Linear Programming

**Case Study The Diet Problem**

The Galaxy Nutrition health-food mega-store chain provides free online nutritional advice and support to its customers. As Web site technical consultant, you are planning to construct an interactive Web page to assist customers prepare a diet tailored to their nutritional and budgetary requirements. Ideally, the customer would select foods to consider and specify nutritional and/or budgetary constraints, and the tool should return the optimal diet meeting those requirements. You would also like the Web page to allow the customer to decide whether, for instance, to find the cheapest possible diet meeting the requirements, the diet with the lowest number of calories, or the diet with the least total carbohydrates. **How do you go about constructing such a Web page?**

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**Web Site**

www.FiniteMath.org

At the Web site you will find:

- Section by section tutorials
- A detailed chapter summary
- A true/false quiz
- Additional review exercises
- A linear programming grapher
- A pivot and Gauss-Jordan tool
- A simplex method tool
Introduction

In this chapter we begin to look at one of the most important types of problems for business and the sciences: finding the largest or smallest possible value of some quantity (such as profit or cost) under certain constraints (such as limited resources). We call such problems **optimization** problems because we are trying to find the best, or optimum, value. The optimization problems we look at in this chapter involve linear functions only and are known as **linear programming** (LP) problems. One of the main purposes of calculus, which you may study later, is to solve nonlinear optimization problems.

Linear programming problems involving only two unknowns can usually be solved by a graphical method that we discuss in Sections 4.1 and 4.2. When there are three or more unknowns, we must use an algebraic method, as we had to do for systems of linear equations. The method we use is called the **simplex method**. Invented in 1947 by George B. Dantzig✱ (1914–2005), the simplex method is still the most commonly used technique to solve LP problems in real applications, from finance to the computation of trajectories for guided missiles.

The simplex method can be used for hand calculations when the numbers are fairly small and the unknowns are few. Practical problems often involve large numbers and many unknowns, however. Problems such as routing telephone calls or airplane flights, or allocating resources in a manufacturing process can involve tens of thousands of unknowns. Solving such problems by hand is obviously impractical, and so computers are regularly used. Although computer programs most often use the simplex method, mathematicians are always seeking faster methods. The first radically different method of solving LP problems was the **ellipsoid algorithm** published in 1979 by the Soviet mathematician Leonid G. Khachiyan² (1952–2005). In 1984, Narendra Karmarkar (1957–), a researcher at Bell Labs, created a more efficient method now known as **Karmarkar’s algorithm**. Although these methods (and others since developed) can be shown to be faster than the simplex method in the worst cases, it seems to be true that the simplex method is still the fastest in the applications that arise in practice.

Calculators and spreadsheets are very useful aids in the simplex method. In practice, software packages do most of the work, so you can think of what we teach you here as a peek inside a “black box.” What the software cannot do for you is convert a real situation into a mathematical problem, so the most important lessons to get out of this chapter are (1) how to recognize and set up a linear programming problem, and (2) how to interpret the results.

4.1 Graphing Linear Inequalities

By the end of the next section, we will be solving linear programming (LP) problems with two unknowns. We use inequalities to describe the constraints in a problem, such as limitations on resources. Recall the basic notation for inequalities.

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**NOTE** Dantzig is the real-life source of the story of the student who, walking in late to a math class, copies down two problems on the board, thinking they’re homework. After much hard work he hands in the solutions, only to discover that he’s just solved two famous unsolved problems. This actually happened to Dantzig in graduate school in 1939.¹


Here are the particular kinds of inequalities in which we’re interested:

**Nonstrict Inequalities**

- \( a \leq b \) means that \( a \) is less than or equal to \( b \).
- \( a \geq b \) means that \( a \) is greater than or equal to \( b \).

<table>
<thead>
<tr>
<th>Quick Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 \leq 99, -2 \leq -2, 0 \leq 3 )</td>
</tr>
<tr>
<td>( 3 \geq 3, 1.78 \geq 1.76, \frac{1}{3} \geq \frac{1}{4} )</td>
</tr>
</tbody>
</table>

There are also the inequalities \(<\) and \(>\), called strict inequalities because they do not permit equality. We do not use them in this chapter.

Following are some of the basic rules for manipulating inequalities. Although we illustrate all of them with the inequality \( \leq \), they apply equally well to inequalities with \( \geq \) and to the strict inequalities \(<\) and \(>\).

**Rules for Manipulating Inequalities**

1. The same quantity can be added to or subtracted from both sides of an inequality:
   - If \( x \leq y \), then \( x + a \leq y + a \) for any real number \( a \).

2. Both sides of an inequality can be multiplied or divided by a positive constant:
   - If \( x \leq y \) and \( a \) is positive, then \( ax \leq ay \).

3. Both sides of an inequality can be multiplied or divided by a negative constant if the inequality is reversed:
   - If \( x \leq y \) and \( a \) is negative, then \( ax \geq ay \).

4. The left and right sides of an inequality can be switched if the inequality is reversed:
   - If \( x \leq y \), then \( y \geq x \); if \( y \geq x \), then \( x \leq y \).

<table>
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<tr>
<td>( x \leq y ) implies ( x - 4 \leq y - 4 )</td>
</tr>
<tr>
<td>( x \leq y ) implies ( 3x \leq 3y )</td>
</tr>
<tr>
<td>( x \leq y ) implies ( -3x \geq -3y )</td>
</tr>
<tr>
<td>( 3x \geq 5y ) implies ( 5y \leq 3x )</td>
</tr>
</tbody>
</table>

Here are the particular kinds of inequalities in which we’re interested:

**Linear Inequalities and Solving Inequalities**

An inequality in the unknown \( x \) is the statement that one expression involving \( x \) is less than or equal to (or greater than or equal to) another. Similarly, we can have an inequality in \( x \) and \( y \), which involves expressions that contain \( x \) and \( y \); an inequality in \( x, y, \) and \( z \); and so on. A linear inequality in one or more unknowns is an inequality of the form

\[
ax \leq b \quad \text{(or } ax \geq b) \quad \text{a and } b \text{ real constants}
\]

\[
ax + by \leq c \quad \text{(or } ax + by \geq c) \quad \text{a, } b, \text{ and } c \text{ real constants}
\]

\[
ax + by + cz \leq d \quad \text{a, } b, \text{ c, and } d \text{ real constants}
\]

\[
ax + by + cz + dw \leq e \quad \text{a, } b, \text{ c, d, and } e \text{ real constants}
\]

and so on.
Solving Linear Inequalities in Two Variables

Our first goal is to solve linear inequalities in two variables—that is, inequalities of the form \( ax + by \leq c \). As an example, let’s solve

\[
2x + 3y \leq 6.
\]

We already know how to solve the equation \( 2x + 3y = 6 \). As we saw in Chapter 1, the solution of this equation may be pictured as the set of all points \((x, y)\) on the straight-line graph of the equation. This straight line has \( x\)-intercept 3 (obtained by putting \( y = 0 \) in the equation) and \( y\)-intercept 2 (obtained by putting \( x = 0 \) in the equation) and is shown in Figure 1.

Notice that, if \((x, y)\) is any point on the line, then \( x \) and \( y \) not only satisfy the equation \( 2x + 3y = 6 \), but they also satisfy the inequality \( 2x + 3y \leq 6 \), because being equal to 6 qualifies as being less than or equal to 6.

**Q**: Do the points on the line give all possible solutions to the inequality?

**A**: No. For example, try the origin, \((0, 0)\). Because \( 2(0) + 3(0) = 0 \leq 6 \), the point \((0, 0)\) is a solution that does not lie on the line. In fact, here is a possibly surprising fact: The solution to any linear inequality in two unknowns is represented by an entire half plane: the set of all points on one side of the line (including the line itself). Thus, because \((0, 0)\) is a solution of \( 2x + 3y \leq 6 \) and is not on the line, every point on the same side of the line as \((0, 0)\) is a solution as well (the colored region below the line in Figure 2 shows which half plane constitutes the solution set).

To see why the solution set of \( 2x + 3y \leq 6 \) is the entire half plane shown, start with any point \( P \) on the line \( 2x + 3y = 6 \). We already know that \( P \) is a solution of \( 2x + 3y \leq 6 \). If we choose any point \( Q \) directly below \( P \), the \( x\)-coordinate of \( Q \) will be the same as that of \( P \), and the \( y\)-coordinate will be smaller. So the value of \( 2x + 3y \) at \( Q \) will be smaller than the value at \( P \), which is 6. Thus, \( 2x + 3y < 6 \) at \( Q \), and so \( Q \) is another solution of the inequality. (See Figure 3.) In other words, every point beneath the line is a solution of \( 2x + 3y \leq 6 \).

On the other hand, any point above the line is directly above a point on the line, and so \( 2x + 3y > 6 \) for such a point. Thus, no point above the line is a solution of \( 2x + 3y \leq 6 \).
The same kind of argument can be used to show that the solution set of every inequality of the form \( ax + by \leq c \) or \( ax + by \geq c \) consists of the half plane above or below the line \( ax + by = c \). The “test-point” procedure we describe below gives us an easy method for deciding whether the solution set includes the region above or below the corresponding line.

Now we are going to do something that will appear backward at first (but makes it simpler to sketch sets of solutions of systems of linear inequalities). For our standard drawing of the region of solutions of \( 2x + 3y \leq 6 \), we are going to shade only the part that we do not want and leave the solution region blank. Think of covering over or “blocking out” the unwanted points, leaving those that we do want in full view (but remember that the points on the boundary line are also points that we want). The result is Figure 4. The reason we do this should become clear in Example 2.

**Sketching the Region Represented by a Linear Inequality in Two Variables**

1. Sketch the straight line obtained by replacing the given inequality with an equality.
2. Choose a test point not on the line; \((0, 0)\) is a good choice if the line does not pass through the origin.
3. If the test point satisfies the inequality, then the set of solutions is the entire region on the same side of the line as the test point. Otherwise, it is the region on the other side of the line. In either case, shade (block out) the side that does not contain the solutions, leaving the solution set unshaded.

**Quick Example**

Here are the three steps used to graph the inequality \( x + 2y \geq 5 \).

1. Sketch the line \( x + 2y = 5 \).
2. Test the point \((0, 0)\)
   \[ 0 + 2(0) \not\geq 5 \]
   Inequality is not satisfied.
3. Because the inequality is not satisfied, shade the region containing the test point.

**EXAMPLE 1 Graphing Single Inequalities**

Sketch the regions determined by each of the following inequalities:

a. \( 3x - 2y \leq 6 \)  
   b. \( 6x \leq 12 + 4y \)  
   c. \( x \leq -1 \)  
   d. \( y \geq 0 \)  
   e. \( x \geq 3y \)
c. The region $x \leq -1$ has as boundary the vertical line $x = -1$. The test point $(0, 0)$ is not in the solution set, as shown in Figure 6.
d. The region $y \geq 0$ has as boundary the horizontal line $y = 0$ (that is, the $x$-axis). We cannot use $(0, 0)$ for the test point because it lies on the boundary line. Instead, we choose a convenient point not on the line $y = 0$—say, $(0, 1)$. Because $1 \geq 0$, this point is in the solution set, giving us the region shown in Figure 7.
e. The line $x \geq 3y$ has as boundary the line $x = 3y$ or, solving for $y$,
$$y = \frac{1}{3}x.$$
This line passes through the origin with slope $1/3$, so again we cannot choose the origin as a test point. Instead, we choose $(0, 1)$. Substituting these coordinates in $x \geq 3y$ gives $0 \geq 3(1)$, which is false, so $(0, 1)$ is not in the solution set, as shown in Figure 8.

