A surface of revolution is formed when a curve is rotated about a line. Such a surface is the lateral boundary of a solid of revolution of the type discussed in Sections 7.2 and 7.3.

We want to define the area of a surface of revolution in such a way that it corresponds to our intuition. If the surface area is $A$, we can imagine that painting the surface would require the same amount of paint as does a flat region with area $A$.

Let’s start with some simple surfaces. The lateral surface area of a circular cylinder with radius $r$ and height $h$ is taken to be $2\pi rh$ because we can imagine cutting the cylinder and unrolling it (as in Figure 1) to obtain a rectangle with dimensions $2\pi r$ and $h$.

Likewise, we can take a circular cone with base radius $r$ and slant height $l$, cut it along the dashed line in Figure 2, and flatten it to form a sector of a circle with radius $l$ and central angle $\theta = \frac{2\pi}{l}$. We know that, in general, the area of a sector of a circle with radius $r$ and angle $\theta$ is $\frac{1}{2}r^2\theta$ (see Exercise 67 in Section 6.2) and so in this case it is

$$A = \frac{1}{2}l^2\theta = \frac{1}{2}l^2\left(\frac{2\pi}{l}\right) = \pi rl$$

Therefore, we define the lateral surface area of a cone to be $A = \pi rl$.

What about more complicated surfaces of revolution? If we follow the strategy we used with arc length, we can approximate the original curve by a polygon. When this polygon is rotated about an axis, it creates a simpler surface whose surface area approximates the actual surface area. By taking a limit, we can determine the exact surface area.

The approximating surface, then, consists of a number of bands, each formed by rotating a line segment about an axis. To find the surface area, each of these bands can be considered a portion of a circular cone, as shown in Figure 3. The area of the band (or frustum of a cone) with slant height $l$ and upper and lower radii $r_1$ and $r_2$ is found by subtracting the areas of two cones:

$$A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi [(r_2 - r_1)l_1 + r_2l]$$

From similar triangles we have

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

which gives

$$r_2l_1 = r_1l_1 + r_1l \quad \text{or} \quad (r_2 - r_1)l_1 = r_1l$$

Putting this in Equation 1, we get

$$A = \pi (r_1l + r_2l)$$

or

$$A = 2\pi rl$$

where $r = \frac{1}{2}(r_1 + r_2)$ is the average radius of the band.
Now we apply this formula to our strategy. Consider the surface shown in Figure 4, which is obtained by rotating the curve \( y = f(x), a \leq x \leq b, \) about the \( x \)-axis, where \( f \) is positive and has a continuous derivative. In order to define its surface area, we divide the interval \([a, b]\) into \( n \) subintervals with endpoints \( x_0, x_1, \ldots, x_n \) and equal width \( \Delta x \), as we did in determining arc length. If \( y_i = f(x_i) \), then the point \( P_i(x_i, y_i) \) lies on the curve. The part of the surface between \( x_i-1 \) and \( x_i \) is approximated by taking the line segment \( P_{i-1}P_i \) and rotating it about the \( x \)-axis. The result is a band with slant height \( l = |P_{i-1}P_i| \) and average radius \( r = \frac{1}{2}(y_i-1 + y_i) \) so, by Formula 2, its surface area is

\[
2\pi \frac{y_i-1 + y_i}{2} |P_{i-1}P_i|
\]

As in the proof of Theorem 7.4.2, we have

\[
|P_{i-1}P_i| = \sqrt{1 + [f'(x_i)]^2} \Delta x
\]

where \( x_i^* \) is some number in \([x_{i-1}, x_i]\). When \( \Delta x \) is small, we have \( y_i = f(x_i) \approx f(x_i^*) \) and also \( y_{i-1} = f(x_{i-1}) = f(x_i^*) \), since \( f \) is continuous. Therefore

\[
2\pi \frac{y_i-1 + y_i}{2} |P_{i-1}P_i| = 2\pi f(x_i) \sqrt{1 + [f'(x_i)]^2} \Delta x
\]

and so an approximation to what we think of as the area of the complete surface of revolution is

\[ \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx \]

This approximation appears to become better as \( n \to \infty \) and, recognizing (3) as a Riemann sum for the function \( g(x) = 2\pi f(x) \sqrt{1 + [f'(x)]^2} \), we have

\[
\lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i) \sqrt{1 + [f'(x_i)]^2} \Delta x = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx
\]

Therefore, in the case where \( f \) is positive and has a continuous derivative, we define the surface area of the surface obtained by rotating the curve \( y = f(x), a \leq x \leq b, \) about the \( x \)-axis as

\[
S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx
\]

With the Leibniz notation for derivatives, this formula becomes

\[ S = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

If the curve is described as \( x = g(y), c \leq y \leq d, \) then the formula for surface area becomes

\[ S = \int_c^d 2\pi y \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \]

and both Formulas 5 and 6 can be summarized symbolically, using the notation for arc
length given in Section 7.4, as

$$S = \int 2\pi y \, ds$$

For rotation about the $y$-axis, the surface area formula becomes

$$S = \int 2\pi x \, ds$$

where, as before, we can use either

$$ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \quad \text{or} \quad ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy$$

These formulas can be remembered by thinking of $2\pi y$ or $2\pi x$ as the circumference of a circle traced out by the point $(x, y)$ on the curve as it is rotated about the $x$-axis or $y$-axis, respectively (see Figure 5).

**EXAMPLE 1** The curve $y = \sqrt{4 - x^2}$, $-1 \leq x \leq 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the $x$-axis. (The surface is a portion of a sphere of radius 2. See Figure 6.)

