trinomial $36x^2 + 43x - 35$ is $4x + 7$. What is the other factor?

91. Factor the right side of the equation $y = 4x^2 + 18x - 10$, and then use the result to find y when x is $\frac{1}{2}$, when x is -5, and when x is 2.

92. Factor the right side of the equation $y = 9x^2 + 33x - 12$, and use the result to find y when x is $\frac{1}{3}$, when x is -4, and when x is 3.

Maintaining Your Skills

93. $(2x-3)(2x+3)$

94. $(4-5x)(4+5x)$

95. $(2x-3)^2$

96. $(4-5x)^2$

97. $(2x-3)(4x^2+6x+9)$

98. $(2x+3)(4x^2-6x+9)$

Getting Ready for the Next Section

For each problem below, place a number or expression inside the parentheses so that the resulting statement is true.

99. $\frac{25}{64} = \left(\quad\right)^2$

100. $\frac{4}{9} = \left(\quad\right)^2$

101. $x^6 = (\quad)^2$

102. $x^8 = (\quad)^2$

103. $16x^4 = (\quad)^2$

104. $81y^4 = (\quad)^2$

Write as a perfect cube.

105. $\frac{1}{8} = \left(\quad\right)^3$

106. $\frac{1}{27} = \left(\quad\right)^3$

107. $x^6 = (\quad)^3$

108. $x^{12} = (\quad)^3$

109. $27x^3 = (\quad)^3$

110. $125y^3 = (\quad)^3$

111. $8y^3 = (\quad)^3$

112. $1000x^3 = (\quad)^3$

Extending the Concepts

Factor completely.

113. $8x^6 + 26x^3y^2 + 15y^4$

114. $24x^4 + 6x^2y^3 - 45y^6$

115. $3x^2 + 295x - 500$

116. $3x^2 + 594x - 1{,}200$

117. $\frac{1}{8}x^2 + x + 2$

118. $\frac{1}{9}x^2 + x + 2$

119. $2x^2 + 1.5x + 0.25$

120. $6x^2 + 2x + 0.16$

5.6 Special Factoring

OBJECTIVES

A Factor perfect square trinomials.

B Factor the difference of two squares.

C Factor the sum or difference of two cubes.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

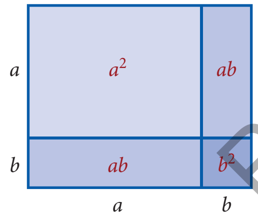
1. What is a perfect square trinomial?
2. Is it possible to factor the sum of two squares?
3. What is the formula for the sum of two cubes?
4. Write a problem that uses the formula for the difference of two cubes.

Suppose a smoothie shop can make x smoothies for a total cost of $C = 10x^2 + 100x + 250$. If we begin by factoring out the greatest common factor of 10, we find that the resulting equation looks like this

$$C = 10(x^2 + 10x + 25)$$

This is a special kind of trinomial called a perfect square trinomial. We will explore these and other kinds of special trinomials in this section.

To find the area of the large square in the margin, we can square the length of its side, giving us $(a + b)^2$. However, we can add the areas of the four smaller figures to arrive at the same result.



Since the area of the large square is the same whether we find it by squaring a side or by adding the four smaller areas, we can write the following relationship:

$$(a + b)^2 = a^2 + 2ab + b^2$$

This is the formula for the square of a binomial. The figure gives us a geometric interpretation for one of the special multiplication formulas. We begin this section by looking at the special multiplication formulas from a factoring perspective.

A Perfect Square Trinomials

We previously listed some special products found in multiplying polynomials. Two of the formulas looked like this:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

If we exchange the left and right sides of each formula, we have two special formulas for factoring:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

The left side of each formula is called a *perfect square trinomial.* The right sides are binomial squares. Perfect square trinomials can always be factored using the usual methods for factoring trinomials. However, if we notice that the first and last terms of a trinomial are perfect squares, it is wise to see whether the trinomial factors as a binomial square before attempting to factor by the usual method.

EXAMPLE 1 Factor $x^2 - 6x + 9$.

SOLUTION Since the first and last terms are perfect squares, we attempt to factor according to the preceding formulas.

$$x^2 - 6x + 9 = (x - 3)^2$$

If we expand $(x - 3)^2$, we have $x^2 - 6x + 9$, indicating we have factored correctly.

EXAMPLES Factor each of the following perfect square trinomials.

SOLUTION

2. $16a^2 + 40ab + 25b^2 = (4a + 5b)^2$

3. $49 - 14t + t^2 = (7 - t)^2$

4. $9x^4 - 12x^2 + 4 = (3x^2 - 2)^2$

5. $(y + 3)^2 + 10(y + 3) + 25 = [(y + 3) + 5]^2 = (y + 8)^2$

EXAMPLE 6 Factor $8x^2 - 24xy + 18y^2$.

SOLUTION We begin by factoring the greatest common factor 2 from each term.

$$\begin{aligned} 8x^2 - 24xy + 18y^2 &= 2(4x^2 - 12xy + 9y^2) \\ &= 2(2x - 3y)^2 \end{aligned}$$

B The Difference of Two Squares

Recall the formula that results in the difference of two squares:

$$(a + b)(a - b) = a^2 - b^2.$$

Writing this as a factoring formula, we have

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLES Each of the following is the difference of two squares. Use the formula $a^2 - b^2 = (a + b)(a - b)$ to factor each one.

SOLUTION

7. $x^2 - 25 = x^2 - 5^2 = (x + 5)(x - 5)$

8. $49 - t^2 = 7^2 - t^2 = (7 + t)(7 - t)$

9. $81a^2 - 25b^2 = (9a)^2 - (5b)^2 = (9a + 5b)(9a - 5b)$

10. $4x^6 - 1 = (2x^3)^2 - 1^2 = (2x^3 + 1)(2x^3 - 1)$

11. $x^2 - \frac{4}{9} = x^2 - \left(\frac{2}{3}\right)^2 = \left(x + \frac{2}{3}\right)\left(x - \frac{2}{3}\right)$

As our next example shows, the difference of two fourth powers can be factored as the difference of two squares.

EXAMPLE 12 Factor $16x^4 - 81y^4$.

SOLUTION The first and last terms are perfect squares. We factor according to the preceding formula.

$$\begin{aligned} 16x^4 - 81y^4 &= (4x^2)^2 - (9y^2)^2 \\ &= (4x^2 + 9y^2)(4x^2 - 9y^2) \end{aligned}$$

NOTE
The sum of two squares never factors into the product of two binomials; that is, if we were to attempt to factor $(4x^2 + 9y^2)$ in the last example, we would be unable to find two binomials (or any other polynomials) whose product is $4x^2 + 9y^2$. The factors do not exist as polynomials.

Notice that the second factor is also the difference of two squares. Factoring completely, we have

$$16x^4 - 81y^4 = (4x^2 + 9y^2)(2x + 3y)(2x - 3y)$$

Here is another example of the difference of two squares.

EXAMPLE 13 Factor $(x - 3)^2 - 25$.

SOLUTION This example has the form $a^2 - b^2$, where a is $x - 3$ and b is 5. We factor it according to the formula for the difference of two squares.

$$\begin{aligned} (x - 3)^2 - 25 &= (x - 3)^2 - 5^2 && \text{Write 25 as } 5^2 \\ &= [(x - 3) + 5][(x - 3) - 5] && \text{Factor} \\ &= (x + 2)(x - 8) && \text{Simplify} \end{aligned}$$

Notice in this example we could have expanded $(x - 3)^2$, subtracted 25, and then factored to obtain the same result.

$$\begin{aligned} (x - 3)^2 - 25 &= x^2 - 6x + 9 - 25 && \text{Expand } (x - 3)^2 \\ &= x^2 - 6x - 16 && \text{Simplify} \\ &= (x - 8)(x + 2) && \text{Factor} \end{aligned}$$

EXAMPLE 14 Factor $x^2 - 10x + 25 - y^2$.

SOLUTION Notice the first three terms form a perfect square trinomial; that is, $x^2 - 10x + 25 = (x - 5)^2$. If we replace the first three terms by $(x - 5)^2$, the expression that results has the form $a^2 - b^2$. We can factor as we did in Example 13.

$$\begin{aligned} x^2 - 10x + 25 - y^2 &= (x^2 - 10x + 25) - y^2 && \text{Group first three terms together} \\ &= (x - 5)^2 - y^2 && \text{This has the form } a^2 - b^2 \\ &= [(x - 5) + y][(x - 5) - y] && \text{Factor according to the formula } a^2 - b^2 = (a + b)(a - b) \\ &= (x - 5 + y)(x - 5 - y) && \text{Simplify} \end{aligned}$$

We could check this result by multiplying the two factors together. (You may want to do that to convince yourself that we have the correct result.)

EXAMPLE 15 Factor $x^3 + 2x^2 - 9x - 18$ completely.

SOLUTION We use factoring by grouping to begin and then factor the difference of two squares.

$$\begin{aligned} x^3 + 2x^2 - 9x - 18 &= x^2(x + 2) - 9(x + 2) \\ &= (x + 2)(x^2 - 9) \\ &= (x + 2)(x + 3)(x - 3) \end{aligned}$$

C The Sum and Difference of Two Cubes

Here are the formulas for factoring the sum and difference of two cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Since these formulas are unfamiliar, it is important that we verify them.

EXAMPLE 16 Verify the two formulas.

SOLUTION We verify the formulas by multiplying the right sides and comparing the results with the left sides.

$$\begin{array}{rrrr} a^2 & - ab & + b^2 & \\ & a & + b & \\ \hline a^3 - a^2b & + ab^2 & & \\ & a^2b & - ab^2 & + b^3 \\ \hline a^3 & & & + b^3 \end{array}$$

The first formula is correct.

$$\begin{array}{rrrr} a^2 & + ab & + b^2 & \\ & a & - b & \\ \hline a^3 + a^2b & + ab^2 & & \\ & -a^2b & - ab^2 & - b^3 \\ \hline a^3 & & & - b^3 \end{array}$$

The second formula is correct.