**Solution**

a. The boundary line $3x - 2y = 6$ has $x$-intercept 2 and $y$-intercept $-3$ (Figure 5). We use $(0, 0)$ as a test point (because it is not on the line). Because $3(0) - 2(0) \leq 6$, the inequality is satisfied by the test point $(0, 0)$, and so it lies inside the solution set. The solution set is shown in Figure 5.
b. The given inequality, $6x \leq 12 + 4y$, can be rewritten in the form $ax + by \leq c$ by subtracting $4y$ from both sides:
$$6x - 4y \leq 12.$$Dividing both sides by 2 gives the inequality $3x - 2y \leq 6$, which we considered in part (a). Now, applying the rules for manipulating inequalities does not affect the set of solutions. Thus, the inequality $6x \leq 12 + 4y$ has the same set of solutions as $3x - 2y \leq 6$. (See Figure 5.)

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**using Technology**

Technology can be used to graph inequalities. Here is an outline (see the Technology Guides at the end of the chapter for additional details on using a TI-83/84 Plus or Excel):

**TI-83/84 Plus**
Solve the inequality for $y$ and enter the resulting function of $x$; for example, $Y_1 = -(2/3)x + 2$.
Position the cursor on the icon to the left of $Y_1$ and press ENTER until you see the kind of shading desired (above or below the line).
[More details on page 343.]

**Excel**
Solve the inequality for $y$ and create a scattergraph using two points on the line. Then use the drawing palette to create a polygon to provide the shading.
[More details on page 345.]

**Web Site**
www.FiniteMath.org
→ Online Utilities
→ Linear Programming

Grapher
Type “graph” and enter one or more inequalities (each one on a new line) as shown:

```
Enter the linear
graph
x + 2y >= 5
```
Adjust the graph window settings, and click “Graph.”
EXAMPLE 2 Graphing Simultaneous Inequalities

Sketch the region of points that satisfy both inequalities:

\[ 2x - 5y \leq 10 \]
\[ x + 2y \leq 8. \]

Solution Each inequality has a solution set that is a half plane. If a point is to satisfy both inequalities, it must lie in both sets of solutions. Put another way, if we cover the points that are not solutions to \( 2x - 5y \leq 10 \) and then also cover the points that are not solutions to \( x + 2y \leq 8 \), the points that remain uncovered must be the points we want, those that are solutions to both inequalities. The result is shown in Figure 9, where the unshaded region is the set of solutions.

As a check, we can look at points in various regions in Figure 9. For example, our graph shows that \((0, 0)\) should satisfy both inequalities, and it does;

\[ 2(0) - 5(0) = 0 \leq 10 \quad \checkmark \]
\[ 0 + 2(0) = 0 \leq 8. \quad \checkmark \]

On the other hand, \((0, 5)\) should fail to satisfy one of the inequalities.

\[ 2(0) - 5(5) = -25 \not\leq 10 \quad \times \]
\[ 0 + 2(5) = 10 > 8 \]

One more: \((5, -1)\) should fail one of the inequalities:

\[ 2(5) - 5(-1) = 15 \not> 10 \quad \times \]
\[ 5 + 2(-1) = 3 \leq 8. \quad \checkmark \]

TECHNOLOGY NOTE

Although these graphs are quite easy to do by hand, the more lines we have to graph the more difficult it becomes to get everything in the right place, and this is where graphing technology can become important. This is especially true when, for instance, three or more lines intersect in points that are very close together and hard to distinguish in hand-drawn graphs.

EXAMPLE 3 Corner Points

Sketch the region of solutions of the following system of inequalities and list the coordinates of all the corner points.

\[ 3x - 2y \leq 6 \]
\[ x + y \geq -5 \]
\[ y \leq 4 \]

Solution Shading the regions that we do not want leaves us with the triangle shown in Figure 10. We label the corner points \(A\), \(B\), and \(C\) as shown.

Each of these corner points lies at the intersection of two of the bounding lines. So, to find the coordinates of each corner point, we need to solve the system of equations given by the two lines. To do this systematically, we make the following table:

<table>
<thead>
<tr>
<th>Point</th>
<th>Lines through Point</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(y = 4) [x + y = -5]</td>
<td>((-9, 4))</td>
</tr>
<tr>
<td>(B)</td>
<td>(y = 4) [3x - 2y = 6]</td>
<td>(\left(\frac{14}{3}, 4\right))</td>
</tr>
<tr>
<td>(C)</td>
<td>(x + y = -5) [3x - 2y = 6]</td>
<td>(\left(-\frac{4}{3}, -\frac{21}{5}\right))</td>
</tr>
</tbody>
</table>
* TECHNOLOGY NOTE

Using the trace feature makes it easy to locate corner points graphically. Remember to zoom in for additional accuracy when appropriate. Of course, you can also use technology to help solve the systems of equations, as we discussed in Chapter 2.

Here, we have solved each system of equations in the middle column to get the point on the right, using the techniques of Chapter 2. You should do this for practice.*

As a partial check that we have drawn the correct region, let us choose any point in its interior—say, (0, 0). We can easily check that (0, 0) satisfies all three given inequalities. It follows that all of the points in the triangular region containing (0, 0) are also solutions.

Take another look at the regions of solutions in Examples 2 and 3 (Figure 11).

Notice that the solution set in Figure 11(a) extends infinitely far to the left, whereas the one in Figure 11(b) is completely enclosed by a boundary. Sets that are completely enclosed are called bounded, and sets that extend infinitely in one or more directions are unbounded. For example, all the solution sets in Example 1 are unbounded.

EXAMPLE 4 Resource Allocation

Socaccio Pistachio Inc. makes two types of pistachio nuts: Dazzling Red and Organic. Pistachio nuts require food color and salt, and the following table shows the amount of food color and salt required for a 1-kilogram batch of pistachios, as well as the total amount of these ingredients available each day.

<table>
<thead>
<tr>
<th>Food Color (g)</th>
<th>Dazzling Red</th>
<th>Organic</th>
<th>Total Available</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>Salt (g)</td>
<td>10</td>
<td>20</td>
<td>220</td>
</tr>
</tbody>
</table>

Use a graph to show the possible numbers of batches of each type of pistachio Socaccio can produce each day. This region (the solution set of a system of inequalities) is called the feasible region.

Solution As we did in Chapter 2, we start by identifying the unknowns: Let $x$ be the number of batches of Dazzling Red manufactured per day and let $y$ be the number of batches of Organic manufactured each day.

Now, because of our experience with systems of linear equations, we are tempted to say: For food color $2x + y = 20$ and for salt, $10x + 20y = 220$. However, no one is saying that Socaccio has to use all available ingredients; the company might choose to use fewer than the total available amounts if this proves more profitable. Thus, $2x + y$ can be anything up to a total of 20. In other words,

$$2x + y \leq 20.$$  

Similarly,

$$10x + 20y \leq 220.$$
Before we go on...

Every point in the feasible region in Example 4 represents a value for \( x \) and a value for \( y \) that do not violate any of the company’s restrictions. For example, the point (5, 6) lies well inside the region, so the company can produce five batches of Dazzling Red nuts and six batches of Organic without exceeding the limitations on ingredients [that is, \( 2(5) + 6 = 16 \leq 20 \) and \( 10(5) + 20(6) = 170 \leq 220 \)]. The corner points A, B, C, and D are significant if the company wishes to realize the greatest profit, as we will see in Section 4.2. We can find the corners as in the following table:

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<tr>
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<tbody>
<tr>
<td>A</td>
<td></td>
<td>(0, 0)</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>(10, 0)</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>(0, 11)</td>
</tr>
<tr>
<td>D</td>
<td>( 2x + y = 20 )</td>
<td>(6, 8)</td>
</tr>
<tr>
<td></td>
<td>( 10x + 20y = 220 )</td>
<td></td>
</tr>
</tbody>
</table>

(We have not listed the lines through the first three corners because their coordinates can be read easily from the graph.) Points on the line segment DB represent use of all the food color (because the segment lies on the line \( 2x + y = 20 \)), and points on the line segment CD represent use of all the salt (because the segment lies on the line \( 10x + 20y = 220 \)). Note that the point D is the only solution that uses all of both ingredients.

There are two more restrictions not explicitly mentioned: Neither \( x \) nor \( y \) can be negative. (The company cannot produce a negative number of batches of nuts.) Therefore, we have the additional restrictions

\[
\begin{align*}
x & \geq 0 \\
y & \geq 0.
\end{align*}
\]

These two inequalities tell us that the feasible region (solution set) is restricted to the first quadrant, because in the other quadrants, either \( x \) or \( y \) or both \( x \) and \( y \) are negative. So instead of shading out all other quadrants, we can simply restrict our drawing to the first quadrant.

The (bounded) feasible region shown in Figure 12 is a graphical representation of the limitations the company faces.

We can find the corners as in the following table:

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**FAQs**

**Q:** How do I know whether to model a situation by a linear inequality like \( 3x + 2y \leq 10 \) or by a linear equation like \( 3x + 2y = 10 \)?

**A:** Here are some key phrases to look for: *at most, up to, no more than, at least, or more, exactly*. Suppose, for instance, that nuts cost 3¢, bolts cost 2¢, \( x \) is the number of nuts you can buy, and \( y \) is the number of bolts you can buy.

- If you have *up to* 10¢ to spend, then \( 3x + 2y \leq 10 \).
- If you must spend *exactly* 10¢, then \( 3x + 2y = 10 \).
- If you plan to spend *at least* 10¢, then \( 3x + 2y \geq 10 \).

The use of inequalities to model a situation is often more realistic than the use of equations; for instance, one cannot always expect to exactly fill all orders, spend the exact amount of one’s budget, or keep a plant operating at exactly 100% capacity.
4.1 Exercises

In Exercises 1–26, sketch the region that corresponds to the given inequalities, say whether the region is bounded or unbounded, and find the coordinates of all corner points (if any).

1. \(2x + y \leq 10\)
2. \(-x - y \leq 12\)
3. \(-x - 2y \leq 8\)
4. \(-x + 2y \geq 4\)
5. \(3x + 2y \geq 5\)
6. \(2x - 3y \leq 7\)
7. \(x \leq 3y\)
8. \(y \geq 3x\)
9. \(\frac{3x - y}{4} \leq 1\)
10. \(\frac{x}{3} + \frac{2y}{3} \geq 2\)
11. \(x \geq -5\)
12. \(y \leq -4\)
13. \(4x - y \leq 8\)
14. \(2x + y \leq 4\)
15. \(x + 2y \leq 2\)
16. \(3x + 2y \leq 6\)
17. \(3x - 2y \leq 6\)
18. \(3x - 2y \geq 6\)
19. \(x \geq 0\)
20. \(x - 2y \geq 0\)
21. \(x + y \geq 5\)
22. \(x \leq 10\)
23. \(x \leq y\)
24. \(y \leq x/2\)
25. \(x - 3y \leq 0\)
26. \(4x - 3y \geq 0\)

HINT [See Example 1.]

11. \(x \geq -5\)
12. \(y \leq -4\)
13. \(4x - y \leq 8\)
14. \(2x + y \leq 4\)
15. \(x + 2y \leq 2\)
16. \(3x + 2y \leq 6\)
17. \(3x - 2y \leq 6\)
18. \(3x - 2y \geq 6\)
19. \(x \geq 0\)
20. \(x - 2y \geq 0\)
21. \(x + y \geq 5\)
22. \(x \leq 10\)
23. \(x \leq y\)
24. \(y \leq x/2\)
25. \(x - 3y \leq 0\)
26. \(4x - 3y \geq 0\)

HINT [See Examples 2 and 3.]