**SOLUTION** We have

$$\frac{dy}{dx} = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2}}$$

and so, by Formula 5, the surface area is

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}} \, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \frac{2}{\sqrt{4 - x^2}} \, dx$$

$$= 4\pi \int_{-1}^{1} 1 \, dx = 4\pi(2) = 8\pi$$
EXAMPLE 2 The arc of the parabola \( y = x^2 \) from \( (1, 1) \) to \( (2, 4) \) is rotated about the y-axis. Find the area of the resulting surface.

**SOLUTION 1** Using  
\[
y = x^2 \quad \text{and} \quad \frac{dy}{dx} = 2x
\]
we have, from Formula 8,
\[
S = \int 2\pi x \, ds \\
= \int_x^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\
= 2\pi \int_x^2 x \sqrt{1 + 4x^2} \, dx
\]
Substituting \( u = 1 + 4x^2 \), we have \( du = 8x \, dx \). Remembering to change the limits of integration, we have
\[
S = \frac{\pi}{4} \int_6^{17} \sqrt{u} \, du = \frac{\pi}{4} \left[ \frac{2}{3}u^{3/2} \right]_6^{17} \\
= \frac{\pi}{6} \left( 17\sqrt{17} - 5\sqrt{5} \right)
\]

**SOLUTION 2** Using  
\[
y = x^2 \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}
\]
we have
\[
S = \int 2\pi x \, ds = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \\
= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}} \, dy = \pi \int_1^4 \sqrt{4y + 1} \, dy \\
= \frac{\pi}{6} \int_0^{17} u \, du \quad \text{(where } u = 1 + 4y \text{)} \\
= \frac{\pi}{6} \left( 17\sqrt{17} - 5\sqrt{5} \right) \quad \text{(as in Solution 1)}
\]

EXAMPLE 3 Find the area of the surface generated by rotating the curve \( y = e^x \), \( 0 \leq x \leq 1 \), about the x-axis.

**SOLUTION** Using Formula 5 with  
\[
y = e^x \quad \text{and} \quad \frac{dy}{dx} = e^x
\]
we have
\[ S = \int_a^b 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 2\pi \int_a^b \sqrt{1 + e^{2x}} \, dx \]
\[ = 2\pi \int_{e^{-1}}^{e^1} \sqrt{1 + u^2} \, du \quad \text{(where } u = e^x) \]
\[ = 2\pi \int_{e^{-1/2}}^{e^{1/2}} \sec^2 \theta \, d\theta \quad \text{(where } u = \tan \theta \text{ and } \alpha = \tan^{-1} e) \]

** Or use Formula 21 in the Table of Integrals.

\[ = 2\pi \cdot \frac{1}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|]_{e^{-1/2}}^{e^{1/2}} \quad \text{(by Example 8 in Section 6.2)} \]
\[ = \pi [\sec \alpha \tan \alpha + \ln(\sec \alpha + \tan \alpha) - \sqrt{2} - \ln(\sqrt{2} + 1)] \]

Since \( \tan \alpha = e \), we have \( \sec^2 \alpha = 1 + \tan^2 \alpha = 1 + e^2 > 1 \)
\[ S = \pi [e \sqrt{1 + e^2} + \ln(e + \sqrt{1 + e^2}) - \sqrt{2} - \ln(\sqrt{2} + 1)] \]

\[ \text{EXERCISES} \]

1–4. Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve about the given axis.
1. \( y = \ln x \), \( 1 \leq x \leq 3 \); \(-\text{axis} \)
2. \( y = \sin 2x \), \( 0 \leq x \leq \pi/2 \); \(-\text{axis} \)
3. \( y = \sec x \), \( 0 \leq x \leq \pi/4 \); \(y\)-axis
4. \( y = e^x \), \( 1 \leq y \leq 2 \); \(y\)-axis

5–12. Find the area of the surface obtained by rotating the curve about the \(x\)-axis.
5. \( y = x^3 \), \( 0 \leq x \leq 1 \)
6. \( y = x^3 + 18x^2 \), \( 2 \leq x \leq 4 \)
7. \( y = \sqrt{4x} \), \( 0 \leq x \leq 1 \)
8. \( y = \cos x \), \( -\pi/2 \leq x \leq \pi/4 \)
9. \( y = \cosh x \), \( 0 \leq x \leq 1 \)
10. \( y = \frac{x^3}{6} + \frac{1}{2x} \), \( 1 \leq x \leq 2 \)
11. \( x = \frac{1}{2}(y^2 + 2)^{1/3} \), \( 1 \leq y \leq 2 \)
12. \( x = 1 + 2y^2 \), \( 1 \leq y \leq 2 \)

13–16. The given curve is rotated about the \(y\)-axis. Find the area of the resulting surface.
13. \( y = \sqrt{x} \), \( 1 \leq y \leq 2 \)
14. \( y = 1 - x^2 \), \( 0 \leq x \leq 1 \)
15. \( x = \sqrt{a^2 - y^2} \), \( 0 \leq y \leq a/2 \)
16. \( x = a \cosh(y/a) \), \( -a \leq y \leq a \)

17–20. Use Simpson’s Rule with \( n = 10 \) to approximate the area of the surface obtained by rotating the curve about the \(x\)-axis. Compare your answer with the value of the integral produced by your calculator.
17. \( y = x^2 \), \( 1 \leq x \leq 3 \)
18. \( y = 1 + x^2 \), \( 1 \leq x \leq 2 \)
19. \( y = \sec x \), \( 0 \leq x \leq \pi/3 \)
20. \( y = \sqrt{1 + e^x} \), \( 0 \leq x \leq 1 \)

21–22. Use either a CAS or a table of integrals to find the exact area of the surface obtained by rotating the given curve about the \(x\)-axis.
21. \( y = 1/x \), \( 1 \leq x \leq 2 \)
22. \( y = \sqrt{x^2 + 1} \), \( 0 \leq x \leq 3 \)

23–24. Use a CAS to find the exact area of the surface obtained by rotating the curve about the \(y\)-axis. If your CAS has trouble evaluating the integral, express the surface area as an integral in the other variable.
23. \( y = x^3 \), \( 0 \leq y \leq 1 \)
24. \( y = \ln(x + 1) \), \( 0 \leq x \leq 1 \)

25. (a) If \( a > 0 \), find the area of the surface generated by rotating the loop of the curve \( 3ay^2 = x(a - x) \) about the \(x\)-axis.
(b) Find the surface area if the loop is rotated about the \(y\)-axis.