Here are some examples using the formulas for factoring the sum and difference of two cubes.

EXAMPLE 17 Factor $64 + t^3$.

SOLUTION The first term is the cube of 4 and the second term is the cube of t. Therefore,

$$\begin{aligned} 64 + t^3 &= 4^3 + t^3 \\ &= (4 + t)(16 - 4t + t^2) \end{aligned}$$

EXAMPLE 18 Factor $27x^3 + 125y^3$.

SOLUTION Writing both terms as perfect cubes, we have

$$\begin{aligned} 27x^3 + 125y^3 &= (3x)^3 + (5y)^3 \\ &= (3x + 5y)(9x^2 - 15xy + 25y^2) \end{aligned}$$

EXAMPLE 19 Factor $a^3 - \frac{1}{8}$.

SOLUTION The first term is the cube of a, whereas the second term is the cube of $\frac{1}{2}$.

$$a^3 - \frac{1}{8} = a^3 - \left(\frac{1}{2}\right)^3$$
$$= \left(a - \frac{1}{2}\right)\left(a^2 + \frac{1}{2}a + \frac{1}{4}\right)$$

EXAMPLE 20 Factor $x^6 - y^6$.

SOLUTION We have a choice of how we want to write the two terms to begin. We can write the expression as the difference of two squares, $(x^3)^2 - (y^3)^2$, or as the difference of two cubes, $(x^2)^3 - (y^2)^3$. It is better to use the difference of two squares if we have a choice.

$$x^6 - y^6 = (x^3)^2 - (y^3)^2$$
$$= (x^3 - y^3)(x^3 + y^3)$$
$$= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$$

Try this example again writing the first line as the difference of two cubes instead of the difference of two squares. It will become apparent why it is better to use the difference of two squares.

Problem Set 5.6

Moving Toward Success

"No one can cheat you out of ultimate success but yourself."

—Ralph Waldo Emerson, 1803–1882, American poet and essayist

1. Placing blame on others if you do poorly will only hurt your success in this class. Why?
2. What is one thing you can do today to perform better in this class?

A Factor each perfect square trinomial. [Examples 1–6]

1. $x^2 - 6x + 9$

2. $x^2 + 10x + 25$

3. $a^2 - 12a + 36$

4. $36 - 12a + a^2$

5. $25 - 10t + t^2$

6. $64 + 16t + t^2$

7. $\frac{1}{9}x^2 + 2x + 9$

8. $\frac{1}{4}x^2 - 2x + 4$

9. $4y^4 - 12y^2 + 9$

10. $9y^4 + 12y^2 + 4$

11. $16a^2 + 40ab + 25b^2$

12. $25a^2 - 40ab + 16b^2$

13. $\frac{1}{25} + \frac{1}{10}t^2 + \frac{1}{16}t^4$

14. $\frac{1}{9} - \frac{1}{3}t^3 + \frac{1}{4}t^6$

15. $y^2 + 3y + \frac{9}{4}$

16. $y^2 - 7y + \frac{49}{4}$

17. $a^2 - a + \frac{1}{4}$

18. $a^2 - 5a + \frac{25}{4}$

19. $x^2 - \frac{1}{2}x + \frac{1}{16}$

20. $x^2 - \frac{3}{4}x + \frac{9}{64}$

21. $t^2 + \frac{2}{3}t + \frac{1}{9}$

22. $t^2 - \frac{4}{5}t + \frac{4}{25}$

23. $16x^2 - 48x + 36$

24. $36x^2 + 48x + 16$

25. $75a^3 + 30a^2 + 3a$

26. $45a^4 - 30a^3 + 5a^2$

27. $(x + 2)^2 + 6(x + 2) + 9$

28. $(x + 5)^2 + 4(x + 5) + 4$

B Factor each as the difference of two squares. Be sure to factor completely. [Examples 7–12]

29. $x^2 - 9$

30. $x^2 - 16$

31. $49x^2 - 64y^2$

32. $81x^2 - 49y^2$

33. $4a^2 - \frac{1}{4}$

34. $25a^2 - \frac{1}{25}$

35. $x^2 - \frac{9}{25}$

36. $x^2 - \frac{25}{36}$

37. $9x^2 - 16y^2$

38. $25x^2 - 49y^2$

39. $250 - 10t^2$

40. $640 - 10t^2$

B Factor each as the difference of two squares. Be sure to factor completely. [Examples 7–12]

41. $x^4 - 81$

42. $x^4 - 16$

43. $9x^6 - 1$

44. $25x^6 - 1$

45. $16a^4 - 81$

46. $81a^4 - 16b^4$

47. $\frac{1}{81} - \frac{y^4}{16}$

48. $\frac{1}{25} - \frac{y^4}{64}$

B Factor completely. [Examples 13–15]

49. $x^6 - y^6$

50. $x^6 - 1$

51. $2a^7 - 128a$

52. $128a^8 - 2a^2$

53. $(x - 2)^2 - 9$

54. $(x + 2)^2 - 9$

55. $(y + 4)^2 - 16$

56. $(y - 4)^2 - 16$

57. $x^2 - 10x + 25 - y^2$

58. $x^2 - 6x + 9 - y^2$

59. $a^2 + 8a + 16 - b^2$

60. $a^2 + 12a + 36 - b^2$

61. $x^2 + 2xy + y^2 - a^2$

62. $a^2 + 2ab + b^2 - y^2$

63. $x^3 + 3x^2 - 4x - 12$

64. $x^3 + 5x^2 - 4x - 20$

65. $x^3 + 2x^2 - 25x - 50$

66. $x^3 + 4x^2 - 9x - 36$

67. $2x^3 + 3x^2 - 8x - 12$

68. $3x^3 + 2x^2 - 27x - 18$

69. $4x^3 + 12x^2 - 9x - 27$

70. $9x^3 + 18x^2 - 4x - 8$

71. $(2x - 5)^2 - 100$

72. $(7a + 5)^2 - 64$

73. $(a - 3)^2 - (4b)^2$

74. $(2x - 5)^2 - (6y)^2$

75. $a^2 - 6a + 9 - 16b^2$

76. $x^2 - 10x + 25 - 9y^2$

77. $x^2(x + 4) - 6x(x + 4) + 9(x + 4)$

78. $x^2(x - 6) + 8x(x - 6) + 16(x - 6)$

C Factor each of the following as the sum or difference of two cubes. [Examples 16–20]

79. $x^3 - y^3$

80. $x^3 + y^3$

81. $a^3 + 8$

82. $a^3 - 8$

83. $27 + x^3$

84. $27 - x^3$

85. $y^3 - 1$

86. $y^3 + 1$

87. $10r^3 - 1{,}250$

88. $10r^3 + 1{,}250$

89. $64 + 27a^3$

90. $27 - 64a^3$

91. $8x^3 - 27y^3$

92. $27x^3 - 8y^3$

93. $t^3 + \frac{1}{27}$

94. $t^3 - \frac{1}{27}$

95. $27x^3 - \dfrac{1}{27}$

96. $8x^3 + \dfrac{1}{8}$

97. $64a^3 + 125b^3$

98. $125a^3 - 27b^3$

99. Find two values of b that will make $9x^2 + bx + 25$ a perfect square trinomial.

100. Find a value of c that will make $49x^2 - 42x + c$ a perfect square trinomial.

Getting Ready for the Next Section

Factor out the greatest common factor.

101. $y^3 + 25y$

102. $y^4 + 36y^2$

103. $2ab^5 + 8ab^4 + 2ab^3$

104. $3a^2b^3 + 6a^2b^2 - 3a^2b$

Factor by grouping.

105. $4x^2 - 6x + 2ax - 3a$

106. $6x^2 - 4x + 3ax - 2a$

Factor the difference of squares.

107. $x^2 - 4$

108. $x^2 - 9$

Factor the perfect square trinomial.

109. $x^2 - 6x + 9$

110. $x^2 - 10x + 25$

Factor.

111. $6a^2 - 11a + 4$

112. $6x^2 - x - 15$

Factor the sum or difference of cubes.

113. $x^3 + 8$

114. $x^3 - 27$

Maintaining Your Skills

Solve each system by using matrices.

115. $\begin{aligned} 2x - 4y &= -2 \\ 3x - 2y &= 9 \end{aligned}$

116. $\begin{aligned} 3x + 5y &= 14 \\ 5x + 2y &= -2 \end{aligned}$

117. $\begin{aligned} 3x + 4y &= 15 \\ 2x - 5z &= -3 \\ 4y - 3z &= 9 \end{aligned}$

118. $\begin{aligned} x + 3y &= 5 \\ 6y + z &= 12 \\ x - 2z &= -10 \end{aligned}$

Extending the Concepts

Factor completely.

119. $a^2 - b^2 + 6b - 9$

120. $a^2 - b^2 - 18b - 81$

121. $(x - 3)^2 - (y + 5)^2$

122. $(a + 7)^2 - (b - 9)^2$

Find k such that each trinomial becomes a perfect square trinomial.

123. $kx^2 - 168xy + 49y^2$

124. $kx^2 + 110xy + 121y^2$

125. $49x^2 + kx + 81$

126. $64x^2 + kx + 169$

5.7 Factoring: A General Review

OBJECTIVES

A Factor a variety of polynomials.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. How do you know when you have factored completely?
2. What is the first step in the strategy for factoring a polynomial?
3. Why is it important to not skip the last step in the strategy for factoring a polynomial?
4. What method of factoring would you use to factor a polynomial with four terms?

The process of factoring has been used by mathematicians for thousands of years. As early as 2000 BC, the Babylonians were factoring polynomials by carving their numeric characters into stone tablets.