11. \(x \geq -5\)
12. \(y \leq -4\)
13. \(4x - y \leq 8\)
14. \(2x + y \leq 4\)
15. \(x + 2y \leq 2\)
16. \(3x + 2y \leq 6\)
17. \(3x - 2y \leq 6\)
18. \(3x - 2y \geq 6\)
19. \(x \geq 0\)
20. \(x - 2y \geq 0\)
21. \(x + y \geq 5\)
22. \(x \leq 10\)
23. \(x \leq y\)
24. \(y \leq x/2\)
25. \(x - 3y \leq 0\)
26. \(4x - 3y \geq 0\)

HINT [See Examples 2 and 3.]

11. \(x \geq -5\)
12. \(y \leq -4\)
13. \(4x - y \leq 8\)
14. \(2x + y \leq 4\)
15. \(x + 2y \leq 2\)
16. \(3x + 2y \leq 6\)
17. \(3x - 2y \leq 6\)
18. \(3x - 2y \geq 6\)
19. \(x \geq 0\)
20. \(x - 2y \geq 0\)
21. \(x + y \geq 5\)
22. \(x \leq 10\)
23. \(x \leq y\)
24. \(y \leq x/2\)
25. \(x - 3y \leq 0\)
26. \(4x - 3y \geq 0\)

HINT [See Examples 2 and 3.]

11. \(x \geq -5\)
12. \(y \leq -4\)
13. \(4x - y \leq 8\)
14. \(2x + y \leq 4\)
15. \(x + 2y \leq 2\)
16. \(3x + 2y \leq 6\)
17. \(3x - 2y \leq 6\)
18. \(3x - 2y \geq 6\)
19. \(x \geq 0\)
20. \(x - 2y \geq 0\)
21. \(x + y \geq 5\)
22. \(x \leq 10\)
23. \(x \leq y\)
24. \(y \leq x/2\)
25. \(x - 3y \leq 0\)
26. \(4x - 3y \geq 0\)

HINT [See Examples 2 and 3.]

In Exercises 27–32, we suggest you use technology. Graph the regions corresponding to the inequalities, and find the coordinates of all corner points (if any) to two decimal places:

27. \(2.1x - 4.3y \geq 9.7\)
28. \(-4.3x + 4.6y \geq 7.1\)
29. \(-0.2x + 0.7y \geq 3.3\)
30. \(0.2x + 0.3y \geq 7.2\)
31. \(1.1x + 3.4y \geq 0\)
32. \(2.5x - 6.7y \leq 0\)
33. \(4.1x - 4.3y \leq 4.4\)
34. \(7.5x - 4.4y \leq 5.7\)
35. \(4.3x + 8.5y \leq 10\)
36. \(6.1x + 6.7y \leq 9.6\)

Applications

33. Resource Allocation You manage an ice cream factory that makes two flavors: Creamy Vanilla and Continental Mocha. Into each quart of Creamy Vanilla go 2 eggs and 3 cups of cream. Into each quart of Continental Mocha go 1 egg and 3 cups of cream. You have in stock 500 eggs and 900 cups of cream. Draw the feasible region showing the number of quarts of vanilla and number of quarts of mocha that can be produced. Find the corner points of the region. HINT [See Example 4.]

34. Resource Allocation Podunk Institute of Technology’s Math Department offers two courses: Finite Math and Applied Calculus. Each section of Finite Math has 60 students, and each section of Applied Calculus has 50. The department is allowed to offer a total of up to 110 sections. Furthermore, no more than 6,000 students want to take a math course. (No student will take more than one math course.) Draw the feasible region that shows the number of sections of each class that can be offered. Find the corner points of the region. HINT [See Example 4.]

35. Nutrition Ruff Inc. makes dog food out of chicken and grain. Chicken has 10 grams of protein and 5 grams of fat per ounce, and grain has 2 grams of protein and 2 grams of fat per ounce. A bag of dog food must contain at least 200 grams of protein and at least 150 grams of fat. Draw the feasible region that shows the number of ounces of chicken and number of ounces of grain Ruff can mix into each bag of dog food. Find the corner points of the region.

36. Purchasing Enormous State University’s Business School is buying computers. The school has two models to choose from: the Pomegranate and the iZac. Each Pomegranate comes with 400 MB of memory and 80 GB of disk space, and each iZac has 300 MB of memory and 100 GB of disk space. For reasons related to its accreditation, the school would like to be able to say that it has a total of at least 48,000 MB of memory and at least 12,800 GB of disk space. Draw the feasible region that shows the number of each kind of computer it can buy. Find the corner points of the region.
37. **Nutrition** Each serving of Gerber Mixed Cereal for Baby contains 60 calories and 11 grams of carbohydrates, and each serving of Gerber Mango Tropical Fruit Dessert contains 80 calories and 21 grams of carbohydrates.

38. **Nutrition** Each serving of Gerber Mixed Cereal for Baby contains 60 calories, 11 grams of carbohydrates, and no vitamin C. Each serving of Gerber Apple Banana Juice contains 60 calories, 15 grams of carbohydrates, and 120 percent of the U.S. Recommended Daily Allowance (RDA) of vitamin C for infants.

41. **Investments** Your portfolio manager has suggested two high-yielding stocks: Consolidated Edison (ED) and General Electric (GE). ED shares cost $40 and yield 6% in dividends. GE shares cost $16 and yield 7.5% in dividends. You have up to $10,000 to invest, and would like to earn at least $600 in dividends. Draw the feasible region that shows how many shares in each company you can buy. Find the corner points of the region. (Round each coordinate to the nearest whole number.)

42. **Investments** Your friend’s portfolio manager has suggested two energy stocks: Exxon Mobil (XOM) and British Petroleum (BP). XOM shares cost $80 and yield 2% in dividends. BP shares cost $50 and yield 7% in dividends. Your friend has up to $40,000 to invest, and would like to earn at least $1,400 in dividends. Draw the feasible region that shows how many shares in each company she can buy. Find the corner points of the region. (Round each coordinate to the nearest whole number.)

43. **Advertising** You are the marketing director for a company that manufactures bodybuilding supplements and you are planning to run ads in *Sports Illustrated* and *GQ Magazine*. Based on readership data, you estimate that each one-page ad in *Sports Illustrated* will be read by 650,000 people in your target group, while each one-page ad in *GQ* will be read by 150,000. You would like your ads to be read by at least three million people in the target group and, to ensure the broadest possible audience, you would like to place at least three full-page ads in each magazine. Draw the feasible region that shows how many pages you can purchase in each magazine. Find the corner points of the region. (Round each coordinate to the nearest whole number.)

44. **Advertising** You are the marketing director for a company that manufactures bodybuilding supplements and you are planning to run ads in *Sports Illustrated* and *Muscle and Fitness*. Based on readership data, you estimate that each one-page ad in *Sports Illustrated* will be read by 650,000 people in your target group, while each one-page ad in *Muscle and Fitness* will be read by 250,000 people in your target group.

You would like your ads to be read by at least four million people in the target group and, to ensure the broadest possible audience, you would like to place at least three full-page ads in each magazine during the year. Draw the feasible region showing how many pages you can purchase in each magazine. Find the corner points of the region. (Round each coordinate to the nearest whole number.)

### COMMUNICATION AND REASONING EXERCISES

45. Find a system of inequalities whose solution set is unbounded.

46. Find a system of inequalities whose solution set is empty.

47. How would you use linear inequalities to describe the triangle with corner points (0, 0), (2, 0), and (0, 1)?

48. Explain the advantage of shading the region of points that do not satisfy the given inequalities. Illustrate with an example.

49. Describe at least one drawback to the method of finding the corner points of a feasible region by drawing its graph, when the feasible region arises from real-life constraints.

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8 Ibid.


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5 Source: Nutrition information printed on the jar.

4 Ibid.


50. Draw several bounded regions described by linear inequalities. For each region you draw, find the point that gives the greatest possible value of \( x + y \). What do you notice?

In Exercises 51–54, you are mixing \( x \) grams of ingredient A and \( y \) grams of ingredient B. Choose the equation or inequality that models the given requirement.

51. There should be at least 3 more grams of ingredient A than ingredient B.
   (A) \( 3x - y \leq 0 \)  
   (B) \( x - 3y \geq 0 \)  
   (C) \( x - y \geq 3 \)  
   (D) \( 3x - y \geq 0 \)

52. The mixture should contain at least 25% of ingredient A by weight.
   (A) \( 4x - y \leq 0 \)  
   (B) \( x - 4y \geq 0 \)  
   (C) \( x - y \geq 4 \)  
   (D) \( 3x - y \geq 0 \)

53. There should be at least 3 parts (by weight) of ingredient A to 2 parts of ingredient B.
   (A) \( 3x - 2y \geq 0 \)  
   (B) \( 2x - 3y \geq 0 \)  
   (C) \( 3x + 2y \geq 0 \)  
   (D) \( 2x + 3y \geq 0 \)

54. There should be no more of ingredient A (by weight) than ingredient B.
   (A) \( x - y = 0 \)  
   (B) \( x - y \leq 0 \)  
   (C) \( x - y \geq 0 \)  
   (D) \( x + y \geq y \)

55. You are setting up a system of inequalities in the unknowns \( x \) and \( y \). The inequalities represent constraints faced by Fly-by-Night Airlines, where \( x \) represents the number of first-class tickets it should issue for a specific flight and \( y \) represents the number of business-class tickets it should issue for that flight. You find that the feasible region is empty. How do you interpret this?

56. In the situation described in the preceding exercise, is it possible instead for the feasible region to be unbounded? Explain your answer.

57. Create an interesting scenario that leads to the following system of inequalities:

\[
20x + 40y \leq 1,000 \\
30x + 20y \leq 1,200 \\
x \geq 0, y \geq 0.
\]

58. Create an interesting scenario that leads to the following system of inequalities:

\[
20x + 40y \geq 1,000 \\
30x + 20y \geq 1,200 \\
x \geq 0, y \geq 0.
\]

### 4.2 Solving Linear Programming Problems Graphically

As we saw in Example 4 in Section 4.1, in some scenarios the possibilities are restricted by a system of linear inequalities. In that example, it would be natural to ask which of the various possibilities gives the company the largest profit. This is a kind of problem known as a **linear programming problem** (commonly referred to as an LP problem).

#### Linear Programming (LP) Problems

A **linear programming problem** in two unknowns \( x \) and \( y \) is one in which we are to find the maximum or minimum value of a linear expression

\[ ax + by \]

called the **objective function**, subject to a number of linear **constraints** of the form

\[ cx + dy \leq e \quad \text{or} \quad cx + dy \geq e. \]

The largest or smallest value of the objective function is called the **optimal value**, and a pair of values of \( x \) and \( y \) that gives the optimal value constitutes an **optimal solution**.
The set of points \((x, y)\) satisfying all the constraints is the \textbf{feasible region} for the problem. Our methods of solving LP problems rely on the following facts:

**Quick Example**

Maximize \(p = x + y\)  
subject to \[
\begin{align*}
  x + 2y &\leq 12 \\
  2x + y &\leq 12 \\
  x \geq 0, y \geq 0.
\end{align*}
\]

See Example 1 for a method of solving this LP problem (that is, finding an optimal solution and value).