26. A group of engineers is building a parabolic satellite dish whose shape will be formed by rotating the curve \( y = ax^2 \) about the \(y\)-axis. If the dish is to have a 10-ft diameter and a maximum depth of 2 ft, find the value of \( a \) and the surface area of the dish.
27. The ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b \]
is rotated about the x-axis to form a surface called an *ellipsoid*. Find the surface area of this ellipsoid.

28. Find the surface area of the torus in Exercise 41 in Section 7.2.

29. If the curve \( y = f(x), \ a \leq x \leq b, \) is rotated about the horizontal line \( y = c, \) where \( f(x) \leq c, \) find a formula for the area of the resulting surface.

30. Use the result of Exercise 29 to set up an integral to find the area of the surface generated by rotating the curve \( y = \sqrt{x}, \ 0 \leq x \leq 4, \) about the line \( y = 4. \) Then use a CAS to evaluate the integral.

31. Find the area of the surface obtained by rotating the circle \( x^2 + y^2 = r^2 \) about the line \( y = r. \)

32. Show that the surface area of a zone of a sphere that lies between two parallel planes is \( S = \pi dh, \) where \( d \) is the diameter of the sphere and \( h \) is the distance between the planes. (Notice that \( S \) depends only on the distance between the planes and not on their location, provided that both planes intersect the sphere.)

33. Formula 4 is valid only when \( f(x) \geq 0. \) Show that when \( f(x) \) is not necessarily positive, the formula for surface area becomes
\[
S = \int_{a}^{b} 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx
\]

34. Let \( L \) be the length of the curve \( y = f(x), \ a \leq x \leq b, \) where \( f \) is positive and has a continuous derivative. Let \( S_i \) be the surface area generated by rotating the curve about the x-axis. If \( c \) is a positive constant, define \( g(x) = f(x) + c \) and let \( S_p \) be the corresponding surface area generated by the curve \( y = g(x), \ a \leq x \leq b. \) Express \( S_p \) in terms of \( S_i \) and \( L \).
ANSWERS

1. \[ \int_1^2 2 \pi \ln x \sqrt{1 + (1/x)^2} \, dx \]
2. \[ \int_0^1 2 \pi x \sqrt{1 + (\sec x \tan x)^2} \, dx \]
3. \[ \pi \left( 145 \sqrt{145} - 1 \right) / 27 \]
4. \[ \pi \left( 37 \sqrt{37} - 17 \sqrt{17} \right) / 6 \]
5. \[ \pi \left( 1 + \frac{1}{2} (e^2 - e^{-2}) \right) \]
6. \[ 21 \pi / 2 \]
7. \[ \pi \left( 145 \sqrt{145} - 10 \sqrt{10} \right) / 27 \]
8. \[ 9.023754 \]
9. \[ 19.13527296 \]
10. \[ \left( \pi / 4 \right) \left[ 4 \ln(\sqrt{17} + 4) - 4 \ln(\sqrt{2} + 1) - \sqrt{17} + 4 \sqrt{2} \right] \]
11. \[ \left( \pi / 6 \right) \left[ \ln(\sqrt{10} + 3) + 3 \sqrt{10} \right] \]
12. \[ \left( \pi a^2 / 3 \right) \]
13. \[ 56 \pi \sqrt{3} a^2 / 45 \]
14. \[ 2 \pi b^2 \left[ b^2 + a^2 b \sin^{-1} \left( \sqrt{a^2 - b^2} / a \right) / \sqrt{a^2 - b^2} \right] \]
15. \[ 4 \pi r^2 \]
SOLUTIONS

1. \( y = \ln x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} \, dx = \sqrt{1 + (1/x)^2} \, dx \Rightarrow S = \int_1^\pi 2\pi (\ln x) \sqrt{1 + (1/x)^2} \, dx \) [by (7)]

2. \( y = \sin^2 x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} \, dx = \sqrt{1 + (2\sin x \cos x)^2} \, dx \Rightarrow S = \int_0^{\pi/2} 2\pi \sin^2 x \sqrt{1 + (2\sin x \cos x)^2} \, dx \) [by (7)]

3. \( y = \sec x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} \, dx = \sqrt{1 + (\sec x \tan x)^2} \, dx \Rightarrow S = \int_0^{\pi/4} 2\pi x \sqrt{1 + (\sec x \tan x)^2} \, dx \) [by (8)]

4. \( y = e^x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} \, dx = \sqrt{1 + e^{2x}} \, dx \Rightarrow S = \int_0^{\ln 2} 2\pi x \sqrt{1 + e^{2x}} \, dx \) [by (8)]
   or \( \int_0^2 2\pi (\ln y) \sqrt{1 + (1/y)^2} \, dy \) [by (6)]