A Factoring Review

In this section, we will review the different methods of factoring that we have presented in the previous sections of this chapter. This section is important because it will give you an opportunity to factor a variety of polynomials.

We begin this section by listing the steps that can be used to factor polynomials of any type.

Strategy To Factor a Polynomial

Step 1: If the polynomial has a greatest common factor other than 1, then factor out the greatest common factor.

Step 2: If the polynomial has two terms (it is a binomial), then see if it is the difference of two squares or the sum or difference of two cubes, and then factor accordingly. Remember, if it is the sum of two squares it will not factor.

Step 3: If the polynomial has three terms (a trinomial), then it is either a perfect square trinomial, which will factor into the square of a binomial, or it is not a perfect square trinomial, in which case we try to write it as the product of two binomials using the methods developed in this chapter.

Step 4: If the polynomial has more than three terms, then try to factor it by grouping.

Step 5: As a final check, see if any of the factors you have written can be factored further. If you have overlooked a common factor, you can catch it here.

Here are some examples illustrating how we use the steps in our list. There are no new factoring problems in this section. The problems here are all similar to the problems you have seen before. What is different is that they are not all of the same type.

EXAMPLE 1 Factor $2x^5 - 8x^3$.

SOLUTION First we check to see if the greatest common factor is other than 1. Since the greatest common factor is $2x^3$, we begin by factoring it out. Once we have done so, we notice that the binomial that remains is the difference of two squares, which we factor according to the formula $a^2 - b^2 = (a + b)(a - b)$.

$$2x^5 - 8x^3 = 2x^3(x^2 - 4) \quad \text{Factor out the greatest common factor, } 2x^3$$
$$= 2x^3(x + 2)(x - 2) \quad \text{Factor the difference of two squares}$$

EXAMPLE 2 Factor $3x^4 - 18x^3 + 27x^2$.

SOLUTION Step 1 is to factor out the greatest common factor $3x^2$. After we have done so, we notice that the trinomial that remains is a perfect square trinomial, which will factor as the square of a binomial.

$$3x^4 - 18x^3 + 27x^2 = 3x^2(x^2 - 6x + 9) \quad \text{Factor out } 3x^2$$
$$= 3x^2(x - 3)^2 \quad x^2 - 6x + 9 \text{ is the square of } x - 3$$

EXAMPLE 3 Factor $y^3 + 25y$.

SOLUTION We begin by factoring out the y that is common to both terms. The binomial that remains after we have done so is the sum of two squares, which does not factor, so after the first step, we are finished.

$$y^3 + 25y = y(y^2 + 25)$$

EXAMPLE 4 Factor $6a^2 - 11a + 4$.

SOLUTION Here we have a trinomial that does not have a greatest common factor other than 1. Since it is not a perfect square trinomial, we factor it by trial and error. Without showing all the different possibilities, here is the answer:

$$6a^2 - 11a + 4 = (3a - 4)(2a - 1)$$

EXAMPLE 5 Factor $2x^4 + 16x$.

SOLUTION This binomial has a greatest common factor of $2x$. The binomial that remains after the $2x$ has been factored from each term is the sum of two cubes, which we factor according to the formula $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$$\begin{aligned} 2x^4 + 16x &= 2x(x^3 + 8) && \text{Factor 2x from each term} \\ &= 2x(x + 2)(x^2 - 2x + 4) && \text{The sum of two cubes} \end{aligned}$$

EXAMPLE 6 Factor $2ab^5 + 8ab^4 + 2ab^3$.

SOLUTION The greatest common factor is $2ab^3$. We begin by factoring it from each term. After that we find that the trinomial that remains cannot be factored further.

$$2ab^5 + 8ab^4 + 2ab^3 = 2ab^3(b^2 + 4b + 1)$$

EXAMPLE 7 Factor $4x^2 - 6x + 2ax - 3a$.

SOLUTION Our polynomial has four terms, so we factor by grouping.

$$\begin{aligned} 4x^2 - 6x + 2ax - 3a &= 2x(2x - 3) + a(2x - 3) \\ &= (2x - 3)(2x + a) \end{aligned}$$

Problem Set 5.7

Moving Toward Success

"If you don't like something, change it. If you can't change it, change your attitude. Don't complain."

—Maya Angelou, 1928–present, American author and poet

1. How do you plan on staying positive before, during, and after a test?
2. If you receive a poor grade on a test, what do you plan to do to perform better on the next test?

A Factor each of the following polynomials completely. Once you are finished factoring, none of the factors you obtain should be factorable. Also, note that the even-numbered problems are not necessarily similar to the odd-numbered problems that precede them in this problem set. [Examples 1–7]

1. $x^2 - 81$

2. $x^2 - 18x + 81$

3. $x^2 + 2x - 15$

4. $15x^2 + 13x - 6$

5. $x^2(x + 2) + 6x(x + 2) + 9(x + 2)$

6. $12x^2 - 11x + 2$

7. $x^2y^2 + 2y^2 + x^2 + 2$

8. $21y^2 - 25y - 4$

9. $2a^3b + 6a^2b + 2ab$

10. $6a^2 - ab - 15b^2$

11. $x^2 + x + 1$

12. $x^2y + 3y + 2x^2 + 6$

13. $12a^2 - 75$

14. $18a^2 - 50$

15. $9x^2 - 12xy + 4y^2$

16. $x^3 - x^2$

17. $25 - 10t + t^2$

18. $t^2 + 4t + 4 - y^2$

19. $4x^3 + 16xy^2$

20. $16x^2 + 49y^2$

21. $2y^3 + 20y^2 + 50y$

22. $x^2 + 5bx - 2ax - 10ab$

23. $a^7 + 8a^4b^3$

24. $5a^2 - 45b^2$

25. $t^2 + 6t + 9 - x^2$

26. $36 + 12t + t^2$

27. $x^3 + 5x^2 - 9x - 45$

28. $x^3 + 5x^2 - 16x - 80$

29. $5a^2 + 10ab + 5b^2$

30. $3a^3b^2 + 15a^2b^2 + 3ab^2$

31. $x^2 + 49$

32. $16 - x^4$

33. $3x^2 + 15xy + 18y^2$

34. $3x^2 + 27xy + 54y^2$

35. $9a^2 + 2a + \frac{1}{9}$

36. $18 - 2a^2$

37. $x^2(x - 3) - 14x(x - 3) + 49(x - 3)$

38. $x^2 + 3ax - 2bx - 6ab$

39. $x^2 - 64$

40. $9x^2 - 4$

41. $8 - 14x - 15x^2$

42. $5x^4 + 14x^2 - 3$

43. $49a^7 - 9a^5$

44. $a^6 - b^6$

45. $r^2 - \frac{1}{25}$

46. $27 - r^3$

47. $49x^2 + 9y^2$

48. $12x^4 - 62x^3 + 70x^2$

49. $100x^2 - 100x - 600$

50. $100x^2 - 100x - 1{,}200$

51. $25a^3 + 20a^2 + 3a$

52. $16a^5 - 54a^2$

53. $3x^4 - 14x^2 - 5$

54. $8 - 2x - 15x^2$

55. $24a^5b - 3a^2b$

56. $18a^4b^2 - 24a^3b^3 + 8a^2b^4$

57. $64 - r^3$

58. $r^2 - \frac{1}{9}$

59. $20x^4 - 45x^2$

60. $16x^3 + 16x^2 + 3x$

61. $400t^2 - 900$

62. $900 - 400t^2$

63. $16x^5 - 44x^4 + 30x^3$

64. $16x^2 + 16x - 1$

65. $y^6 - 1$

66. $25y^7 - 16y^5$

67. $50 - 2a^2$

68. $4a^2 + 2a + \frac{1}{4}$

69. $12x^4y^2 + 36x^3y^3 + 27x^2y^4$

70. $16x^3y^2 - 4xy^2$

71. $x^2 - 4x + 4 - y^2$

72. $x^2 - 12x + 36 - b^2$

73. $a^2 - \frac{4}{3}ab + \frac{4}{9}b^2$

74. $a^2 + \frac{3}{2}ab + \frac{9}{16}b^2$

75. $x^2 - \frac{4}{5}xy + \frac{4}{25}y^2$

76. $x^2 - \frac{8}{7}xy + \frac{16}{49}y^2$

77. $a^2 - \frac{5}{3}ab + \frac{25}{36}b^2$

78. $a^2 + \frac{5}{4}ab + \frac{25}{64}b^2$

79. $x^2 - \frac{8}{5}xy + \frac{16}{25}y^2$

80. $a^2 + \frac{3}{5}ab + \frac{9}{100}b^2$

81. $2x^2(x + 2) - 13x(x + 2) + 15(x + 2)$

82. $5x^2(x - 4) - 14x(x - 4) - 3(x - 4)$

83. $(x - 4)^3 + (x - 4)^4$

84. $(2x - 7)^5 + (2x - 7)^6$

85. $2y^3 - 54$

86. $81 + 3y^3$

87. $2a^3 - 128b^3$

88. $128a^3 + 2b^3$

89. $2x^3 + 432y^3$

90. $432x^3 - 2y^3$

Maintaining Your Skills

The following problems are taken from the book *Algebra for the Practical Man*, written by J. E. Thompson and published by D. Van Nostrand Company in 1931.

91. A man spent \$112.80 for 108 geese and ducks, each goose costing 14 dimes and each duck 6 dimes. How many of each did he buy?

92. If 15 pounds of tea and 10 pounds of coffee together cost \$15.50, while 25 pounds of tea and 13 pounds of coffee at the same prices cost \$24.55, find the price per pound of each.

93. A number of oranges at the rate of three for \$0.10 and apples at \$0.15 a dozen cost, together, \$6.80. Five times as many oranges and one-fourth as many apples at the same rates would have cost \$25.45. How many of each were bought?