The set of points \((x, y)\) satisfying all the constraints is the \textbf{feasible region} for the problem. Our methods of solving LP problems rely on the following facts:

**Fundamental Theorem of Linear Programming**

- If an LP problem has optimal solutions, then at least one of these solutions occurs at a corner point of the feasible region.
- Linear programming problems with bounded, nonempty feasible regions always have optimal solutions.

Let’s see how we can use this to solve an LP problem, and then we’ll discuss why it’s true.

**EXAMPLE 1 Solving an LP Problem**

Maximize \(p = x + y\)  
subject to \[
\begin{align*}
  x + 2y &\leq 12 \\
  2x + y &\leq 12 \\
  x \geq 0, y \geq 0.
\end{align*}
\]

**Solution** We begin by drawing the feasible region for the problem. We do this using the techniques of Section 4.1, and we get Figure 13.

Each \textbf{feasible point} (point in the feasible region) gives an \(x\) and a \(y\) satisfying the constraints. The question now is, which of these points gives the largest value of the objective function \(p = x + y\)? The Fundamental Theorem of Linear Programming tells us that the largest value must occur at one (or more) of the corners of the feasible region. In the following table, we list the coordinates of each corner point and we compute the value of the objective function at each corner.

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Lines through Point</th>
<th>Coordinates</th>
<th>(p = x + y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td></td>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(B)</td>
<td></td>
<td>(6, 0)</td>
<td>6</td>
</tr>
<tr>
<td>(C)</td>
<td></td>
<td>(0, 6)</td>
<td>6</td>
</tr>
<tr>
<td>(D)</td>
<td>(x + 2y = 12)</td>
<td>(4, 4)</td>
<td>8</td>
</tr>
</tbody>
</table>

Now we simply pick the one that gives the largest value for \(p\), which is \(D\). Therefore, the optimal value of \(p\) is 8, and an optimal solution is \((4, 4)\).
Now we owe you an explanation of why one of the corner points should be an optimal solution. The question is, which point in the feasible region gives the largest possible value of \( p = x + y \)?

Consider first an easier question: Which points result in a particular value of \( p \)? For example, which points result in \( p = 2 \)? These would be the points on the line \( x + y = 2 \), which is the line labeled \( p = 2 \) in Figure 14.

Now suppose we want to know which points make \( p = 4 \): These would be the points on the line \( x + y = 4 \), which is the line labeled \( p = 4 \) in Figure 14. Notice that this line is parallel to but higher than the line \( p = 2 \). (If \( p \) represented profit in an application, we would call these *isoprofit lines*, or *constant-profit lines*.) Imagine moving this line up or down in the picture. As we move the line down, we see smaller values of \( p \), and as we move it up, we see larger values. Several more of these lines are drawn in Figure 14. Look, in particular, at the line labeled \( p = 10 \). This line does not meet the feasible region, meaning that no feasible point makes \( p \) as large as 10. Starting with the line \( p = 2 \), as we move the line up, increasing \( p \), there will be a last line that meets the feasible region. In the figure it is clear that this is the line \( p = 8 \), and this meets the feasible region in only one point, which is the corner point \( D \). Therefore, \( D \) gives the greatest value of \( p \) of all feasible points.

If we had been asked to maximize some other objective function, such as \( p = x + 3y \), then the optimal solution might be different. Figure 15 shows some of the isoprofit lines for this objective function. This time, the last point that is hit as \( p \) increases is \( C \), not \( D \). This tells us that the optimal solution is \((0, 6)\), giving the optimal value \( p = 18 \).

This discussion should convince you that the optimal value in an LP problem will always occur at one of the corner points. By the way, it is possible for the optimal value to occur at two corner points and at all points along an edge connecting them. (Do you see why?) We will see this in Example 3(b).

Here is a summary of the method we have just been using.

**Graphical Method for Solving Linear Programming Problems in Two Unknowns (Bounded Feasible Regions)**

1. Graph the feasible region and check that it is bounded.
2. Compute the coordinates of the corner points.
3. Substitute the coordinates of the corner points into the objective function to see which gives the maximum (or minimum) value of the objective function.
4. Any such corner point is an optimal solution.

**Note** If the feasible region is unbounded, this method will work only if there are optimal solutions; otherwise, it will not work. We will show you a method for deciding this on page 280.

**APPLICATIONS**

**EXAMPLE 2 Resource Allocation**

Acme Babyfoods mixes two strengths of apple juice. One quart of Beginner’s juice is made from 30 fluid ounces of water and 2 fluid ounces of apple juice concentrate. One quart of Advanced juice is made from 20 fluid ounces of water and 12 fluid ounces of
concentrate. Every day Acme has available 30,000 fluid ounces of water and 3,600 fluid ounces of concentrate. Acme makes a profit of 20¢ on each quart of Beginner’s juice and 30¢ on each quart of Advanced juice. How many quarts of each should Acme make each day to get the largest profit? How would this change if Acme made a profit of 40¢ on Beginner’s juice and 20¢ on Advanced juice?

Solution  Looking at the question that we are asked, we see that our unknown quantities are

\[ x = \text{number of quarts of Beginner’s juice made each day} \]
\[ y = \text{number of quarts of Advanced juice made each day}. \]

(In this context, \( x \) and \( y \) are often called the decision variables, because we must decide what their values should be in order to get the largest profit.) We can write down the data given in the form of a table (the numbers in the first two columns are amounts per quart of juice):

<table>
<thead>
<tr>
<th>( \text{Beginner’s, } x )</th>
<th>( \text{Advanced, } y )</th>
<th>( \text{Available} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water (ounces)</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>Concentrate (ounces)</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>Profit (¢)</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

Because nothing in the problem says that Acme must use up all the water or concentrate, just that it can use no more than what is available, the first two rows of the table give us two inequalities:

\[ 30x + 20y \leq 30,000 \]
\[ 2x + 12y \leq 3,600. \]

Dividing the first inequality by 10 and the second by 2 gives

\[ 3x + 2y \leq 3,000 \]
\[ x + 6y \leq 1,800. \]

We also have that \( x \geq 0 \) and \( y \geq 0 \) because Acme can’t make a negative amount of juice. To finish setting up the problem, we are asked to maximize the profit, which is

\[ p = 20x + 30y. \]

Expressed in ¢

This gives us our LP problem:

Maximize \( p = 20x + 30y \)

subject to \( 3x + 2y \leq 3,000 \)
\( x + 6y \leq 1,800 \)
\( x \geq 0, y \geq 0. \)

The (bounded) feasible region is shown in Figure 16.

The corners and the values of the objective function are listed in the following table:

<table>
<thead>
<tr>
<th>( \text{Point} )</th>
<th>( \text{Lines through Point} )</th>
<th>( \text{Coordinates} )</th>
<th>( p = 20x + 30y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td></td>
<td>( (0, 0) )</td>
<td>0</td>
</tr>
<tr>
<td>( B )</td>
<td></td>
<td>( (1,000, 0) )</td>
<td>20,000</td>
</tr>
<tr>
<td>( C )</td>
<td></td>
<td>( (0, 300) )</td>
<td>9,000</td>
</tr>
<tr>
<td>( D )</td>
<td>( 3x + 2y = 3,000 )</td>
<td>( (900, 150) )</td>
<td>22,500</td>
</tr>
<tr>
<td></td>
<td>( x + 6y = 1,800 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
We are seeking to maximize the objective function \( p \), so we look for corner points that give the maximum value for \( p \). Because the maximum occurs at the point \( D \), we conclude that the (only) optimal solution occurs at \( D \). Thus, the company should make 900 quarts of Beginner’s juice and 150 quarts of Advanced juice, for a largest possible profit of 22,500¢, or $225.

If, instead, the company made a profit of 40¢ on each quart of Beginner’s juice and 20¢ on each quart of Advanced juice, then we would have \( p = 40x + 20y \). This gives the following table:

<table>
<thead>
<tr>
<th>Point</th>
<th>Lines through Point</th>
<th>Coordinates</th>
<th>( p = 40x + 20y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( (0, 0) )</td>
<td>( 0 )</td>
<td></td>
</tr>
<tr>
<td>( B )</td>
<td>( (1,000, 0) )</td>
<td>( 40,000 )</td>
<td></td>
</tr>
<tr>
<td>( C )</td>
<td>( (0, 300) )</td>
<td>( 6,000 )</td>
<td></td>
</tr>
<tr>
<td>( D )</td>
<td>( 3x + 2y = 3,000 ) ( x + 6y = 1,800 )</td>
<td>( (900, 150) )</td>
<td>( 39,000 )</td>
</tr>
</tbody>
</table>

We can see that, in this case, Acme should make 1,000 quarts of Beginner’s juice and no Advanced juice, for a largest possible profit of 40,000¢, or $400.

**Before we go on...** Notice that, in the first version of the problem in Example 2, the company used all the water and juice concentrate:

- **Water**: \( 30(900) + 20(150) = 30,000 \)
- **Concentrate**: \( 2(900) + 12(150) = 3,600 \).

In the second version, it used all the water but not all the concentrate:

- **Water**: \( 30(100) + 20(0) = 30,000 \)
- **Concentrate**: \( 2(100) + 12(0) = 200 < 3,600 \).

**EXAMPLE 3 Investments**

The Solid Trust Savings & Loan Company has set aside $25 million for loans to home buyers. Its policy is to allocate at least $10 million annually for luxury condominiums. A government housing development grant it receives requires, however, that at least one-third of its total loans be allocated to low-income housing.

**a.** Solid Trust’s return on condominiums is 12% and its return on low-income housing is 10%. How much should the company allocate for each type of housing to maximize its total return?

**b.** Redo part (a), assuming that the return is 12% on both condominiums and low-income housing.

**Solution**

**a.** We first identify the unknowns: Let \( x \) be the annual amount (in millions of dollars) allocated to luxury condominiums, and let \( y \) be the annual amount allocated to low-income housing.

We now look at the constraints. The first constraint is mentioned in the first sentence: The total the company can invest is $25 million. Thus,

\[ x + y \leq 25. \]
(The company is not required to invest all of the $25 million; rather, it can invest *up to* $25 million.) Next, the company has allocated at least $10 million to condos. Rephrasing this in terms of the unknowns, we get

_The amount allocated to condos is at least $10 million._

The phrase “is at least” means $\geq$. Thus, we obtain a second constraint:

$$x \geq 10.$$ 

The third constraint is that at least one-third of the total financing must be for low-income housing. Rephrasing this, we say:

_The amount allocated to low-income housing is at least one-third of the total._

Because the total investment will be $x + y$, we get

$$y \geq \frac{1}{3}(x + y).$$

We put this in the standard form of a linear inequality as follows:

$$3y \geq x + y \quad \text{Multiply both sides by 3.}$$

$$-x + 2y \geq 0. \quad \text{Subtract } x + y \text{ from both sides.}$$

There are no further constraints.

Now, what about the return on these investments? According to the data, the annual return is given by

$$p = 0.12x + 0.10y.$$ 

We want to make this quantity $p$ as large as possible. In other words, we want to

Maximize 

$$p = 0.12x + 0.10y$$

subject to

$$x + y \leq 25$$

$$x \geq 10$$

$$-x + 2y \geq 0$$

$$x \geq 0, y \geq 0.$$ 

(Do you see why the inequalities $x \geq 0$ and $y \geq 0$ are slipped in here?) The feasible region is shown in Figure 17.