5. \( y = x^3 \Rightarrow y' = 3x^2 \). So
   \[ S = \frac{\pi}{3} \int_{x_1}^{x_2} \sqrt{1 + (y')^2} \, dx = \frac{\pi}{3} \int_{x_1}^{x_2} \sqrt{1 + 9x^4} \, dx = \frac{\pi}{3} \left[ \frac{2x^{3/2}}{3} \right]_{x_1}^{x_2} = \frac{\pi}{3} \left( \sqrt{27} - \sqrt{2} \right) = \frac{\pi}{3} \left( \sqrt{27} - \sqrt{2} \right) \]

6. The curve \( y = x^2 + 18 \) is symmetric about the x-axis, so we only use its top half, given by
   \[ y = 3\sqrt{x - 2}. \text{ } dy/dx = \frac{3}{2\sqrt{x - 2}}, \text{ so } 1 + (dy/dx)^2 = 1 + \frac{9}{4(x - 2)}. \text{ Thus,} \]
   \[ S = \int_2^6 2\pi x \sqrt{1 + (y')^2} \, dx = \int_2^6 2\pi x \sqrt{1 + 9x^4} \, dx = \left[ \pi \left( \frac{2x^{3/2}}{3} \right) \right]_2^6 = \pi \left( \frac{2\sqrt{27}}{3} \right) = \pi \left( \frac{2\sqrt{27}}{3} \right) = \pi \left( \sqrt{27} - \sqrt{3} \right) \]

7. \( y = \sqrt{x} \Rightarrow 1 + \left( dy/dx \right)^2 = 1 + \left( \frac{1}{2\sqrt{x}} \right)^2 = 1 + \frac{1}{4x}. \text{ So} \)
   \[ S = \int_1^4 2\pi y \sqrt{1 + (dy/dx)^2} \, dx = \int_1^4 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx = 2\pi \int_1^4 \sqrt{x + \frac{1}{4}} \, dx \]
   \[ = 2\pi \left[ \frac{2}{3} \left( x + \frac{1}{4} \right)^{3/2} \right]_1^4 = \frac{2\pi}{3} \left[ \left( 4 + \frac{1}{4} \right)^{3/2} - \left( 1 + \frac{1}{4} \right)^{3/2} \right] = \frac{2\pi}{3} \left( \frac{37\sqrt{47} - 19\sqrt{217}}{2} \right) \]

8. \( y = \cos 2x \Rightarrow ds = \sqrt{1 + (dy/dx)^2} \, dx = \sqrt{1 + (-2\sin 2x)^2} \, dx \Rightarrow \)
   \[ S = \int_0^{\pi/6} 2\pi \cos 2x \sqrt{1 + 4\sin^2 2x} \, dx = 2\pi \int_0^{\pi/6} \sqrt{1 + u^2} \, du \] \quad \text{[} u = 2\sin 2x, \ du = 4\cos 2x \, dx \]
   \[ = \int_0^{\pi/6} \left[ \frac{1}{2} u \sqrt{1 + u^2} + \frac{1}{2} \ln(u + \sqrt{1 + u^2}) \right]_0^{\pi/6} = \frac{\pi}{3} \left[ \left( \pi/2 \right)^2 + 2 + \frac{1}{2} \ln(\sqrt{3} + 2) \right] = \frac{\pi^2}{3} + \frac{\pi}{6} \ln(2 + \sqrt{3}) \]
9. \( y = \cosh x \Rightarrow 1 + (dy/dx)^2 = 1 + \sinh^2 x = \cosh^2 x \). So
\[
S = 2 \pi \int_0^1 \cosh x \cosh x \, dx = 2 \pi \int_0^1 \frac{1}{2}(1 + \cosh 2x) \, dx = \pi \left[ x + \frac{1}{2} \sinh 2x \right]_0^1
= \pi \left[ 1 + \frac{1}{2} \sinh 2 \right] \quad \text{or} \quad \pi \left[ 1 + \frac{1}{2} (e^x - e^{-x}) \right]
\]

10. \( y = \frac{x^3}{6} + \frac{1}{2x} \Rightarrow \frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} \Rightarrow
\sqrt{1 + (dy/dx)^2} = \sqrt{\frac{x^4}{4} + 1 + \frac{1}{4x^4}} = \sqrt{\frac{x^2}{2} + \frac{1}{2x^2}} = \frac{x^2}{2} + \frac{1}{2x^2} \Rightarrow
\]
\[
S = 2 \pi \int_{1/2}^1 2\pi \left( \frac{x^3}{6} + \frac{1}{2x} \right) \left( \frac{x^2}{2} + \frac{1}{2x^2} \right) \, dx = 2 \pi \int_{1/2}^1 \left( \frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x} \right) \, dx
= 2 \pi \left[ \frac{x^6}{72} + \frac{x^2}{6} + \frac{x^2}{8} \right]_{1/2}^{1/2}
= 2 \pi \left[ \left( \frac{1}{6} - \frac{1}{2} \right) - \left( \frac{1}{72} + \frac{1}{12} - \frac{1}{2} \right) \right] = 2 \pi \left( \frac{263}{72} \right) = \frac{263}{36} \pi
\]

11. \( x = \frac{1}{2} (y^2 + 2)^{3/2} \Rightarrow \frac{dx}{dy} = \frac{3}{4} (y^2 + 2)^{1/2} (2y) = \frac{3}{2} y \sqrt{y^2 + 2} = 1 + (dx/dy)^2 = 1 + y^2 (y^2 + 2) = (y^2 + 1)^2 \). So
\[
S = 2 \pi \int_0^2 y(y^2 + 1) \, dy = 2 \pi \left[ \frac{y^3}{3} + \frac{y^2}{2} \right]_0^2 = 2 \pi (4 - \frac{1}{2} - \frac{1}{2}) = 2 \pi
\]