94. An estate is divided among three persons: *A*, *B*, and *C*. *A* 's share is three times that of *B* and *B* 's share is twice that of *C*. If *A* receives \$9,000 more than *C*, how much does each receive?

Getting Ready for the Next Section

Simplify.

95. $x^2 + (x + 1)^2$

96. $x^2 + (x + 3)^2$

97. $\dfrac{16t^2 - 64t + 48}{16}$

98. $\dfrac{100p^2 - 1{,}300p + 4{,}000}{100}$

Factor each of the following.

99. $x^2 - 2x - 24$

100. $x^2 - x - 6$

101. $2x^3 - 5x^2 - 3x$

102. $3x^3 - 5x^2 - 2x$

103. $x^3 + 2x^2 - 9x - 18$

104. $x^3 + 5x^2 - 4x - 20$

Solve.

105. $x - 6 = 0$

106. $x + 4 = 0$

107. $2x + 1 = 0$

108. $3x + 1 = 0$

5.8 Solving Equations by Factoring

OBJECTIVES

A Solve equations by factoring.

B Apply the Blueprint for Problem Solving to solve application problems whose solutions involve quadratic equations.

C Solve problems that contain formulas that are quadratic.

TICKET TO SUCCESS

Keep these questions in mind as you read through the section. Then respond in your own words and in complete sentences.

1. What is standard form?
2. What is the zero-factor property?
3. Briefly explain the strategy to solve an equation by factoring.
4. Explain the Pythagorean theorem in words.

A producer of novelty sunglasses finds that the number of glasses sold N is related to the retail price p by the formula $N = 20p - p^2$. What price should they charge if they want to sell the 96 glasses they have in stock? Solving equations like this requires the use of our new factoring skills. Let's put those skills to work.

In this section, we will use our knowledge of factoring to solve equations. Most of the equations we will solve in this section are *quadratic equations.* Here is the definition of a quadratic equation.

Definition

Any equation that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c are constants and a is not 0 ($a \neq 0$) is called a **quadratic equation**. The form $ax^2 + bx + c = 0$ is called **standard form** for quadratic equations.

Each of the following is a quadratic equation:

$$2x^2 = 5x + 3 \qquad 5x^2 = 75 \qquad 4x^2 - 3x + 2 = 0$$

NOTE
The third equation is clearly a quadratic equation since it is in standard form. (Notice that a is 4, b is -3, and c is 2.) The first two equations are also quadratic because they could be put in the form $ax^2 + bx + c = 0$ by using the addition property of equality.

Notation For a quadratic equation written in standard form, the first term ax^2 is called the *quadratic term;* the second term bx is the *linear term;* and the third term c is called the *constant term.*

In the past we have noticed that the number 0 is a special number. There is another property of 0 that is the key to solving quadratic equations. It is called the *zero-factor property.*

NOTE
What the zero-factor property says in words is that we can't multiply and get 0 without multiplying by 0; that is, if we multiply two numbers and get 0, then one or both of the original two numbers we multiplied must have been 0.

Zero-Factor Property
For all real numbers r and s,

$$r \cdot s = 0 \quad \text{if and only if} \quad r = 0 \quad \text{or} \quad s = 0 \quad \text{(or both)}$$

A Solving Equations by Factoring

EXAMPLE 1 Solve $x^2 - 2x - 24 = 0$.

SOLUTION We begin by factoring the left side as $(x - 6)(x + 4)$ and get

$$(x - 6)(x + 4) = 0$$

Now both $(x - 6)$ and $(x + 4)$ represent real numbers. We notice that their product is 0. By the zero-factor property, one or both of them must be 0.

$$x - 6 = 0 \quad \text{or} \quad x + 4 = 0$$

We have used factoring and the zero-factor property to rewrite our original second-degree equation as two first-degree equations connected by the word *or*. Completing the solution, we solve the two first-degree equations.

$$\begin{aligned} x - 6 &= 0 \quad \text{or} \quad x + 4 = 0 \\ x &= 6 \quad \text{or} \quad x = -4 \end{aligned}$$

We check our solutions in the original equation as follows:

NOTE
We are placing a question mark over the equal sign because we don't know yet if the expression on the left will be equal to the expression on the right.

Check $x = 6$	Check $x = -4$
$6^2 - 2(6) - 24 \stackrel{?}{=} 0$	$(-4)^2 - 2(-4) - 24 \stackrel{?}{=} 0$
$36 - 12 - 24 \stackrel{?}{=} 0$	$16 + 8 - 24 \stackrel{?}{=} 0$
$0 = 0$	$0 = 0$

In both cases the result is a true statement, which means that both 6 and -4 are solutions to the original equation.

Although the next equation is not quadratic, the method we use is similar.

EXAMPLE 2 Solve $\frac{1}{3}x^3 = \frac{5}{6}x^2 + \frac{1}{2}x$.

SOLUTION We can simplify our work if we clear the equation of fractions. Multiplying both sides by the LCD, 6, we have

$$\mathbf{6} \cdot \frac{1}{3}x^3 = \mathbf{6} \cdot \frac{5}{6}x^2 + \mathbf{6} \cdot \frac{1}{2}x$$

$$2x^3 = 5x^2 + 3x$$

Next we add $-5x^2$ and $-3x$ to each side so that the right side will become 0.

$$2x^3 - 5x^2 - 3x = 0 \qquad \text{Standard form}$$

We factor the left side and then use the zero-factor property to set each factor to 0.

$$\begin{aligned} x(2x^2 - 5x - 3) &= 0 && \text{Factor out the greatest common factor} \\ x(2x + 1)(x - 3) &= 0 && \text{Continue factoring} \\ x = 0 \quad \text{or} \quad 2x + 1 = 0 \quad \text{or} \quad x - 3 &= 0 && \text{Zero-factor property} \end{aligned}$$

Solving each of the resulting equations, we have

$$x = 0 \quad \text{or} \quad x = -\frac{1}{2} \quad \text{or} \quad x = 3$$

To generalize the preceding example, here are the steps used in solving a quadratic equation by factoring.

Strategy To Solve an Equation by Factoring

Step 1: Write the equation in standard form.

Step 2: Factor the left side.

Step 3: Use the zero-factor property to set each factor equal to 0.

Step 4: Solve the resulting linear equations.

EXAMPLE 3 Solve $100x^2 = 300x$.

SOLUTION We begin by writing the equation in standard form and factoring:

$$\begin{aligned} 100x^2 &= 300x \\ 100x^2 - 300x &= 0 && \text{Standard form} \\ 100x(x - 3) &= 0 && \text{Factor} \end{aligned}$$

Using the zero-factor property to set each factor to 0, we have

$$\begin{aligned} 100x = 0 \quad &\text{or} \quad x - 3 = 0 \\ x = 0 \quad &\text{or} \quad x = 3 \end{aligned}$$

The two solutions are 0 and 3.

EXAMPLE 4 Solve $(x - 2)(x + 1) = 4$.

SOLUTION We begin by multiplying the two factors on the left side. (Notice that it would be incorrect to set each of the factors on the left side equal to 4. The fact that the product is 4 does not imply that either of the factors must be 4.)

$$\begin{aligned} (x - 2)(x + 1) &= 4 \\ x^2 - x - 2 &= 4 && \text{Multiply the left side} \\ x^2 - x - 6 &= 0 && \text{Standard form} \\ (x - 3)(x + 2) &= 0 && \text{Factor} \\ x - 3 = 0 \quad \text{or} \quad x + 2 &= 0 && \text{Zero-factor property} \\ x = 3 \quad \text{or} \quad x &= -2 \end{aligned}$$

EXAMPLE 5 Solve $x^3 + 2x^2 - 9x - 18 = 0$ for x.

SOLUTION We start with factoring by grouping.

$$\begin{aligned} x^3 + 2x^2 - 9x - 18 &= 0 \\ x^2(x + 2) - 9(x + 2) &= 0 \\ (x + 2)(x^2 - 9) &= 0 \end{aligned}$$

$$(x + 2)(x - 3)(x + 3) = 0 \quad \text{The difference of two squares}$$

$$x + 2 = 0 \quad \text{or} \quad x - 3 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{Set factors to 0}$$

$$x = -2 \quad \text{or} \quad x = 3 \quad \text{or} \quad x = -3$$

We have three solutions: -2, 3, and -3.

B Applications

EXAMPLE 6 The sum of the squares of two consecutive integers is 25. Find the two integers.

SOLUTION We apply the Blueprint for Problem Solving to solve this application problem. Remember, step 1 in the blueprint is done mentally.

Step 1: Read and list.
Known items: Two consecutive integers. If we add their squares, the result is 25.
Unknown items: The two integers

Step 2: Assign a variable and translate information.
Let x = the first integer; then $x + 1$ = the next consecutive integer.

Step 3: Reread and write an equation.
Since the sum of the squares of the two integers is 25, the equation that describes the situation is

$$x^2 + (x + 1)^2 = 25$$

NOTE
The common factor can be divided out, since 0 divided by any number is 0.

Step 4: Solve the equation.

$$x^2 + (x + 1)^2 = 25$$
$$x^2 + (x^2 + 2x + 1) = 25$$
$$2x^2 + 2x - 24 = 0$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \quad \text{or} \quad x = 3$$

Step 5: Write the answer.
If $x = -4$, then $x + 1 = -3$. If $x = 3$, then $x + 1 = 4$. The two integers are -4 and -3, or the two integers are 3 and 4.

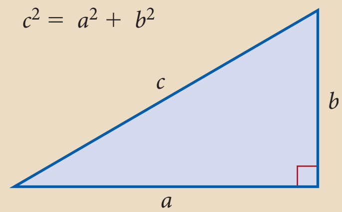
Step 6: Reread and check.
The two integers in each pair are consecutive integers, and the sum of the squares of either pair is 25.