We now make a table that gives the return on investment at each corner point:

<table>
<thead>
<tr>
<th>Point</th>
<th>Lines through Point</th>
<th>Coordinates</th>
<th>$p = 0.12x + 0.10y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$x = 10$ $x + y = 25$</td>
<td>(10, 15)</td>
<td>2.7</td>
</tr>
<tr>
<td>$B$</td>
<td>$x + y = 25$ $-x + 2y = 0$</td>
<td>(50/3, 25/3)</td>
<td>2.833</td>
</tr>
<tr>
<td>$C$</td>
<td>$x = 10$ $-x + 2y = 0$</td>
<td>(10, 5)</td>
<td>1.7</td>
</tr>
</tbody>
</table>

From the table, we see that the values of $x$ and $y$ that maximize the return are $x = 50/3$ and $y = 25/3$, which give a total return of $2.833$ million. In other words, the most profitable course of action is to invest $16.667$ million in loans for condominiums and $8.333$ million in loans for low-income housing, giving a maximum annual return of $2.833$ million.
Looking at the table, we see that a curious thing has happened: We get the same maximum annual return at both $A$ and $B$. Thus, we could choose either option to maximize the annual return. In fact, any point along the line segment $AB$ will yield an annual return of $3 million. For example, the point $(12, 13)$ lies on the line segment $AB$ and also yields an annual revenue of $3 million. This happens because the “isoreturn” lines are parallel to that edge.

b. The LP problem is the same as for part (a) except for the objective function:

Maximize $p = 0.12x + 0.12y$

subject to

$x + y \leq 25$

$x \geq 10$

$-x + 2y \geq 0$

$x \geq 0, y \geq 0$.

Here are the values of $p$ at the three corners:

<table>
<thead>
<tr>
<th>Point</th>
<th>Coordinates</th>
<th>$p = 0.12x + 0.12y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>(10, 15)</td>
<td>3</td>
</tr>
<tr>
<td>$B$</td>
<td>$(50/3, 25/3)$</td>
<td>3</td>
</tr>
<tr>
<td>$C$</td>
<td>(10, 5)</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Looking at the table, we see that a curious thing has happened: We get the same maximum annual return at both $A$ and $B$. Thus, we could choose either option to maximize the annual return. In fact, any point along the line segment $AB$ will yield an annual return of $3 million. For example, the point $(12, 13)$ lies on the line segment $AB$ and also yields an annual revenue of $3 million. This happens because the “isoreturn” lines are parallel to that edge.

Before we go on... What breakdowns of investments would lead to the lowest return for parts (a) and (b)?

The preceding examples all had bounded feasible regions. If the feasible region is unbounded, then, provided there are optimal solutions, the fundamental theorem of linear programming guarantees that the above method will work. The following procedure determines whether or not optimal solutions exist and finds them when they do.

Solving Linear Programming Problems in Two Unknowns (Unbounded Feasible Regions)

If the feasible region of an LP problem is unbounded, proceed as follows:

1. Draw a rectangle large enough so that all the corner points are inside the rectangle (and not on its boundary):
In the next two examples, we work with unbounded feasible regions.

EXAMPLE 4 Cost

You are the manager of a small store that specializes in hats, sunglasses, and other accessories. You are considering a sales promotion of a new line of hats and sunglasses. You will offer the sunglasses only to those who purchase two or more hats, so you will sell at least twice as many hats as pairs of sunglasses. Moreover, your supplier tells you that, due to seasonal demand, your order of sunglasses cannot exceed 100 pairs. To ensure that the sale items fill out the large display you have set aside, you estimate that you should order at least 210 items in all.

a. Assume that you will lose $3 on every hat and $2 on every pair of sunglasses sold. Given the constraints above, how many hats and pairs of sunglasses should you order to lose the least amount of money in the sales promotion?

b. Suppose instead that you lose $1 on every hat sold but make a profit of $5 on every pair of sunglasses sold. How many hats and pairs of sunglasses should you order to make the largest profit in the sales promotion?

c. Now suppose that you make a profit of $1 on every hat sold but lose $5 on every pair of sunglasses sold. How many hats and pairs of sunglasses should you order to make the largest profit in the sales promotion?

Solution

a. The unknowns are:

\[ x = \text{number of hats you order} \]
\[ y = \text{number of pairs of sunglasses you order}. \]
The corner point that gives the minimum value of the objective function $c$ is $B$. Because $B$ is one of the corner points of the original feasible region, we conclude that our linear programming problem has an optimal solution at $B$. Thus, the combination that gives the smallest loss is 140 hats and 70 pairs of sunglasses.
b. The LP problem is the following:

Maximize \( p = -x + 5y \)
subject to \( x - 2y \geq 0 \)
\( y \leq 100 \)
\( x + y \geq 210 \)
\( x \geq 0, y \geq 0. \)

Because most of the work is already done for us in part (a), all we need to do is change the objective function in the table that lists the corner points:

<table>
<thead>
<tr>
<th>Point</th>
<th>Lines through Point</th>
<th>Coordinates</th>
<th>( p = -x + 5y ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( x + y = 210 ) ( x - 2y = 0 )</td>
<td>(210, 0)</td>
<td>-210</td>
</tr>
<tr>
<td>( B )</td>
<td>( x + y = 210 ) ( x - 2y = 0 )</td>
<td>(140, 70)</td>
<td>210</td>
</tr>
<tr>
<td>( C )</td>
<td>( x - 2y = 0 ) ( y = 100 )</td>
<td>(200, 100)</td>
<td>300</td>
</tr>
<tr>
<td>( D )</td>
<td>( x - 2y = 0 ) ( y = 100 )</td>
<td>(300, 100)</td>
<td>200</td>
</tr>
<tr>
<td>( E )</td>
<td>( x - 2y = 0 ) ( y = 100 )</td>
<td>(300, 0)</td>
<td>-300</td>
</tr>
</tbody>
</table>

The corner point that gives the maximum value of the objective function \( p \) is \( C \). Because \( C \) is one of the corner points of the original feasible region, we conclude that our LP problem has an optimal solution at \( C \). Thus, the combination that gives the largest profit ($300) is 200 hats and 100 pairs of sunglasses.

c. The objective function is now \( p = x - 5y \), which is the negative of the objective function used in part (b). Thus, the table of values of \( p \) is the same as in part (b), except that it has opposite signs in the \( p \) column. This time we find that the maximum value of \( p \) occurs at \( E \). However, \( E \) is not a corner point of the original feasible region, so the LP problem has no optimal solution. Referring to Figure 18, we can make the objective \( p \) as large as we like by choosing a point far to the right in the unbounded feasible region. Thus, the objective function is unbounded; that is, it is possible to make an arbitrarily large profit.

EXAMPLE 5 Resource Allocation

You are composing a very avant-garde ballade for violins and bassoons. In your ballade, each violinist plays a total of two notes and each bassoonist only one note. To make your ballade long enough, you decide that it should contain at least 200 instrumental notes. Furthermore, after playing the requisite two notes, each violinist will sing one soprano note, while each bassoonist will sing three soprano notes.* To make the ballade sufficiently interesting, you have decided on a minimum of 300 soprano notes. To give your composition a sense of balance, you wish to have no more than three times as many bassoonists as violinists. Violinists charge $200 per performance and bassoonists $400 per performance. How many of each should your ballade call for in order to minimize personnel costs?

* Whether or not these musicians are capable of singing decent soprano notes will be left to chance. You reason that a few bad notes will add character to the ballade.
From the table we see that the minimum cost occurs at \( B \), a corner point of the original feasible region. The linear programming problem thus has an optimal solution, and the minimum cost is $44,000 per performance, employing 60 violinists and 80 bassoonists. (Quite a wasteful ballade, one might say.)
Recognizing a Linear Programming Problem, Setting Up Inequalities, and Dealing with Unbounded Regions

Q: How do I recognize when an application leads to an LP problem as opposed to a system of linear equations?

A: Here are some cues that suggest an LP problem:

- Key phrases suggesting inequalities rather than equalities, like at most, up to, no more than, at least, and or more.
- A quantity that is being maximized or minimized (this will be the objective). Key phrases are maximum, minimum, most, least, largest, greatest, smallest, as large as possible, and as small as possible.

Q: How do I deal with tricky phrases like “there should be no more than twice as many nuts as bolts” or “at least 50% of the total should be bolts”?

A: The easiest way to deal with phrases like this is to use the technique we discussed in Chapter 2: reword the phrases using “the number of . . .”, as in

The number of nuts \( x \) is no more than twice the number of bolts \( y \)

\[ x \leq 2y \]

The number of bolts is at least 50% of the total

\[ y \geq 0.50(x + y) \]

Q: Do I always have to add a rectangle to deal with unbounded regions?

A: Under some circumstances, you can tell right away whether optimal solutions exist, even when the feasible region is unbounded.

Note that the following apply only when we have the constraints \( x \geq 0 \) and \( y \geq 0 \).

1. If you are minimizing \( c = ax + by \) with \( a \) and \( b \) non-negative, then optimal solutions always exist. (Examples 4(a) and 5 are of this type.)
2. If you are maximizing \( p = ax + by \) with \( a \) and \( b \) non-negative (and not both zero), then there is no optimal solution unless the feasible region is bounded.

Do you see why statements (1) and (2) are true?
7. Maximize \( p = 3x + 2y \)
subject to
\[
\begin{align*}
0.2x + 0.1y & \leq 1 \\
0.15x + 0.3y & \leq 1.5 \\
10x + 10y & \leq 60 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

8. Maximize \( p = x + 2y \)
subject to
\[
\begin{align*}
30x + 20y & \leq 600 \\
0.1x + 0.4y & \leq 4 \\
0.2x + 0.3y & \leq 4.5 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

9. Minimize \( c = 0.2x + 0.3y \)
subject to
\[
\begin{align*}
0.2x + 0.1y & \geq 1 \\
0.15x + 0.3y & \geq 1.5 \\
10x + 10y & \geq 80 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

10. Minimize \( c = 0.4x + 0.1y \)
subject to
\[
\begin{align*}
30x + 20y & \geq 600 \\
0.1x + 0.4y & \geq 4 \\
0.2x + 0.3y & \geq 4.5 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

11. Maximize and minimize \( p = x + 2y \)
subject to
\[
\begin{align*}
x + y & \geq 2 \\
x + y & \leq 10 \\
x - y & \leq 2 \\
x - y & \geq -2.
\end{align*}
\]

12. Maximize and minimize \( p = 2x - y \)
subject to
\[
\begin{align*}
x - y & \leq 2 \\
x - y & \geq -2 \\
x & \leq 10, y & \geq 10.
\end{align*}
\]

13. Maximize \( p = 2x + 3y \)
subject to
\[
\begin{align*}
0.1x + 0.2y & \geq 1 \\
2x + y & \geq 10 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

14. Maximize \( p = 3x + 2y \)
subject to
\[
\begin{align*}
0.1x + 0.1y & \geq 0.2 \\
y & \leq 10 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

15. Minimize \( c = 2x + 4y \)
subject to
\[
\begin{align*}
0.1x + 0.1y & \geq 1 \\
x + 2y & \geq 14 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

16. Maximize \( p = 2x + 3y \)
subject to
\[
\begin{align*}
-x + y & \geq 10 \\
x + 2y & \leq 12 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

17. Minimize \( c = 3x - 3y \)
subject to
\[
\begin{align*}
x & \leq \frac{y}{4} \\
y & \leq \frac{2x}{3} \\
x + y & \geq 5 \\
x + 2y & \leq 10 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

18. Minimize \( c = -x + 2y \)
subject to
\[
\begin{align*}
y & \leq \frac{2x}{3} \\
x & \leq 3y \\
y & \geq 4 \\
x & \geq 6 \\
x + y & \leq 16.
\end{align*}
\]

19. Maximize \( p = x + y \)
subject to
\[
\begin{align*}
x + 2y & \geq 10 \\
x + 2y & \leq 10 \\
x + y & \geq 10 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

20. Minimize \( c = 3x + y \)
subject to
\[
\begin{align*}
10x + 20y & \geq 100 \\
0.3x + 0.1y & \geq 1 \\
x & \geq 0, y & \geq 0.
\end{align*}
\]

**APPLICATIONS**

21. **Resource Allocation** You manage an ice cream factory that makes two flavors: Creamy Vanilla and Continental Mocha. Into each quart of Creamy Vanilla go 2 eggs and 3 cups of cream. Into each quart of Continental Mocha go 1 egg and 3 cups of cream. You have in stock 500 eggs and 900 cups of cream. You make a profit of $3 on each quart of Creamy Vanilla and $2 on each quart of Continental Mocha. How many quarts of each flavor should you make in order to earn the largest profit? Hint [See Example 2.]