12. \( x = 1 + 2y^2 \Rightarrow 1 + (dx/dy)^2 = 1 + (4y)^2 = 1 + 16y^2 \Rightarrow
\)
\[
S = 2 \pi \int_0^2 y \sqrt{1 + 16y^2} \, dy = \pi \left[ \frac{2}{3} \sqrt{16y^2 + 1} \right]_0^2 = \frac{2}{3} \pi \left[ \sqrt{16y^2 + 1} \right]_0^2
= \frac{2}{3} \pi (145 \sqrt{145} - 10 \sqrt{10})
\]

13. \( y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2} (\sqrt{x})^{-1/2} = 1 + 9y^4 \). So
\[
S = 2 \pi \int_0^1 y \sqrt{1 + 9y^4} \, dy = \pi \left[ \frac{3}{2} \sqrt{1 + 9y^4} \right]_0^1 = \frac{3}{2} \pi \left( \sqrt{1 + 9(1)} \right) = \frac{3}{2} \pi \sqrt{10}
\]

14. \( x = 1 + \frac{y}{x} \Rightarrow 1 + (dy/dx)^2 = 1 + 4x^2 \Rightarrow
\]
\[
S = 2 \pi \int_0^1 x \sqrt{1 + 4x^2} \, dx = \frac{8 \pi}{3} \int_0^1 8x \sqrt{4x^2 + 1} \, dx = \frac{8}{3} \pi \left[ \frac{x}{2} (4x^2 + 1)^{3/2} \right]_0^1 = \frac{8}{3} \pi (5 \sqrt{5} - 1)
\]

15. \( x = \sqrt{a^2 - y^2} \Rightarrow \frac{dx}{dy} = \frac{1}{2}(a^2 - y^2)^{-1/2} (-2y) = -y/\sqrt{a^2 - y^2} \Rightarrow
\[
1 + (dx/dy)^2 = 1 + \frac{y^2}{a^2 - y^2} = \frac{a^2 - y^2}{a^2 - y^2} + \frac{y^2}{a^2 - y^2} = \frac{a^2}{a^2 - y^2} \Rightarrow
\]
\[
S = \int_0^{a/2} 2 \pi \sqrt{a^2 - y^2} \frac{0}{\sqrt{a^2 - y^2}} \, dy = 2 \pi \int_0^{a/2} 0 \, dy = 2 \pi a \left[ y \right]_0^{a/2} = 2 \pi a \left( \frac{a}{2} - 0 \right) = \pi a^2 \). Note that this is
\[
\frac{1}{2} \text{ the surface area of a sphere of radius } a, \text{ and the length of the interval } y = 0 \text{ to } y = a/2 \text{ is } \frac{1}{2} \text{ the length of the interval } y = -a \text{ to } y = a.
\]
16. \( x = a \cosh(y/a) \Rightarrow 1 + (dx/dy)^2 = 1 + \sinh^2(y/a) = \cosh^2(y/a). \) So

\[
S = 2\pi \int_{-a}^{a} a \cosh(y/a) \cosh\left(\frac{y}{a}\right) dy = 4\pi a \int_{0}^{a} \cosh^2\left(\frac{y}{a}\right) dy = 2\pi a \int_{0}^{a} \left[ 1 + \cosh\left(\frac{2y}{a}\right) \right] dy
\]

\[
= 2\pi a \left[ y + \frac{a}{2} \sinh\left(\frac{2y}{a}\right) \right]_{0}^{a} = 2\pi a \left[ a + \frac{a}{4} \sinh 2 \right] = 2\pi a^2 [1 + \frac{1}{4} \sinh 2] \quad \text{or} \quad \pi a^2 (e^2 + 4 - e^{-2})
\]

17. \( y = \ln x \Rightarrow dy/dx = 1/x \Rightarrow 1 + (dy/dx)^2 = 1 + 1/x^2 \Rightarrow S = \int_{1}^{3} 2\pi \ln x \sqrt{1 + 1/x^2} dx. \)

Let \( f(x) = \ln x \sqrt{1 + 1/x^2}. \) Since \( n = 10, \Delta x = \frac{3-1}{10} = \frac{1}{5}. \) Then

\[
S \approx S_{10} = 2\pi \cdot \frac{1}{10} \left[ f(1) + 4f(1.2) + 2f(1.4) + \cdots + 2f(2.6) + 4f(2.8) + f(3) \right] \approx 9.02377
\]

The value of the integral produced by a calculator is 9.024262 (to six decimal places).

18. \( y = x + \sqrt{x} \Rightarrow dy/dx = 1 + \frac{1}{2} x^{-1/2} \Rightarrow 1 + (dy/dx)^2 = 2 + x^{-1/2} \Rightarrow S = \int_{1}^{2} 2\pi (x + \sqrt{x}) \sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}} dx. \)

Let \( f(x) = (x + \sqrt{x}) \sqrt{2 + \frac{1}{\sqrt{x}} + \frac{1}{4x}}. \) Since \( n = 10, \Delta x = \frac{2-1}{10} = \frac{1}{10}. \) Then

\[
S \approx S_{10} = 2\pi \cdot \frac{1}{10} \left[ f(1) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.9) + f(2) \right] \approx 29.506566
\]

The value of the integral produced by a calculator is 29.506566 (to six decimal places).

19. \( y = \sec x \Rightarrow dy/dx = \sec x \tan x \Rightarrow 1 + (dy/dx)^2 = 1 + \sec^2 x \tan^2 x \Rightarrow S = \int_{0}^{\pi/3} 2\pi \sec x \sqrt{1 + \sec^2 x \tan^2 x} dx. \)

Let \( f(x) = \sec x \sqrt{1 + \sec^2 x \tan^2 x}. \) Since \( n = 10, \Delta x = \frac{\pi/3 - 0}{10} = \frac{\pi}{30}. \) Then

\[
S \approx S_{10} = \frac{\pi}{15} \sum_{k=1}^{10} f(0) + 4f(\frac{\pi}{30}) + 2f(\frac{2\pi}{30}) + \cdots + 2f(\frac{8\pi}{30}) + 4f(\frac{9\pi}{30}) + f(\frac{\pi}{3}) \approx 13.527296
\]

The value of the integral produced by a calculator is 13.516987 (to six decimal places).