Another application of quadratic equations involves the Pythagorean theorem, an important theorem from geometry. The theorem gives the relationship between the sides of any right triangle (a triangle with a 90-degree angle). We state it here without proof.

Pythagorean Theorem

In any right triangle, the square of the length of the longest side (hypotenuse) is equal to the sum of the squares of the length of the other two sides (legs).

$$c^2 = a^2 + b^2$$



EXAMPLE 7 The lengths of the three sides of a right triangle are given by three consecutive integers. Find the lengths of the three sides.

SOLUTION ***Step 1: Read and list.***

Known items: A right triangle. The three sides are three consecutive integers.

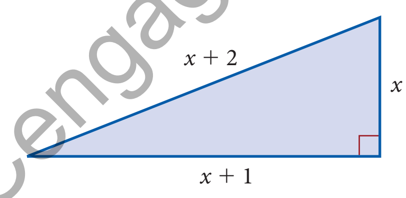
Unknown items: The three sides

Step 2: Assign a variable and translate information.

Let x = first integer (shortest side).

Then $x + 1$ = next consecutive integer

$x + 2$ = last consecutive integer (longest side)



Step 3: Reread and write an equation.

By the Pythagorean theorem, we have

$$(x + 2)^2 = (x + 1)^2 + x^2$$

Step 4: Solve the equation.

$$x^2 + 4x + 4 = x^2 + 2x + 1 + x^2$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \quad \text{or} \quad x = -1$$

Step 5: Write the answer.

Since x is the length of a side in a triangle, it must be a positive number. Therefore, $x = -1$ cannot be used.

The shortest side is 3. The other two sides are 4 and 5.

Step 6: Reread and check.

The three sides are given by consecutive integers. The square of the longest side is equal to the sum of the squares of the two shorter sides.

EXAMPLE 8 Two boats leave from an island port at the same time. One travels due north at a speed of 12 miles per hour, and the other travels due west at a speed of 16 miles per hour. How long until the distance between the boats is 60 miles?

SOLUTION ***Step 1: Read and list.***

Known items: The speed and direction of both boats. The distance between the boats.

Unknown items: The distance traveled by each boat, and the time.

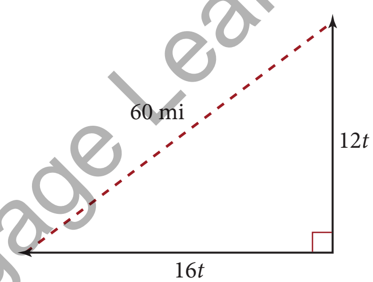
Step 2: Assign a variable and translate information.

Let t = the time.

Then $12t$ = the distance traveled by boat going north

$16t$ = the distance traveled by boat going west

If we draw a diagram for the problem, we see that the distances traveled by the two boats form the legs of a right triangle. The hypotenuse of the triangle will be the distance between the boats, which is 60 miles.



Step 3: Reread and write an equation.

By the Pythagorean theorem, we have

$$(16t)^2 + (12t)^2 = 60^2$$

Step 4: Solve the equation.

$$256t^2 + 144t^2 = 3600$$

$$400t^2 = 3600$$

$$400t^2 - 3600 = 0$$

$$t^2 - 9 = 0$$

$$(t + 3)(t - 3) = 0$$

$$t = -3 \quad \text{or} \quad t = 3$$

Step 5: Write the answer.

Because t is measuring time, it must be a positive number. Therefore, $t = -3$ cannot be used.

The two boats will be 60 miles apart after 3 hours.

Step 6: Reread and check.

The boat going north will travel $12 \cdot 3 = 36$ miles in 3 hours, and the boat going west wil travel $16 \cdot 3 = 48$ miles. The distance between them after 3 hours will be 60 miles ($48^2 + 36^2 = 60^2$).

C Formulas

Our next two examples involve formulas that are quadratic.

EXAMPLE 9 If an object is projected into the air with an initial vertical velocity of v feet/second, its height h, in feet, above the ground after t seconds will be given by

$$h = vt - 16t^2$$

Find t if $v = 64$ feet/second and $h = 48$ feet.

SOLUTION Substituting $v = 64$ and $h = 48$ into the preceding formula, we have

$$48 = 64t - 16t^2$$

which is a quadratic equation. We write it in standard form and solve by factoring.

$$\begin{aligned} 16t^2 - 64t + 48 &= 0 \\ t^2 - 4t + 3 &= 0 \quad \text{Divide each side by 16} \\ (t - 1)(t - 3) &= 0 \\ t - 1 = 0 \quad \text{or} \quad t - 3 &= 0 \\ t = 1 \quad \text{or} \quad t &= 3 \end{aligned}$$

Here is how we interpret our results: If an object is projected upward with an initial vertical velocity of 64 feet/second, it will be 48 feet above the ground after 1 second and after 3 seconds; that is, it passes 48 feet going up and also coming down.

EXAMPLE 10 A manufacturer of headphones knows that the number of headphones she can sell each week is related to the price of the headphones by the equation $x = 1{,}300 - 100p$, where x is the number of headphones and p is the price per set. What price should she charge for each set of headphones if she wants the weekly revenue to be $4,000?

SOLUTION The formula for total revenue is $R = xp$. Since we want R in terms of p, we substitute $1{,}300 - 100p$ for x in the equation $R = xp$.

If $\quad R = xp$

and $\quad x = 1{,}300 - 100p$

then $\quad R = (1{,}300 - 100p)p$

We want to find p when R is 4,000. Substituting 4,000 for R in the formula gives us

$$4{,}000 = (1{,}300 - 100p)p$$

$$4{,}000 = 1{,}300p - 100p^2$$

which is a quadratic equation. To write it in standard form, we add $100p^2$ and $-1{,}300p$ to each side, giving us

$$\begin{aligned} 100p^2 - 1{,}300p + 4{,}000 &= 0 \\ p^2 - 13p + 40 &= 0 \quad \text{Divide each side by 100} \\ (p - 5)(p - 8) &= 0 \\ p - 5 = 0 \quad \text{or} \quad p - 8 &= 0 \\ p = 5 \quad \text{or} \quad p &= 8 \end{aligned}$$

If she sells the headphones for $5 each or for $8 each she will have a weekly revenue of $4,000.

Problem Set 5.8

Moving Toward Success

"You don't save a pitcher for tomorrow. Tomorrow it may rain."

—Leo Durocher, 1905–1991, Hall of Fame American baseball manager and player

1. How can distraction impede your success in this class?
2. How do questions like "Why am I taking this class?" and "When am I ever going to use this stuff?" distract you?

A Solve each equation. [Examples 1–5]

1. $x^2 - 5x - 6 = 0$
2. $x^2 + 5x - 6 = 0$
3. $x^3 - 5x^2 + 6x = 0$
4. $x^3 + 5x^2 + 6x = 0$
5. $3y^2 + 11y - 4 = 0$
6. $3y^2 - y - 4 = 0$
7. $60x^2 - 130x + 60 = 0$
8. $90x^2 + 60x - 80 = 0$
9. $\frac{1}{10}t^2 - \frac{5}{2} = 0$
10. $\frac{2}{7}t^2 - \frac{7}{2} = 0$
11. $100x^4 = 400x^3 + 2{,}100x^2$
12. $100x^4 = -400x^3 + 2{,}100x^2$
13. $\frac{1}{5}y^2 - 2 = -\frac{3}{10}y$
14. $\frac{1}{2}y^2 + \frac{5}{3} = \frac{17}{6}y$
15. $9x^2 - 12x = 0$
16. $4x^2 + 4x = 0$
17. $0.02r + 0.01 = 0.15r^2$
18. $0.02r - 0.01 = -0.08r^2$
19. $9a^3 = 16a$
20. $16a^3 = 25a$
21. $-100x = 10x^2$
22. $800x = 100x^2$
23. $(x + 6)(x - 2) = -7$
24. $(x - 7)(x + 5) = -20$
25. $(y - 4)(y + 1) = -6$
26. $(y - 6)(y + 1) = -12$
27. $(x + 1)^2 = 3x + 7$
28. $(x + 2)^2 = 9x$
29. $(2r + 3)(2r - 1) = -(3r + 1)$
30. $(3r + 2)(r - 1) = -(7r - 7)$
31. $x^3 + 3x^2 - 4x - 12 = 0$
32. $x^3 + 5x^2 - 4x - 20 = 0$
33. $x^3 + 2x^2 - 25x - 50 = 0$
34. $x^3 + 4x^2 - 9x - 36 = 0$
35. $2x^3 + 3x^2 - 8x - 12 = 0$
36. $3x^3 + 2x^2 - 27x - 18 = 0$
37. $4x^3 + 12x^2 - 9x - 27 = 0$
38. $9x^3 + 18x^2 - 4x - 8 = 0$

A Problems 39–48 are problems you will see later in the book. Solve each equation. [Examples 1–5]

39. $3x^2 + x = 10$
40. $y^2 + y - 20 = 2y$
41. $12(x + 3) + 12(x - 3) = 3(x^2 - 9)$
42. $8(x + 2) + 8(x - 2) = 3(x^2 - 4)$
43. $(y + 3)^2 + y^2 = 9$
44. $(2y + 4)^2 + y^2 = 4$
45. $(x + 3)^2 + 1^2 = 2$
46. $(x - 3)^2 + (-1)^2 = 10$
47. $(x + 2)(x) = 2^3$
48. $(x + 3)(x) = 2^2$
49. Let $f(x) = \left(x + \frac{3}{2}\right)^2$. Find all values for the variable x, for which $f(x) = 0$.
50. Let $f(x) = \left(x - \frac{5}{2}\right)^2$. Find all values for the variable x, for which $f(x) = 0$.
51. Let $f(x) = (x - 3)^2 - 25$. Find all values for the variable x, for which $f(x) = 0$.
52. Let $f(x) = 9x^3 + 18x^2 - 4x - 8$. Find all values for the variable x, for which $f(x) = 0$.