22. **Resource Allocation** Podunk Institute of Technology’s Math Department offers two courses: Finite Math and Applied Calculus. Each section of Finite Math has 60 students, and each section of Applied Calculus has 50. The department is allowed to offer a total of up to 110 sections. Furthermore, no more than 6,000 students want to take a math course (no student will take more than one math course). Suppose the university makes a profit of $100,000 on each section of Finite Math and $50,000 on each section of Applied Calculus (the
profit is the difference between what the students are charged and what the professors are paid). How many sections of each course should the department offer to make the largest profit? HINT (See Example 2.)

23. **Nutrition** Ruff, Inc. makes dog food out of chicken and grain. Chicken has 10 grams of protein and 5 grams of fat per ounce, and grain has 2 grams of protein and 2 grams of fat per ounce. A bag of dog food must contain at least 200 grams of protein and at least 150 grams of fat. If chicken costs 10¢ per ounce and grain costs 1¢ per ounce, how many ounces of each should Ruff use in each bag of dog food in order to minimize cost? HINT (See Example 4.)

24. **Purchasing** Enormous State University’s Business School is buying computers. The school has two models from which to choose, the Pomegranate and the iZac. Each Pomegranate comes with 400 MB of memory and 80 GB of disk space; each iZac has 300 MB of memory and 100 GB of disk space. For reasons related to its accreditation, the school would like to be able to say that it has a total of at least 48,000 MB of memory and at least 12,800 GB of disk space. If the Pomegranate and the iZac cost $2,000 each, how many of each should the school buy to keep the cost as low as possible? HINT (See Example 4.)

25. **Nutrition** Each serving of Gerber Mixed Cereal for Baby contains 60 calories and 11 grams of carbohydrates. Each serving of Gerber Mango Tropical Fruit Dessert contains 80 calories and 21 grams of carbohydrates. If the cereal costs 30¢ per serving and the dessert costs 50¢ per serving, and you want to provide your child with at least 140 calories and at least 32 grams of carbohydrates, how can you do so at the least cost? (Fractions of servings are permitted.)

26. **Nutrition** Each serving of Gerber Mixed Cereal for Baby contains 60 calories, 10 grams of carbohydrates, and no vitamin C. Each serving of Gerber Apple Banana Juice contains 60 calories, 15 grams of carbohydrates, and 120 percent of the U.S. Recommended Daily Allowance (RDA) of vitamin C for infants. The cereal costs 10¢ per serving and the juice costs 30¢ per serving. If you want to provide your child with at least 120 calories, at least 25 grams of carbohydrates, and at least 60 percent of the U.S. RDA of vitamin C for infants, how can you do so at the least cost? (Fractions of servings are permitted.)

27. **Energy Efficiency** You are thinking of making your home more energy efficient by replacing some of the light bulbs with compact fluorescent bulbs, and insulating part or all of your exterior walls. Each compact fluorescent light bulb costs $4 and saves you an average of $2 per year in energy costs, and each square foot of wall insulation costs $1 and saves you an average of $0.20 per year in energy costs. Your home has 60 light fittings and 1,100 sq. ft. of uninsulated exterior wall. You can spend no more than $1,200 and would like to save as much per year in energy costs as possible. How many compact fluorescent light bulbs and how many square feet of insulation should you purchase? How much will you save in energy costs per year?

28. **Energy Efficiency** (Compare with the preceding exercise.) You are thinking of making your mansion more energy efficient by replacing some of the light bulbs with compact fluorescent bulbs, and insulating part or all of your exterior walls. Each compact fluorescent light bulb costs $4 and saves you an average of $2 per year in energy costs, and each square foot of wall insulation costs $1 and saves you an average of $0.20 per year in energy costs. Your mansion has 200 light fittings and 3,000 sq. ft. of uninsulated exterior wall. To impress your friends, you would like to spend as much as possible, but save no more than $800 per year in energy costs (you are proud of your large utility bills). How many compact fluorescent light bulbs and how many square feet of insulation should you purchase? How much will you save in energy costs per year?

**Creatine Supplements** Exercises 29 and 30 are based on the following data on three bodybuilding supplements. (Figures shown correspond to a single serving.)

<table>
<thead>
<tr>
<th>Supplement</th>
<th>Creatine (g)</th>
<th>Carbohydrates (g)</th>
<th>Taurine (g)</th>
<th>Alpha Lipoic Acid (mg)</th>
<th>Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cell-Tech®</td>
<td>10</td>
<td>75</td>
<td>2</td>
<td>200</td>
<td>2.20</td>
</tr>
<tr>
<td>RiboForce HP®</td>
<td>5</td>
<td>15</td>
<td>1</td>
<td>0</td>
<td>1.60</td>
</tr>
<tr>
<td>Creatine Transport®</td>
<td>5</td>
<td>35</td>
<td>1</td>
<td>100</td>
<td>0.60</td>
</tr>
</tbody>
</table>

29. You are thinking of combining Cell-Tech and RiboForce HP to obtain a 10-day supply that provides at least 80 grams of creatine and at least 10 grams of taurine, but no more than 750 grams of carbohydrates and no more than 1,000 milligrams of alpha lipoic acid. How many servings of each supplement should you combine to meet your specifications at the least cost?

30. You are thinking of combining Cell-Tech and Creatine Transport to obtain a 10-day supply that provides at least 80 grams of creatine and at least 10 grams of taurine, but no more than 600 grams of carbohydrates and no more than 2,000 milligrams of alpha lipoic acid. How many servings of each supplement should you combine to meet your specifications at the least cost?

31. **Resource Allocation** Your salami manufacturing plant can order up to 1,000 pounds of pork and 2,400 pounds of beef per day for use in manufacturing its two specialties: “Count Dracula Salami” and “Frankenstein Sausage.” Production of

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11 Source: Nutrition information printed on the containers.
13 Source: Nutritional information supplied by the manufacturers (www.netrition.com). Cost per serving is approximate.
the Count Dracula variety requires 1 pound of pork and 3 pounds of beef for each salami, while the Frankensteina variety requires 2 pounds of pork and 2 pounds of beef for every sausage. In view of your heavy investment in advertising Count Dracula Salami, you have decided that at least one-third of the total production should be Count Dracula. On the other hand, due to the health-conscious consumer climate, your Frankensteina Sausage (sold as having less beef) is earning your company a profit of $3 per sausage, while sales of the Count Dracula variety are down and it is earning your company only $1 per salami. Given these restrictions, how many of each kind of sausage should you produce to maximize profits, and what is the maximum possible profit? HINT [See Example 3.]

32. Project Design The Megabuck Hospital Corp. is to build a state-subsidized nursing home catering to homeless patients as well as high-income patients. State regulations require that every subsidized nursing home must house a minimum of 1,000 homeless patients and no more than 750 high-income patients in order to qualify for state subsidies. The overall capacity of the hospital is to be 2,100 patients. The board of directors, under pressure from a neighborhood group, insists that the number of homeless patients should not exceed twice the number of high-income patients. Due to the state subsidy, the hospital will make an average profit of $10,000 per month for every homeless patient it houses, whereas the profit per high-income patient is estimated at $8,000 per month. How many of each type of patient should it house in order to maximize profit? HINT [See Example 3.]

33. Television Advertising In February 2008, each episode of “American Idol” was typically seen by 28.5 million viewers, while each episode of “Back to You” was seen by 12.3 million viewers. Your marketing services firm has been hired to promote Bald No More’s hair replacement process by buying at least 30 commercial spots during episodes of “American Idol” and “Back to You.” The cable company running “American Idol” has quoted a price of $3,000 per spot, while the cable company showing “Back to You” has quoted a price of $1,000 per spot. Bald No More’s advertising budget for TV commercials is $120,000, and it would like no more than 50% of the total number of spots to appear on “Back to You.” How many spots should you purchase on each show to reach the most homes?

**Investigations** Exercises 35 and 36 are based on the following data on four stocks.17

<table>
<thead>
<tr>
<th>Stock</th>
<th>Price</th>
<th>Dividend Yield</th>
<th>Earnings per Share</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATRI (Atrion)</td>
<td>$90</td>
<td>1%</td>
<td>$7.50</td>
</tr>
<tr>
<td>BBY (Best Buy)</td>
<td>$20</td>
<td>2%</td>
<td>3.20</td>
</tr>
<tr>
<td>IMCL (ImClone Systems)</td>
<td>$70</td>
<td>0%</td>
<td>$-0.70</td>
</tr>
<tr>
<td>TR (Tootsie Roll)</td>
<td>$25</td>
<td>1%</td>
<td>0.80</td>
</tr>
</tbody>
</table>

35. You are planning to invest up to $10,000 in ATRI and BBY shares. You want your investment to yield at least $120 in dividends and you want to maximize the total earnings. How many shares of each company should you purchase?

36. You are planning to invest up to $43,000 in IMCL and TR shares. For tax reasons, you want your investment to yield no more than $10 in dividends. You want to maximize the total earnings. How many shares of each company should you purchase?

37. Investments Your portfolio manager has suggested two high-yielding stocks: Consolidated Edison (ED) and General Electric (GE). ED shares cost $40, yield 6% in dividends, and have a risk index of 2.0 per share. GE shares cost $16, yield 7.5% in dividends, and have a risk index of 3.0 per share. You have up to $10,000 to invest, and would like to earn at least $600 in dividends. How many shares of each stock should you purchase to meet your requirements and minimize the total risk index for your portfolio? What is the minimum total risk index?

38. Investments Your friend’s portfolio manager has suggested two energy stocks: Exxon Mobil (XOM) and British Petroleum (BP). XOM shares cost $80, yield 2% in dividends, and have a risk index of 2.5. BP shares cost $50, yield 7% in dividends, and have a risk index of 4.5. Your friend has up to $40,000

19 Ibid.
40. **Planning** My friends: I, the mighty Brutus, have decided to prepare for retirement by instructing young warriors in the arts of battle and diplomacy. For each hour spent in battle instruction, I have decided to charge 50 ducats. For each hour in diplomacy instruction I shall charge 40 ducats. Due to my advancing years, I can spend no more than 50 hours per week instructing the youths, although the great Jove knows that they are sorely in need of instruction! Due to my fondness for physical pursuits, I have decided to spend no more than one-third of the total time in diplomatic instruction. However, the present border crisis with the Gauls is a sore indication of our poor abilities as diplomats. As a result, I have decided to spend at least 10 hours per week instructing in diplomacy. Finally, to complicate things further, there is the matter of Scarlet Brew: I have estimated that each hour of battle instruction will require 10 gallons of Scarlet Brew to quench my students’ thirst, and that each hour of diplomacy instruction, being less physically demanding, requires half that amount. Because my harvest of red berries has far exceeded my expectations, I estimate that I’ll have to use at least 400 gallons per week in order to avoid storing the fine brew at my expectations, I estimate that I’ll have to use at least 600 gallons per week in order to avoid storing the fine brew at great expense. Given all these restrictions, how many hours per week should I spend in each type of instruction to maximize my income?