20. \( y = (1 + e^x)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2} (1 + e^x)^{-1/2} \cdot e^x = e^x \frac{e^x}{2(1 + e^x)^{1/2}} \Rightarrow
\]

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{e^{2x}}{4(1 + e^x)} = \frac{4 + 4e^x + e^{2x}}{4(1 + e^x)} \Rightarrow
\]

\[
S = \int_{0}^{1} 2\pi \sqrt{1 + e^{2x} - \frac{e^x + 2}{2\sqrt{e^x + 2}}} dx = \pi \int_{0}^{1} (e^{x} + 2) dx = \pi \left[ e^{x} + 2x \right]_{0}^{1} = \pi [(e + 2) - (1 + 0)] = \pi(e + 1).
\]

Let \( f(x) = \frac{1}{4} (e^x + 2). \) Since \( n = 10, \Delta x = \frac{1-0}{10} = \frac{1}{10}. \) Then

\[
S \approx S_{10} = 2\pi \cdot \frac{1}{10} \left[ f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1) \right] \approx 11.681330.
\]

The value of the integral produced by a calculator is 11.681327 (to six decimal places).
21. \( y = \frac{1}{x} \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + (-1/x^2)^2} \, dx = \sqrt{1 + 1/x^4} \, dx \Rightarrow \\
S = \int_1^2 2\pi \cdot \frac{2\pi}{x} \left(\sqrt{1 + \frac{1}{x^4}}\right) \, dx = 2\pi \int_1^2 \frac{\sqrt{x^4 + 1}}{x^3} \, dx = 2\pi \int_1^2 \frac{\sqrt{u^2 + 1}}{u^2} \left(\frac{2}{u}\right) \, du \quad [u = x^2, \, du = 2x \, dx] \\
= \pi \int_1^2 \frac{\sqrt{1 + u^2}}{u^2} \, du \approx \pi \left[ -\frac{x\sqrt{1 + x^2}}{2} + \frac{\ln\left(x + \sqrt{1 + x^2}\right)}{x}\right]_1^2 \\
= \pi \left[ -\frac{\sqrt{5}}{2} + \frac{\ln\left(4 + \sqrt{17}\right)}{2} - \frac{\sqrt{2}}{2} - \frac{\ln\left(1 + \sqrt{2}\right)}{2}\right] = \pi \left[ \sqrt{2} - \frac{\sqrt{17}}{2} + \frac{\ln\left(4 + \sqrt{17}\right)}{4} \right]

22. \( y = \sqrt{x^2 + 1} \Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}} \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + \frac{x^2}{x^2 + 1}} \, dx = \\
S = \int_0^3 2\pi \sqrt{x^2 + 1} \sqrt{1 + \frac{x^2}{x^2 + 1}} \, dx = 2\pi \int_0^3 \sqrt{2x^2 + 1} \, dx = 2\sqrt{2\pi} \int_0^3 \sqrt{\frac{2x^2}{x^2 + 1}} \, dx \\
\approx 2\sqrt{2\pi} \left[ \frac{x}{\sqrt{2}} \sqrt{x^2 + 1} + \frac{1}{2} \ln\left(x + \sqrt{x^2 + 1}\right) \right]_0^3 \\
= 2\sqrt{2}\pi \left[ \frac{3\sqrt{11}}{2} + \frac{1}{2} \ln\left(3\sqrt{2} + \sqrt{17}\right) \right] = 3\sqrt{2}\pi + \frac{1}{2} \ln\left(3\sqrt{2} + \sqrt{17}\right)

23. \( y = x^3 \) and \( 0 \leq y \leq 1 \Rightarrow \frac{dy}{dx} = 3x^2 \) and \( 0 \leq x \leq 1 \Rightarrow \\
S = \int_0^1 2\pi x \sqrt{1 + (3x^2)^2} \, dx = 2\pi \int_0^1 \left(1 + u^2\right)^{1/2} \, du \quad [u = 3x^2, \, du = 6x \, dx] \\
= \frac{\pi}{3} \int_0^1 \left(1 + u^2\right)^{1/2} \, du \approx \frac{\pi}{3} \left[ \frac{1}{2} u \sqrt{1 + u^2} + \frac{1}{2} \ln\left(u + \sqrt{1 + u^2}\right) \right]_0^1 \\
= \pi \left[ \frac{1}{2} (1 + \ln\sqrt{2}) - \frac{1}{2} \ln\sqrt{2} \right] = \pi \left[ \frac{1}{2} \ln\left(1 + \sqrt{2}\right) \right]