Let $f(x) = x^2 + 6x + 3$. Find all values for the variable x, for which $f(x) = g(x)$.

53. $g(x) = -6$

54. $g(x) = 19$

55. $g(x) = 10$

56. $g(x) = -2$

Let $h(x) = x^2 - 5x$. Find all values for the variable x, for which $h(x) = f(x)$.

57. $f(x) = 0$

58. $f(x) = -6$

59. $f(x) = 2x + 8$

60. $f(x) = -2x + 10$

61. Solve each equation.

a. $9x - 25 = 0$

b. $9x^2 - 25 = 0$

c. $9x^2 - 25 = 56$

d. $9x^2 - 25 = 30x - 50$

62. Solve each equation.

a. $5x - 6 = 0$

b. $(5x - 6)^2 = 0$

c. $25x^2 - 36 = 0$

d. $25x^2 - 36 = 28$

A Applying the Concepts [Examples 6–8]

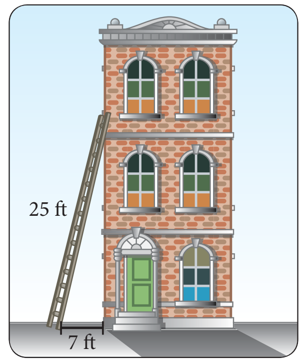
63. Distance Two cyclists leave from an intersection at the same time. One travels due north at a speed of 15 miles per hour, and the other travels due east at a speed of 20 miles per hour. How long until the distance between the two cyclists is 75 miles?

64. Distance Two airplanes leave from an airport at the same time. One travels due south at a speed of 480 miles per hour, and the other travels due west at a speed of 360 miles per hour. How long until the distance between the two airplanes is 2400 miles?

65. Consecutive Integers The square of the sum of two consecutive integers is 81. Find the two integers.

66. Consecutive Integers Find two consecutive even integers whose sum squared is 100.

67. Right Triangle A 25-foot ladder is leaning against a building. The base of the ladder is 7 feet from the side of the building. How high does the ladder reach along the side of the building?



68. Right Triangle Noreen wants to place a 13-foot ramp against the side of her house so that the top of the ramp rests on a ledge that is 5 feet above the ground. How far will the base of the ramp be from the house?

69. Right Triangle The lengths of the three sides of a right triangle are given by three consecutive even integers. Find the lengths of the three sides.

70. Right Triangle The longest side of a right triangle is 3 less than twice the shortest side. The third side measures 12 inches. Find the length of the shortest side.

71. Geometry The length of a rectangle is 2 feet more than 3 times the width. If the area is 16 square feet, find the width and the length.

72. Geometry The length of a rectangle is 4 yards more than twice the width. If the area is 70 square yards, find the width and the length.

73. Geometry The base of a triangle is 2 inches more than 4 times the height. If the area is 36 square inches, find the base and the height.

74. Geometry The height of a triangle is 4 feet less than twice the base. If the area is 48 square feet, find the base and the height.

75. Projectile Motion If an object is thrown straight up into the air with an initial velocity of 32 feet per second, then its height above the ground at any time t is given by the formula $h = 32t - 16t^2$. Find the times at which the object is on the ground by letting $h = 0$ in the equation and solving for t.

76. **Projectile Motion** An object is projected into the air with an initial velocity of 64 feet per second. Its height at any time t is given by the formula $h = 64t - 16t^2$. Find the times at which the object is on the ground.

C The formula $h = vt - 16t^2$ gives the height h, in feet, of an object projected into the air with an initial vertical velocity v, in feet per second, after t seconds.
[Examples 9–10]

77. **Projectile Motion** If an object is projected upward with an initial velocity of 48 feet per second, at what times will it reach a height of 32 feet above the ground?

78. **Projectile Motion** If an object is projected upward into the air with an initial velocity of 80 feet per second, at what times will it reach a height of 64 feet above the ground?

79. **Projectile Motion** An object is projected into the air with a vertical velocity of 24 feet per second. At what times will the object be on the ground? (It is on the ground when h is 0.)

80. **Projectile Motion** An object is projected into the air with a vertical velocity of 20 feet per second. At what times will the object be on the ground?

81. **Height of a Bullet** A bullet is fired into the air with an initial upward velocity of 80 feet per second from the top of a building 96 feet high. The equation that gives the height of the bullet at any time t is $h = 96 + 80t - 16t^2$. At what times will the bullet be 192 feet in the air?

82. **Height of an Arrow** An arrow is shot into the air with an upward velocity of 48 feet per second from a hill 32 feet high. The equation that gives the height of the arrow at any time t is $h = 32 + 48t - 16t^2$. Find the times at which the arrow will be 64 feet above the ground.

83. **Price and Revenue** A company that manufactures typewriter ribbons knows that the number of ribbons x it can sell each week is related to the price per ribbon p by the equation $x = 1{,}200 - 100p$. At what price should it sell the ribbons if it wants the weekly revenue to be \$3,200? (*Remember:* The equation for revenue is $R = xp$.)

84. **Price and Revenue** A company manufactures CDs for home computers. It knows from past experience that the number of CDs x it can sell each day is related to the price per CD p by the equation $x = 800 - 100p$. At what price should it sell its CDs if it wants the daily revenue to be \$1,200?

85. **Price and Revenue** The relationship between the number of calculators x a company sells per day and the price of each calculator p is given by the equation $x = 1{,}700 - 100p$. At what price should the calculators be sold if the daily revenue is to be \$7,000?

86. **Price and Revenue** The relationship between the number of pencil sharpeners x a company can sell each week and the price of each sharpener p is given by the equation $x = 1{,}800 - 100p$. At what price should the sharpeners be sold if the weekly revenue is to be \$7,200?

Maintaining Your Skills

Solve each system.

87. $\begin{aligned} 2x - 5y &= -8 \\ 3x + y &= 5 \end{aligned}$

88. $\begin{aligned} 4x - 7y &= -2 \\ -5x + 6y &= -3 \end{aligned}$

89. $\begin{aligned} \frac{1}{3}x - \frac{1}{6}y &= 3 \\ -\frac{1}{5}x + \frac{1}{4}y &= 0 \end{aligned}$

90. $\begin{aligned} 2x - 5y &= 14 \\ y &= 3x + 8 \end{aligned}$

91. $\begin{aligned} 2x - y + z &= 9 \\ x + y - 3z &= -2 \\ 3x + y - z &= 6 \end{aligned}$

92. **Number Problem** A number is 1 less than twice another. Their sum is 14. Find the two numbers.

93. **Investing** John invests twice as much money at 6% as he does at 5%. If his investments earn a total of \$680 in 1 year, how much does he have invested at each rate?

94. **Speed of a Boat** A boat can travel 20 miles downstream in 2 hours. The same boat can travel 18 miles upstream in 3 hours. What is the speed of the boat in still water, and what is the speed of the current?

Graph the solution set for each system.

95. $\begin{aligned} 3x + 2y &< 6 \\ -2x + 3y &< 6 \end{aligned}$

96. $\begin{aligned} y &\le x + 3 \\ y &> x - 4 \end{aligned}$

97. $\begin{aligned} x &\le 4 \\ y &< 2 \end{aligned}$

98. $\begin{aligned} 2x + y &< 4 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$

Chapter 5 Summary

Properties of Exponents [5.1]

EXAMPLES

1. These expressions illustrate the properties of exponents.

a. $x^2 \cdot x^3 = x^{2+3} = x^5$

b. $(x^2)^3 = x^{2 \cdot 3} = x^6$

c. $(3x)^2 = 3^2 \cdot x^2 = 9x^2$

d. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

e. $\left(\frac{x}{5}\right)^2 = \frac{x^2}{5^2} = \frac{x^2}{25}$

f. $\frac{x^7}{x^5} = x^{7-5} = x^2$

g. $3^1 = 3$
$3^0 = 1$

If a and b represent real numbers and r and s represent integers, then

1. $a^r \cdot a^s = a^{r+s}$

2. $(a^r)^s = a^{r \cdot s}$

3. $(ab)^r = a^r \cdot b^r$

4. $a^{-r} = \frac{1}{a^r} \qquad (a \neq 0)$

5. $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad (b \neq 0)$

6. $\frac{a^r}{a^s} = a^{r-s} \qquad (a \neq 0)$

7. $a^1 = a$
$a^0 = 1 \qquad (a \neq 0)$

Scientific Notation [5.1]

2. $49{,}800{,}000 = 4.98 \times 10^7$
$0.00462 = 4.62 \times 10^{-3}$

A number is written in scientific notation when it is written as the product of a number between 1 and 10 and an integer power of 10; that is, when it has the form

$$n \times 10^r$$

where $1 \leq n < 10$ and r = an integer.

Addition of Polynomials [5.2]

3. $(3x^2 + 2x - 5) + (4x^2 - 7x + 2)$
$= 7x^2 - 5x - 3$

To add two polynomials, simply combine the coefficients of similar terms.

Negative Signs Preceding Parentheses [5.2]

4. $-(2x^2 - 8x - 9)$
$= -2x^2 + 8x + 9$

If there is a negative sign directly preceding the parentheses surrounding a polynomial, we may remove the parentheses and preceding negative sign by changing the sign of each term within the parentheses. (This procedure is actually just another application of the distributive property.)

Multiplication of Polynomials [5.3]

5. $(3x - 5)(x + 2)$
$= 3x^2 + 6x - 5x - 10$
$= 3x^2 + x - 10$

To multiply two polynomials, multiply each term in the first by each term in the second.