40. **Planning** Repeat the preceding exercise with the following changes: I would like to spend no more than half the total time in diplomatic instruction, and I must use at least 600 gallons of Scarlet Brew.

41. **Resource Allocation** One day, Gillian the magician summoned the wisest of her women. “Devoted followers,” she began, “I have a quandary: As you well know, I possess great expertise in sleep spells and shock spells, but unfortunately, these are proving to be a drain on my aural energy resources; each sleep spell costs me 500 pico-shirleys of aural energy, while each shock spell requires 750 pico-shirleys. Clearly, I would like to hold my overall expenditure of aural energy to a minimum, and still meet my commitments in protecting the Sisterhood from the ever-present threat of trolls. Specifically, I have estimated that each sleep spell keeps us safe for an average of two minutes, while every shock spell protects us for about three minutes. We certainly require enough protection to last 24 hours of each day, and possibly more, just to be safe. At the same time, I have noticed that each of my sleep spells can immobilize three trolls at once, while one of my typical shock spells (having a narrower range) can immobilize only two trolls at once. We are faced, my sisters, with an onslaught of 1,200 trolls per day! Finally, as you are no doubt aware, the bylaws dictate that for a Magician of the Order to remain in good standing, the number of shock spells must be between one-quarter and one-third the number of shock and sleep spells combined. What do I do, oh Wise Ones?”

42. **Risk Management** The Grand Vizier of the Kingdom of Um is being blackmailed by numerous individuals and is having a very difficult time keeping his blackmailers from going public. He has been keeping them at bay with two kinds of payoff: gold bars from the Royal Treasury and political favors. Through bitter experience, he has learned that each payoff in gold gives him peace for an average of about 1 month, while each political favor seems to earn him about a month and a half of reprieve. To maintain his flawless reputation in the Court, he feels he cannot afford any revelations about his tainted past to come to light within the next year. Thus it is imperative that his blackmailers be kept at bay for 12 months. Furthermore, he would like to keep the number of gold payoffs at no more than one-quarter of the combined number of payoffs because the outward flow of gold bars might arouse suspicion on the part of the Royal Treasurer. The Grand Vizier feels that he can do no more than seven political favors per year without arousing undue suspicion in the Court. The gold payoffs tend to deplete his travel budget. (The treasury has been subsidizing his numerous trips to the Himalayas.) He estimates that each gold bar removed from the treasury will cost him four trips. On the other hand because the administering of political favors tends to cost him valuable travel time, he suspects that each political favor will cost him about two trips. Now, he would obviously like to keep his blackmailers silenced and lose as few trips as possible. What is he to do? How many trips will he lose in the next year?

43. **Management**<sup>20</sup> You are the service manager for a supplier of closed-circuit television systems. Your company can provide up to 160 hours per week of technical service for your customers, although the demand for technical service far exceeds this amount. As a result, you have been asked to develop a model to allocate service technicians’ time between new customers (those still covered by service contracts) and old customers (whose service contracts have expired). To ensure that new customers are satisfied with your company’s service, the sales department has instituted a policy that at least 100 hours per week be allocated to servicing new customers. At the same time, your superiors have informed you that the company expects your department to generate at least $1,200 per week in revenues. Technical service time for new customers generates an average of $10 per hour (because much of the service is still under warranty) and for old customers generates $30 per hour. How many hours per week should you allocate to each type of customer to generate the most revenue?

---

44. Scheduling21 The Scottsville Textile Mill produces several different fabrics on eight dobby looms which operate 24 hours per day and are scheduled for 30 days in the coming month. The Scottsville Textile Mill will produce only Fabric 1 and Fabric 2 during the coming month. Each dobby loom can turn out 4.63 yards of either fabric per hour. Assume that there is a monthly demand of 16,000 yards of Fabric 1 and 12,000 yards of Fabric 2. Profits are calculated as 33¢ per yard for each fabric produced on the dobby looms.

a. Will it be possible to satisfy total demand?

b. In the event that total demand is not satisfied, the Scottsville Textile Mill will need to purchase the fabrics from another mill to make up the shortfall. Its profits on resold fabrics ordered from another mill amount to 20¢ per yard for Fabric 1 and 16¢ per yard for Fabric 2. How many yards of each fabric should it produce to maximize profits?

COMMUNICATION AND REASONING EXERCISES

45. If a linear programming problem has a bounded, nonempty feasible region, then optimal solutions

(A) must exist (B) may or may not exist (C) cannot exist

46. If a linear programming problem has an unbounded, nonempty feasible region, then optimal solutions

(A) must exist (B) may or may not exist (C) cannot exist

47. What can you say if the optimal value occurs at two adjacent corner points?

48. Describe at least one drawback to using the graphical method to solve a linear programming problem arising from a real-life situation.

49. Create a linear programming problem in two variables that has no optimal solution.

50. Create a linear programming problem in two variables that has more than one optimal solution.

51. Create an interesting scenario leading to the following linear programming problem:

Maximize \( p = 10x + 10y \)

subject to

\[ 20x + 40y \leq 1,000 \]
\[ 30x + 20y \leq 1,200 \]
\[ x \geq 0, y \geq 0. \]

52. Create an interesting scenario leading to the following linear programming problem:

Minimize \( c = 10x + 10y \)

subject to

\[ 20x + 40y \geq 1,000 \]
\[ 30x + 20y \geq 1,200 \]
\[ x \geq 0, y \geq 0. \]

53. Use an example to show why there may be no optimal solution to a linear programming problem if the feasible region is unbounded.

54. Use an example to illustrate why, in the event that an optimal solution does occur despite an unbounded feasible region, that solution corresponds to a corner point of the feasible region.

55. You are setting up an LP problem for Fly-by-Night Airlines with the unknowns \( x \) and \( y \), where \( x \) represents the number of first-class tickets it should issue for a specific flight and \( y \) represents the number of business-class tickets it should issue for that flight, and the problem is to maximize profit. You find that there are two different corner points that maximize the profit. How do you interpret this?

56. In the situation described in the preceding exercise, you find that there are no optimal solutions. How do you interpret this?

57. Consider the following example of a nonlinear programming problem: Maximize \( p = xy \) subject to \( x \geq 0, y \geq 0, x + y \leq 2 \). Show that \( p \) is zero on every corner point, but is greater than zero at many noncorner points.

58. Solve the nonlinear programming problem in Exercise 57.


4.3 The Simplex Method: Solving Standard Maximization Problems

The method discussed in Section 4.2 works quite well for LP problems in two unknowns, but what about three or more unknowns? Because we need an axis for each unknown, we would need to draw graphs in three dimensions (where we have \( x \)-, \( y \)-, and \( z \)-coordinates) to deal with problems in three unknowns, and we would have to draw in hyperspace to answer questions involving four or more unknowns. Given the state of technology as this book is being written, we can’t easily do this. So we need another method for solving
LP problems that will work for any number of unknowns. One such method, called the simplex method, has been the method of choice since it was invented by George Dantzig in 1947. To illustrate it best, we first use it to solve only so-called standard maximization problems.

**General Linear Programming (LP) Problem**

A linear program problem in \( n \) unknowns \( x_1, x_2, \ldots, x_n \) is one in which we are to find the maximum or minimum value of a linear objective function

\[
a_1 x_1 + a_2 x_2 + \cdots + a_n x_n,
\]

where \( a_1, a_2, \ldots, a_n \) are numbers, subject to a number of linear constraints of the form

\[
b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \leq c \quad \text{or} \quad b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \geq c,
\]

where \( b_1, b_2, \ldots, b_n, c \) are numbers.

**Standard Maximization Problem**

A standard maximization problem is an LP problem in which we are required to maximize (not minimize) an objective function of the form

\[
p = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n
\]

subject to the constraints

\[
x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.
\]

and further constraints of the form

\[
b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \leq c
\]

with \( c \) non-negative. It is important that the inequality here be \( \leq \), not \( = \) or \( \geq \).

**Note** As in the chapter on linear equations, we will almost always use \( x, y, z, \ldots \) for the unknowns. Subscripted variables \( x_1, x_2, \ldots \) are very useful names when you start running out of letters of the alphabet, but we should not find ourselves in that predicament.

**Quick Examples**

1. Maximize

\[
p = 2x - 3y + 3z
\]

subject to

\[
2x + z \leq 7
\]

\[
-x + 3y - 6z \leq 6
\]

\[
x \geq 0, y \geq 0, z \geq 0.
\]

This is a standard maximization problem.

2. Maximize

\[
p = 2x_1 + x_2 - x_3 + x_4
\]

subject to

\[
x_1 - 2x_2 + x_4 \leq 0
\]

\[
3x_1 \leq 1
\]

\[
x_2 + x_3 \leq 2
\]

\[
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.
\]

This is a standard maximization problem.
The idea behind the simplex method is this: In any linear programming problem, there is a feasible region. If there are only two unknowns, we can draw the region; if there are three unknowns, it is a solid region in space; and if there are four or more unknowns, it is an abstract higher-dimensional region. But it is a faceted region with corners (think of a diamond), and it is at one of these corners that we will find the optimal solution. Geometrically, what the simplex method does is to start at the corner where all the unknowns are 0 (possible because we are talking of standard maximization problems) and then walk around the region, from corner to adjacent corner, always increasing the value of the objective function, until the best corner is found. In practice, we will visit only a small number of the corners before finding the right one. Algebraically, as we are about to see, this walking around is accomplished by matrix manipulations of the same sort as those used in the chapter on systems of linear equations.

We describe the method while working through an example.

### Example 1 Meet the Simplex Method

Maximize \( p = 3x + 2y + z \)

subject to

\[
\begin{align*}
2x + 2y + z &\leq 10 \\
x + 2y + 3z &\leq 15 \\
x &\geq 0, \ y \geq 0, \ z \geq 0.
\end{align*}
\]

The inequality \( 2x + z \geq 7 \) cannot be written in the required form. If we reverse the inequality by multiplying both sides by \(-1\), we get \(-2x - z \leq -7\), but a negative value on the right side is not allowed.

The idea behind the simplex method is this: In any linear programming problem, there is a feasible region. If there are only two unknowns, we can draw the region; if there are three unknowns, it is a solid region in space; and if there are four or more unknowns, it is an abstract higher-dimensional region. But it is a faceted region with corners (think of a diamond), and it is at one of these corners that we will find the optimal solution. Geometrically, what the simplex method does is to start at the corner where all the unknowns are 0 (possible because we are talking of standard maximization problems) and then walk around the region, from corner to adjacent corner, always increasing the value of the objective function, until the best corner is found. In practice, we will visit only a small number of the corners before finding the right one. Algebraically, as we are about to see, this walking around is accomplished by matrix manipulations of the same sort as those used in the chapter on systems of linear equations.

We describe the method while working through an example.