24. \( y = \ln\left(\frac{1}{x}\right) \), \( 0 \leq x \leq 1 \), \( ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + \left(\frac{-1}{x^2}\right)^2} \, dx, \) so \\
S = \int_1^2 2\pi x \sqrt{1 + \left(\frac{1}{x^2}\right)^2} \, dx = \int_1^2 2\pi\left(\frac{1}{x^2} - 1\right) \sqrt{1 + \left(\frac{x}{x^2}\right)^2} \, dx = \int_1^2 \frac{2\pi}{x^2} \cdot \sqrt{1 + \frac{1}{x^2}} \, dx - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u^2} \, du \\
= 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u^2} \, du - 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u^2} \, du = 2\pi \int_1^2 \frac{\sqrt{1 + u^2}}{u^2} \, du - 2\pi \int_1^2 \frac{1 + u^2}{u^2} \, du \\
\approx 2\pi \left[ \frac{1}{2} \ln\left(2 + \sqrt{5}\right) - \frac{1}{2} \ln\left(1 + \sqrt{2}\right) \right] - 2\pi \left[ \sqrt{5} - \ln\left(1 + \sqrt{5}\right) \right] \\
= 2\pi \left[ \frac{1}{2} \ln\left(2 + \sqrt{5}\right) + \ln\left(\frac{1 + \sqrt{5}}{\sqrt{2}}\right) + \frac{1}{2} \ln\left(1 + \sqrt{2}\right) \right]
25. Since \( a > 0 \), the curve \( 3ay^2 = x(a^2-x^2) \) only has points with 
\[ x \geq 0. \] 
\((3ay^2 \geq 0 \Rightarrow x(a-x)^2 \geq 0 \Rightarrow x \geq 0.)\) The 
curve is symmetric about the \( x \)-axis (since the equation is 
unchanged when \( y \) is replaced by \(-y\)). \( y = 0 \) when \( x = 0 \) or \( a \), 
so the curve’s loop extends from \( x = 0 \) to \( x = a \).

\[
\frac{d}{dx}(3ay^2) = \frac{d}{dx}[x(a-x)^2] \Rightarrow \quad 6ay \frac{dy}{dx} = x \cdot 2(a-x)(-1) + (a-x)^2 \Rightarrow \quad \frac{dy}{dx} = \frac{(a-x)[-2x+a-x]}{6ay}
\]

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{(a-x)^2(a-3x)^2}{36a^2} = \frac{(a-x)^2(a-3x)^2}{36a^2} \left[ \frac{3a}{x(a-x)^2} \right] = \frac{(a-x)^2}{12ax} \quad \text{for} \quad x \neq 0.
\]

\[
(a) \quad S = \int_{x=0}^{a} 2\pi y \, ds = 2\pi \int_{0}^{a} \frac{\sqrt{x(a-x)}}{\sqrt{3a}} \frac{a+3x}{\sqrt{12ax}} \, dx = 2\pi \int_{0}^{a} \frac{(a-x)(a+3x)}{6a} \, dx
\]

\[
= \frac{\pi}{3a} \int_{0}^{a} (a^2 + 2ax - 3x^2) \, dx = \frac{\pi}{3a} \left[ a^2x + ax^2 - x^3 \right]_0^a = \frac{\pi}{3a} \left( a^3 + a^3 - a^3 \right) = \frac{\pi}{3a} a^3 = \frac{\pi a^3}{3}
\]

Note that we have rotated the top half of the loop about the \( y \)-axis, this generates the full 

surface.

(b) We must rotate the full loop about the \( y \)-axis, so we want double the area obtained by rotating the top half of 
the loop:

\[
S = 2 \cdot 2\pi \int_{x=0}^{a} x \, ds = 4\pi \int_{0}^{a} a \frac{x + 3x}{2ax} = \frac{4\pi}{2} \int_{0}^{a} x^{1/2}(a + 3x) \, dx
\]

\[
= \frac{2\pi}{\sqrt{3a}} \int_{0}^{a} (ax^{3/2} + 3x^{1/2}) \, dx = \frac{2\pi}{\sqrt{3a}} \left[ \frac{2}{5}x^{5/2} + \frac{6}{5}x^{3/2} \right]_0^a = \frac{2\pi}{\sqrt{3a}} \left( \frac{2}{5}a^{5/2} + \frac{6}{5}a^{3/2} \right)
\]

\[
= \frac{2\pi}{\sqrt{3}} \left( \frac{2a^2}{5} + \frac{6a}{5} \right) \frac{a^2}{45} = \frac{56\pi \sqrt{3} a^2}{45}
\]

26. In general if the parabola \( y = a^2c^2 - c^2 \leq x \leq c \) is rotated about the \( y \)-axis, the surface area it generates is

\[
2\pi \int_{x=0}^{c} x \, dx + (2ac)^2 \, dx = 2\pi \int_{0}^{2ac} \frac{u}{\sqrt{1+u^2}} \frac{1}{2a} \, du \left[ \frac{u = 2ax}{da = 2a \, dx} \right]
\]

\[
= \frac{\pi}{4a^2} \int_{0}^{2ac} (1 + u^2)^{1/2} 2u \, du = \frac{\pi}{4a^2} \left[ \frac{2}{3}(1 + u^2)^{3/2} \right]_{0}^{2ac}
\]

\[
= \frac{\pi}{6ac} \left( 1 + 4a^2c^2 \right)^{3/2} - 1
\]

Here \( 2c = 10 \text{ ft and } ac^2 = 2 \text{ ft}, \) so \( c = 5 \text{ and } a = \frac{2}{5}. \) Thus, the surface area is

\[
S = \frac{\pi}{6} \left[ \frac{625}{4} \left( 1 + 4 \cdot \frac{4}{25} \cdot 25 \right)^{3/2} - 1 \right] = \frac{625\pi}{24} \left( 1 + \frac{16\cdot 25}{25} \right)^{3/2} - 1 = \frac{625\pi}{24} \left( \frac{41\sqrt{11} - 125}{125} \right) \approx \frac{41\sqrt{11} - 125}{125} \approx 90.01 \text{ ft}^2
\]
The analogue of $f(x^2)$ in the derivation of (4) is now $c - f(x^2)$, so

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi[c - f(x_i^2)] \sqrt{1 + [f'(x_i^2)]^2} \Delta x = \int_{a}^{b} 2\pi[c - f(x)] \sqrt{1 + [f'(x)]^2} \, dx.$$ 