Special Products [5.3]

6. The following are examples of the three special products:
$(x + 3)^2 = x^2 + 6x + 9$
$(5 - x)^2 = 25 - 10x + x^2$
$(x + 7)(x - 7) = x^2 - 49$

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a + b)(a - b) = a^2 - b^2$$

Business Applications [5.2, 5.3, 5.4]

7. A company makes x items each week and sells them for p dollars each, according to the equation
$p = 35 - 0.1x$. Then, the revenue is
$$R = x(35 - 0.1x) = 35x - 0.1x^2$$
If the total cost to make all x items is $C = 8x + 500$, then the profit gained by selling the x items is
$$P = 35x - 0.1x^2 - (8x + 500)$$
$$= -500 + 27x - 0.1x^2$$

If a company manufacturers and sells x items at p dollars per item, then the revenue R is given by the formula

$$R = xp$$

If the total cost to manufacture all x items is C, then the profit obtained from selling all x items is

$$P = R - C$$

Greatest Common Factor [5.4]

8. The greatest common factor of $10x^5 - 15x^4 + 30x^3$ is $5x^3$. Factoring it out of each term, we have
$$5x^3(2x^2 - 3x + 6)$$

The greatest common factor of a polynomial is the largest monomial (the monomial with the largest coefficient and highest exponent) that divides each term of the polynomial. The first step in factoring a polynomial is to factor the greatest common factor (if it is other than 1) out of each term.

Factoring Trinomials [5.5]

9. $x^2 + 5x + 6 = (x + 2)(x + 3)$
$x^2 - 5x + 6 = (x - 2)(x - 3)$
$x^2 + x - 6 = (x - 2)(x + 3)$
$x^2 - x - 6 = (x + 2)(x - 3)$

We factor a trinomial by writing it as the product of two binomials. (This refers to trinomials whose greatest common factor is 1.) Each factorable trinomial has a unique set of factors. Finding the factors is sometimes a matter of trial and error.

Special Factoring [5.6]

10. Here are some binomials that have been factored this way.
$x^2 + 6x + 9 = (x + 3)^2$
$x^2 - 6x + 9 = (x - 3)^2$
$x^2 - 9 = (x + 3)(x - 3)$
$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$
$x^3 + 27 = (x + 3)(x^2 - 3x + 9)$

$a^2 + 2ab + b^2 = (a + b)^2$ Perfect square trinomials
$a^2 - 2ab + b^2 = (a - b)^2$
$a^2 - b^2 = (a - b)(a + b)$ Difference of two squares
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ Difference of two cubes
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ Sum of two cubes

To Factor Polynomials in General [5.7]

11. Factor completely.

a. $3x^3 - 6x^2 = 3x^2(x - 2)$

b. $x^2 - 9 = (x + 3)(x - 3)$
$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$
$x^3 + 27 = (x + 3)(x^2 - 3x + 9)$

c. $x^2 - 6x + 9 = (x - 3)^2$
$6x^2 - 7x - 5 = (2x + 1)(3x - 5)$

d. $x^2 + ax + bx + ab$
$= x(x + a) + b(x + a)$
$= (x + a)(x + b)$

Step 1: If the polynomial has a greatest common factor other than 1, then factor out the greatest common factor.

Step 2: If the polynomial has two terms (it is a binomial), then see if it is the difference of two squares, or the sum or difference of two cubes, and then factor accordingly. Remember, if it is the sum of two squares it will not factor.

Step 3: If the polynomial has three terms (a trinomial), then it is either a perfect square trinomial, which will factor into the square of a binomial, or it is not a perfect square trinomial, in which case you use one of the methods developed in Section 5.5.

Step 4: If the polynomial has more than three terms, then try to factor it by grouping.

Step 5: As a final check, see if any of the factors you have written can be factored further. If you have overlooked a common factor, you can catch it here.

To Solve an Equation by Factoring [5.8]

12. Solve $x^2 - 5x = -6$.

$$x^2 - 5x + 6 = 0$$
$$(x - 3)(x - 2) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 3 \quad \text{or} \quad x = 2$$

Step 1: Write the equation in standard form.

Step 2: Factor the left side.

Step 3: Use the zero-factor property to set each factor equal to zero.

Step 4: Solve the resulting linear equations.

COMMON MISTAKES

When we subtract one polynomial from another, it is common to forget to add the opposite of each term in the second polynomial. For example

$$(6x - 5) - (3x + 4) = 6x - 5 - 3x + 4 \qquad \text{Mistake}$$

This mistake occurs if the negative sign outside the second set of parentheses is not distributed over all terms inside the parentheses. To avoid this mistake, remember: the opposite of a sum is the sum of the opposites, or,

$$-(3x + 4) = -3x + (-4)$$

Chapter 5 Review

Simplify each of the following. [5.1]

1. $x^3 \cdot x^7$

2. $(5x^3)^2$

3. $(2x^3y)^2(-2x^4y^2)^3$

Write with positive exponents, and then simplify. [5.1]

4. 2^{-3}

5. $\left(\frac{2}{3}\right)^{-2}$

6. $2^{-2} + 4^{-1}$

Write in scientific notation. [5.1]

7. 34,500,000

8. 0.00357

Write in expanded form. [5.1]

9. 4.45×10^4

10. 4.45×10^{-4}

Simplify each expression. All answers should contain positive exponents only. (Assume all variables are nonnegative.) [5.1]

11. $\dfrac{a^{-4}}{a^5}$

12. $\dfrac{(4x^2)(-3x^3)^2}{(12x^{-2})^2}$

13. $\dfrac{x^n x^{3n}}{x^{4n-2}}$

Simplify each expression as much as possible. Write all answers in scientific notation. [5.1]

14. $(2 \times 10^3)(4 \times 10^{-5})$

15. $\dfrac{(600{,}000)(0.000008)}{(4{,}000)(3{,}000{,}000)}$

Simplify by combining similar terms. [5.2]

16. $(6x^2 - 3x + 2) - (4x^2 + 2x - 5)$

17. $(x^3 - x) - (x^2 + x) + (x^3 - 3) - (x^2 + 1)$

18. Subtract $2x^2 - 3x + 1$ from $3x^2 - 5x - 2$.

19. Simplify $-3[2x - 4(3x + 1)]$.

20. Find the value of $2x^2 - 3x + 1$ when x is -2.

Multiply. [5.3]

21. $3x(4x^2 - 2x + 1)$

22. $2a^2b^3(a^2 + 2ab + b^2)$

23. $(6 - y)(3 - y)$

24. $(2x^2 - 1)(3x^2 + 4)$

25. $2t(t + 1)(t - 3)$

26. $(x + 3)(x^2 - 3x + 9)$

27. $(2x - 3)(4x^2 + 6x + 9)$

28. $(a^2 - 2)^2$

29. $(3x + 5)^2$

30. $(4x - 3y)^2$

31. $\left(x - \dfrac{1}{3}\right)\left(x + \dfrac{1}{3}\right)$

32. $(2a + b)(2a - b)$

33. $(x - 1)^3$

34. $(x^m + 2)(x^m - 2)$

Factor out the greatest common factor. [5.4]

35. $6x^4y - 9xy^4 + 18x^3y^3$

36. $4x^2(x + y)^2 - 8y^2(x + y)^2$

Factor by grouping. [5.4, 5.6]

37. $8x^2 + 10 - 4x^2y - 5y$

38. $x^3 + 8b^2 - x^3y^2 - 8y^2b^2$

Factor completely. [5.4, 5.5]

39. $x^2 - 5x + 6$

40. $2x^3 + 4x^2 - 30x$

41. $20a^2 - 41ab + 20b^2$

42. $6x^4 - 11x^3 - 10x^2$

43. $24x^2y - 6xy - 45y$

Factor completely. [5.6]

44. $x^4 - 16$

45. $3a^4 + 18a^2 + 27$

46. $a^3 - 8$

47. $5x^3 + 30x^2y + 45xy^2$

48. $3a^3b - 27ab^3$

49. $x^2 - 10x + 25 - y^2$

50. $36 - 25a^2$

51. $x^3 + 4x^2 - 9x - 36$

Solve each equation. [5.8]

52. $x^2 + 5x + 6 = 0$

53. $\frac{5}{6}y^2 = \frac{1}{4}y + \frac{1}{3}$

54. $9x^2 - 25 = 0$

55. $5x^2 = -10x$

56. $(x + 2)(x - 5) = 8$

57. $x^3 + 4x^2 - 9x - 36 = 0$

Solve each application. In each case be sure to show the equation used. [5.8]

58. Consecutive Numbers The product of two consecutive even integers is 80. Find the two integers.

59. Consecutive Numbers The sum of the squares of two consecutive integers is 41. Find the two integers.

60. Geometry The lengths of the three sides of a right triangle are given by three consecutive integers. Find the three sides.

61. Geometry The lengths of three sides of a right triangle are given by three consecutive even integers. Find the three sides.

Chapter 5 Cumulative Review

Simplify each of the following.

1. $(-5)^3$

2. $(-5)^{-3}$

3. $|-32 - 41| - 13$

4. $2^3 + 2(6^2 - 4^2)$

5. $96 \div 8 \cdot 4$

6. $55 \div 5 \cdot 11$

7. $\dfrac{3^3 + 5}{-2(5^2 - 3^2)}$

8. $\dfrac{2^3(17 - 2 \cdot 4)}{7^2 - 13}$

Let $P(x) = 11.5x - 0.01x^2$. Find the following.

9. $P(1)$

10. $P(-1)$

Let $Q(x) = \dfrac{3}{2}x^2 + \dfrac{3}{4}x - 5$. Find the following.

11. $Q(4)$

12. $Q(-4)$

Simplify each of the following expressions.

13. $6\left(x - \dfrac{1}{6}\right)$

14. $15\left(\dfrac{4}{3}x - \dfrac{3}{5}y\right)$

15. $(3x - 2) - (5x + 2)$

16. $3a - 2[4 - (a + 3)]$

17. $(3x^3 - 2) - (5x^2 + 2) + (5x^3 - 6) - (2x^2 + 4)$

18. $\left(\dfrac{5}{6}x^2 - \dfrac{2}{3}x\right) - \left(\dfrac{5}{3}x^2 + \dfrac{1}{2}x\right) + \left(\dfrac{5}{2}x^2 - \dfrac{1}{6}x\right) - \left(\dfrac{4}{3}x^2 + \dfrac{3}{2}x\right)$

Multiply or divide as indicated.

19. $(3x^2y)^4$

20. $(-3a^2b)(5a^3b^4)$

21. $\dfrac{32x^2y^6}{-16xy^2}$

22. $\dfrac{(-3a^2b)^4}{(a^3b^4)^2}$

23. $(3x - 2)(5x + 2)$

24. $5x(3x^2 + 2x + 7)$

25. $\left(x - \dfrac{2}{3}\right)^2$

26. $\left(t + \dfrac{2}{3}\right)^3$

27. $(4 \times 10^3)(7 \times 10^6)$

28. $\dfrac{8 \times 10^6}{2 \times 10^4}$

Factor each of the following expressions.

29. $x^2 - xy - ax + ay$

30. $6a^2 + a - 35$

31. $(x + 3)^2 - 25$

32. $\dfrac{1}{8} + t^3$

Solve the following equations.

33. $3x + 2 = 8$

34. $|2x - 4| + 7 = 13$

35. $\dfrac{3}{4}x - 7 = -34$

36. $-\dfrac{x}{2} = -\dfrac{3}{4} - \dfrac{3}{2}x$

37. $25a^3 = 36a$

38. $(x + 3)^2 = -2x - 7$

Solve each system.

39. $x - y = -4$
$x + 3y = 4$

40. $y = \dfrac{2}{3}x - 3$
$y = -\dfrac{3}{2}x + 23$

Find the slope of the line through the given points.

41. $(-2, 1), (5, -4)$

42. $(-10, -6), (-4, -4)$

Let $f(x) = \dfrac{3x - 5}{2}$ and $g(x) = \dfrac{2x + 5}{3}$. Find the following.

43. $f(-7)$

44. $g(-13)$

45. $(g \circ f)(2)$

46. $(f \circ g)(3)$

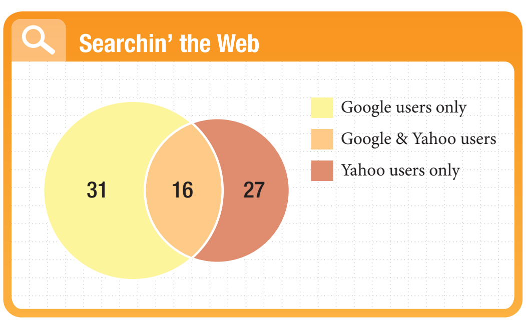
47. y varies inversely with the square of x. If $y = 4$ when $x = 4$, find y when $x = 6$.

48. z varies jointly with x and the cube of y. If $z = -48$ when $x = 3$ and $y = 2$, find z when $x = 2$ and $y = 3$.

49. Geometry The height of a triangle is 5 feet less than 2 times the base. If the area is 75 square feet, find the base and height.

50. Geometry Find all three angles in a triangle if the smallest angle is one-sixth the largest angle and the remaining angle is 20 degrees more than the smallest angle.

The results of a survey of 80 internet users is displayed here as a Venn diagram. Use this information to answer Questions 51 and 52.



51. How many of the internet users surveyed use either Yahoo or Google?

52. How many of the internet users surveyed use neither Yahoo nor Google?

Chapter 5 Test

Simplify. All answers should contain positive exponents only. (Assume all variables are nonzero.) [5.1]

1. $\left(\frac{3}{4}\right)^{-2}$

2. 2^{-5}

3. $x^4 \cdot x^7 \cdot x^{-3}$

4. $\frac{a^{-5}}{a^{-7}}$

5. $(2x^2y)^3(2x^3y^4)^2$

6. $\frac{(2ab^3)^{-2}}{(a^{-4}b^3)^4}$

Write each number in scientific notation. [5.1]

7. 6,530,000

8. 0.00087

Perform the indicated operations, and write your answers in scientific notation. [5.1]

9. $(2.9 \times 10^{12})(3 \times 10^{-5})$

10. $\frac{(6 \times 10^{-4})(4 \times 10^{9})}{8 \times 10^{-3}}$

Let $P(x) = 15x - 0.01x^2$ and find the following.

11. $P(10)$

12. $P(-10)$

Simplify the following expressions. [5.2]

13. $\left(\frac{3}{4}x^3 - x^2 - \frac{3}{2}\right) - \left(\frac{1}{4}x^2 + 2x - \frac{1}{2}\right)$

14. $3 - 4[2x - 3(x + 6)]$

Multiply. [5.3]

15. $(3y - 7)(2y + 5)$

16. $(2x - 5)(x^2 + 4x - 3)$

17. $(8 - 3t^3)^2$

18. $(1 - 6y)(1 + 6y)$

19. $2x(x - 3)(2x + 5)$

20. $\left(5t^2 - \frac{1}{2}\right)\left(2t^2 + \frac{1}{5}\right)$

Factor the following expressions. [5.4, 5.5, 5.6, 5.7]

21. $x^2 + x - 12$

22. $12x^4 + 26x^2 - 10$

23. $16a^4 - 81y^4$

24. $7ax^2 - 14ay - b^2x^2 + 2b^2y$

25. $t^3 + \frac{1}{8}$

26. $4a^5b - 24a^4b^2 - 64a^3b^3$

27. $x^2 - 10x + 25 - b^2$

28. $81 - x^4$

Solve each equation. [5.8]

29. $\frac{1}{5}x^2 = \frac{1}{3}x + \frac{2}{15}$

30. $100x^3 = 500x^2$

31. $(x + 1)(x + 2) = 12$

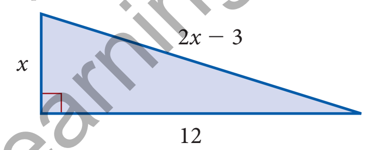
32. $x^3 + 2x^2 - 16x - 32 = 0$

Let $f(x) = x^2 - 5x + 6$. Find all values for the variable x for which $f(x) = g(x)$. [5.8]

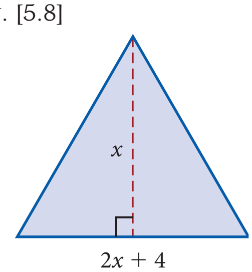
33. $g(x) = 0$

34. $g(x) = x - 2$

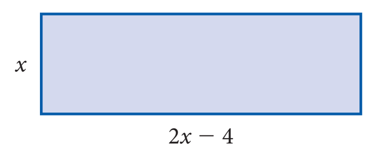
35. Find the value of the variable x in the following figure. [5.8]



36. The area of the figure below is 35 square inches. Find the value of x. [5.8]



37. The area of the figure below is 48 square centimeters. Find the value of x. [5.8]



Profit, Revenue, and Cost A company making ceramic coffee cups finds that it can sell x cups per week at p dollars each, according to the formula $p = 25 - 0.2x$. If the total cost to produce and sell x coffee cups is $C = 2x + 100$, find [5.2, 5.3]

38. an equation for the revenue that gives the revenue in terms of x.

39. the profit equation.

40. the revenue brought in by selling 100 coffee cups.

41. the cost of producing 100 coffee cups.

42. the profit obtained by making and selling 100 coffee cups.

Chapter 5 Projects

EXPONENTS AND POLYNOMIALS

GROUP PROJECT

Discovering Pascal's Triangle

Number of People 3

Time Needed 20 minutes

Equipment Paper and pencils

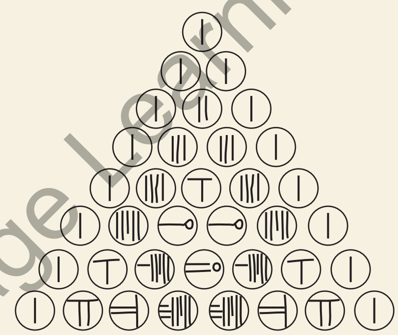
Background The triangular array of numbers shown here is known as Pascal's triangle, after the French philosopher Blaise Pascal (1623–1662).

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
```

Procedure Look at Pascal's triangle and discover how the numbers in each row of the triangle are obtained from the numbers in the row above it.

1. Once you have discovered how to extend the triangle, write the next two rows.

2. Pascal's triangle can be linked to the Fibonacci sequence by rewriting Pascal's triangle so that the 1s on the left side of the triangle line up under one another and the other columns are equally spaced to the right of the first column. Rewrite Pascal's triangle as indicated and then look along the diagonals of the new array until you discover how the Fibonacci sequence can be obtained from it.



Pascal's triangle in Japanese (1781)

3. The diagram above shows Pascal's triangle as written in Japanese in 1781. Use your knowledge of Pascal's triangle to translate the numbers written in Japanese into our number system. Then write down the Japanese numbers from 1 to 20.

RESEARCH PROJECT

Binomial Expansions

The title on the following diagram is *Binomial Expansions* because each line gives the expansion of the binomial $x + y$ raised to a whole-number power.

Binomial Expansions

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 =$$

$$(x + y)^5 =$$

The fourth row in the diagram was completed by expanding $(x + y)^3$ using the methods developed in this chapter. Next, complete the diagram by expanding the binomials $(x + y)^4$ and $(x + y)^5$ using the multiplication procedures you have learned in this chapter. Finally, study the completed diagram until you see patterns that will allow you to continue the diagram one more row without using multiplication. (One pattern that you will see is Pascal's triangle, which we mentioned in the preceding group project.) When you are finished, write an essay in which you describe what you have done and the results you have obtained.