28. The upper half of the torus is generated by rotating the curve $(x - R)^2 + y^2 = r^2$, $y > 0$, about the $y$-axis.

$$y \frac{dy}{dx} = -(x - R) \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{y}{x - R}\right)^2 = \frac{y^2 + (x - R)^2}{y^2} = \frac{r^2}{r^2 - (x - R)^2}.$$ Thus,

$$S = 2\pi \int_{R - r}^{R + r} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 4\pi \int_{-r}^{r} \frac{\sqrt{x}}{\sqrt{x^2 - (x - R)^2}} \, dx$$

$$= 4\pi \int_{-r}^{r} \frac{u \, du}{\sqrt{r^2 - u^2}} + 4\pi r R \int_{-r}^{r} \frac{du}{\sqrt{r^2 - u^2}}$$

$$= 4\pi r \int_{-r}^{r} \frac{du}{\sqrt{r^2 - u^2}} + 4\pi R r \int_{-r}^{r} \frac{du}{\sqrt{r^2 - u^2}}$$

$$= 4\pi r \cdot 0 + 8\pi R r \int_{0}^{r} \frac{du}{\sqrt{r^2 - u^2}}$$

$$= 8\pi R r \left[\sin^{-1}\left(\frac{u}{r}\right)\right]_{0}^{r} = 8\pi R r \left(\frac{\pi}{2}\right) = 4\pi^2 R r$$

29. y = $x^{1/4}$ \Rightarrow \quad \frac{dy}{dx} = \frac{1}{4} x^{-3/4} \Rightarrow \quad 1 + (\frac{dy}{dx})^2 = 1 + 1/4x$, so by Exercise 29,

$$S = \int_{0}^{4} 2\pi(4 - \sqrt{x}) \sqrt{1 + 1/4x} \, dx.$$ Using a CAS, we get $S = 2\pi \ln(\sqrt{17} + 4) + \pi (31 \sqrt{17} + 1) \approx 80.6095.$
31. For the upper semicircle, \( f(x) = \sqrt{r^2 - x^2} \), \( f'(x) = -x/\sqrt{r^2 - x^2} \). The surface area generated is

\[
S_1 = \int_{-r}^{r} 2\pi \left( r - \sqrt{r^2 - x^2} \right) \sqrt{1 + \left( \frac{x}{\sqrt{r^2 - x^2}} \right)^2} \, dx = 4\pi \int_{0}^{r} \left( r - \sqrt{r^2 - x^2} \right) \frac{r}{\sqrt{r^2 - x^2}} \, dx
\]

\[
= 4\pi \int_{0}^{r} \left( \frac{r^2}{\sqrt{r^2 - x^2}} - r \right) \, dx
\]

For the lower semicircle, \( f(x) = -\sqrt{r^2 - x^2} \) and \( f'(x) = x/\sqrt{r^2 - x^2} \), so \( S_2 = 4\pi \int_{0}^{r} \left( \frac{r^2}{\sqrt{r^2 - x^2}} + r \right) \, dx \).

Thus, the total area is \( S = S_1 + S_2 = 8\pi \int_{0}^{r} \left( \frac{r^2}{\sqrt{r^2 - x^2}} \right) \, dx = 8\pi \left[ r^2 \sin^{-1} \left( \frac{x}{r} \right) \right]_0^r = 8\pi r^2 \left( \frac{\pi}{2} \right) = 2\pi r^2 \).

32. Take the sphere \( x^2 + y^2 + z^2 = \frac{1}{4}d^2 \) and let the intersecting planes be \( y = c \) and \( y = c + h \), where \( -\frac{1}{2}d \leq c \leq \frac{1}{2}d - h \). The sphere intersects the \( xy \)-plane in the circle \( x^2 + y^2 = \frac{1}{4}d^2 \). From this equation, we get \( x \frac{dx}{dy} + y = 0 \), so \( \frac{dx}{dy} = -\frac{y}{x} \). The desired surface area is

\[
S = 2\pi \int x \, ds = 2\pi \int_{c}^{c+h} x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = 2\pi \int_{c}^{c+h} x \sqrt{1 + \left( \frac{-y}{x} \right)^2} \, dy = 2\pi \int_{c}^{c+h} \sqrt{x^2 + y^2} \, dy
\]

\[
= 2\pi \int_{c}^{c+h} d \, dy = \pi d \int_{c}^{c+h} dy = \pi dh
\]

33. In the derivation of (4), we computed a typical contribution to the surface area to be \( 2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \), the area of a frustum of a cone. When \( f(x) \) is not necessarily positive, the approximations \( y_i = f(x_i) \approx f(x_i^*) \) and \( y_{i-1} = f(x_{i-1}) \approx f(x_{i-1}^*) \) must be replaced by \( |f_{\Delta f}(x_i)| \approx |f(x_i^*)| \) and \( |f_{\Delta f}(x_{i-1})| \approx |f(x_{i-1}^*)| \). Thus,

\[
2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1}P_i| \approx 2\pi \frac{|f(x_{i-1}^*)| + |f(x_i^*)|}{2} \Delta x. \]

Continuing with the rest of the derivation as before, we obtain \( S = \int_{a}^{b} |f(x)| \sqrt{1 + [f'(x)]^2} \, dx \).

34. Since \( g(x) = f(x) + c \), we have \( g'(x) = f'(x) \). Thus,

\[
S = \int_{a}^{b} 2\pi g(x) \sqrt{1 + [g'(x)]^2} \, dx = \int_{a}^{b} 2\pi |f(x) + c| \sqrt{1 + [f'(x)]^2} \, dx
\]

\[
= \int_{a}^{b} 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx + 2\pi c \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx = S_j + 2\pi c L
\]