**Example 1 Meet the Simplex Method**

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subject to

\[
\begin{align*}
2x + 2y + z &\leq 10 \\
x + 2y + 3z &\leq 15 \\
x &\geq 0, \ y \geq 0, \ z \geq 0.
\end{align*}
\]

**Solution**

**Step 1 Convert to a system of linear equations.** The inequalities \( 2x + 2y + z \leq 10 \) and \( x + 2y + 3z \leq 15 \) are less convenient than equations. Look at the first inequality. It says that the left-hand side, \( 2x + 2y + z \), must have some positive number (or zero) added to it if it is to equal 10. Because we don’t yet know what \( x, y, \) and \( z \) are, we are not yet sure what number to add to the left-hand side. So we invent a new unknown, \( s \geq 0 \), called a slack variable, to “take up the slack,” so that

\[
2x + 2y + z + s = 10.
\]

Turning to the next inequality, \( x + 2y + 3z \leq 15 \), we now add a slack variable to its left-hand side, to get it up to the value of the right-hand side. We might have to add a different number than we did the last time, so we use a new slack variable, \( t \geq 0 \), and obtain

\[
x + 2y + 3z + t = 15. \quad \text{Use a different slack variable for each constraint.}
\]
Now we write the system of equations we have (including the one that defines the objective function) in standard form.

\[
\begin{align*}
2x + 2y + z + s &= 10 \\
x + 2y + 3z + t &= 15 \\
-3x - 2y - z + p &= 0
\end{align*}
\]

Note three things: First, all the variables are neatly aligned in columns, as they were in Chapter 2. Second, in rewriting the objective function \( p = 3x + 2y + z \), we have left the coefficient of \( p \) as +1 and brought the other variables over to the same side of the equation as \( p \). This will be our standard procedure from now on. Don’t write \( 3x + 2y + z - p = 0 \) (even though it means the same thing) because the negative coefficients will be important in later steps. Third, the above system of equations has fewer equations than unknowns, and hence cannot have a unique solution.

**Step 2 Set up the initial tableau.** We represent our system of equations by the following table (which is simply the augmented matrix in disguise), called the *initial tableau*:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( s )</th>
<th>( t )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The labels along the top keep track of which columns belong to which variables.

Now notice a peculiar thing. If we rewrite the matrix using the variables \( s \), \( t \), and \( p \) first, we get the following matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 2 & 2 & 1 & 10 \\
0 & 1 & 0 & 1 & 2 & 3 & 15 \\
0 & 0 & 1 & -3 & -2 & -1 & 0
\end{bmatrix}
\]

which is already in reduced form. We can therefore read off the general solution (see Section 2.2) to our system of equations as

\[
\begin{align*}
& s = 10 - 2x - 2y - z \\
& t = 15 - x - 2y - 3z \\
& p = 0 + 3x + 2y + z \\
& x, y, z \text{ arbitrary.}
\end{align*}
\]

Thus, we get a whole family of solutions, one for each choice of \( x \), \( y \), and \( z \). One possible choice is to set \( x \), \( y \), and \( z \) all equal to 0. This gives the particular solution

\[
\begin{align*}
& s = 10, \\
& t = 15, \\
& p = 0, \\
& x = 0, y = 0, z = 0. \\
& \text{Set } x = y = z = 0 \text{ above.}
\end{align*}
\]

This solution is called the *basic solution* associated with the tableau. The variables \( s \) and \( t \) are called the *active* variables, and \( x \), \( y \), and \( z \) are the *inactive* variables. (Other terms used are basic and nonbasic variables.)

We can obtain the basic solution directly from the tableau as follows.

- The active variables correspond to the cleared columns (columns with only one nonzero entry).
- The values of the active variables are calculated as shown following.
- All other variables are inactive, and set equal to zero.
This basic solution represents our starting position $x = y = z = 0$ in the feasible region in $xyz$ space.

We now need to move to another corner point. To do so, we choose a pivot in one of the first three columns of the tableau and clear its column. Then we will get a different basic solution, which corresponds to another corner point. Thus, in order to move from corner point to corner point, all we have to do is choose suitable pivots and clear columns in the usual manner.

The next two steps give the procedure for choosing the pivot.

**Step 3 Select the pivot column** (the column that contains the pivot we are seeking).

### Selecting the Pivot Column

Choose the negative number with the largest magnitude on the left-hand side of the bottom row (that is, don’t consider the last number in the bottom row). Its column is the pivot column. (If there are two or more candidates, choose any one.) If all the numbers on the left-hand side of the bottom row are zero or positive, then we are done, and the basic solution is the optimal solution.

Simple enough. The most negative number in the bottom row is $-3$, so we choose the $x$ column as the pivot column:

As an additional aid to recognizing which variables are active and which are inactive, we label each row with the name of the corresponding active variable. Thus, the complete initial tableau looks like this.

<table>
<thead>
<tr>
<th>Inactive</th>
<th>Inactive</th>
<th>Inactive</th>
<th>Active</th>
<th>Active</th>
<th>Active</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>$y = 0$</td>
<td>$z = 0$</td>
<td>$s = \frac{10}{7}$</td>
<td>$t = \frac{15}{7}$</td>
<td>$p = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$s$</th>
<th>$t$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>-3</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

This basic solution represents our starting position $x = y = z = 0$ in the feasible region in $xyz$ space.

*NOTE* Also see Section 2.2 for a discussion of pivots and pivoting.
because \( p \) would increase by 3 units for every 1-unit increase in \( x \). (If we chose to increase \( y \), then \( p \) would increase by only 2 units for every 1-unit increase in \( y \), and if we increased \( z \) instead, \( p \) would grow even more slowly.) In short, choosing the pivot column this way makes it likely that we’ll increase \( p \) as much as possible.

### Step 4  Select the pivot in the pivot column.

#### Selecting the Pivot

1. The pivot must always be a positive number. (This rules out zeros and negative numbers, such as the \(-3\) in the bottom row.)
2. For each positive entry \( b \) in the pivot column, compute the ratio \( a/b \), where \( a \) is the number in the rightmost column in that row. We call this a test ratio.
3. Of these ratios, choose the smallest one. (If there are two or more candidates, choose any one.) The corresponding number \( b \) is the pivot.

In our example, the test ratio in the first row is \( 10/2 = 5 \), and the test ratio in the second row is \( 15/1 = 15 \). Here, 5 is the smallest, so the 2 in the upper left is our pivot.

#### Why select the pivot this way?

The rule given above guarantees that, after pivoting, all variables will be non-negative in the basic solution. In other words, it guarantees that we will remain in the feasible region. We will explain further after finishing this example.

### Step 5  Use the pivot to clear the column in the normal manner and then relabel the pivot row with the label from the pivot column.

It is important to follow the exact prescription described in Section 2.2 for formulating the row operations:

\[
a R \rightarrow b R, \quad a \text{ and } b \text{ both positive}
\]

Row to change  
\[ \uparrow \quad \uparrow \]  
Pivot row

All entries in the last column should remain non-negative after pivoting. Furthermore, because the \( x \) column (and no longer the \( s \) column) will be cleared, \( x \) will become an active variable. In other words, the \( s \) on the left of the pivot will be replaced by \( x \). We call \( s \) the departing, or exiting variable and \( x \) the entering variable for this step.
This gives

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t</td>
<td>0</td>
<td>2</td>
<td>5</td>
<td>−1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

This is the second tableau.

**Step 6 Go to Step 3.** But wait! According to Step 3, we are finished because there are no negative numbers in the bottom row. Thus, we can read off the answer. Remember, though, that the solution for \( x \), the first active variable, is not just \( x = 10 \), but is \( x = 10/2 = 5 \) because the pivot has not been reduced to a 1. Similarly, \( t = 20/2 = 10 \) and \( p = 30/2 = 15 \). All the other variables are zero because they are inactive. Thus, the solution is as follows: \( p \) has a maximum value of 15, and this occurs when \( x = 5, y = 0 \) and \( z = 0 \). (The slack variables then have the values \( s = 0 \) and \( t = 10 \).)

**Q:** Why can we stop when there are no negative numbers in the bottom row? Why does this tableau give an optimal solution?

**A:** The bottom row corresponds to the equation \( 2y + z + 3s + 2p = 30 \), or

\[
p = 15 - y - \frac{1}{2}z - \frac{3}{2}s.
\]

Think of this as part of the general solution to our original system of equations, with \( y, z, \) and \( s \) as the parameters. Because these variables must be non-negative, the largest possible value of \( p \) in any feasible solution of the system comes when all three of the parameters are 0. Thus, the current basic solution must be an optimal solution.

**NOTE** Calculators or spreadsheets could obviously be a big help in the calculations here, just as in Chapter 2. We’ll say more about that after the next couple of examples.

We owe some further explanation for Step 4 of the simplex method. After Step 3, we knew that \( x \) would be the entering variable, and we needed to choose the departing variable. In the next basic solution, \( x \) was to have some positive value and we wanted this value to be as large as possible (to make \( p \) as large as possible) without making any other variables negative. Look again at the equations written in Step 2:

\[
s = 10 - 2x - 2y - z \\
t = 15 - x - 2y - 3z.
\]

We needed to make either \( s \) or \( t \) into an inactive variable and hence zero. Also, \( y \) and \( z \) were to remain inactive. If we had made \( s \) inactive, then we would have had \( 0 = 10 - 2x \), so \( x = 10/2 = 5 \). This would have made \( t = 15 - 5 = 10 \), which would be fine. On the other hand, if we had made \( t \) inactive, then we would have had \( 0 = 15 - x \), so \( x = 15 \), and this would have made \( s = 10 - 2 \cdot 15 = -20 \), which would not be fine, because slack variables must be non-negative. In other words, we had a choice of making \( x = 10/2 = 5 \) or \( x = 15/1 = 15 \), but making \( x \) larger than 5 would have made another variable negative. We were thus compelled to choose the smaller ratio, 5, and make \( s \) the departing variable. Of course, we do not have to think it through this way every time. We just use the rule stated in Step 4. (For a graphical explanation, see Example 3.)
EXAMPLE 2 Simplex Method

Find the maximum value of $p = 12x + 15y + 5z$, subject to the constraints:

\[
\begin{align*}
2x + 2y + z & \leq 8 \\
x + 4y - 3z & \leq 12 \\
x & \geq 0, \ y & \geq 0, \ z & \geq 0.
\end{align*}
\]

Solution

Following Step 1, we introduce slack variables and rewrite the constraints and objective function in standard form:

\[
\begin{align*}
2x + 2y + z + s & = 8 \\
x + 4y - 3z + t & = 12 \\
-12x - 15y - 5z + p & = 0.
\end{align*}
\]

We now follow with Step 2, setting up the initial tableau:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t</td>
<td>1</td>
<td>4</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>-12</td>
<td>-15</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

For Step 3, we select the column over the negative number with the largest magnitude in the bottom row, which is the $y$ column. For Step 4, finding the pivot, we see that the test ratios are $8/2$ and $12/4$, the smallest being $12/4 = 3$. So we select the pivot in the $t$ row and clear its column:

\[
\begin{align*}
2R_1 - R_2 \\
4R_3 + 15R_2
\end{align*}
\]

The departing variable is $t$ and the entering variable is $y$. This gives the second tableau.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t</td>
<td>1</td>
<td>4</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>-12</td>
<td>-15</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

We now go back to Step 3. Because we still have negative numbers in the bottom row, we choose the one with the largest magnitude (which is $-65$), and thus our pivot column is the $z$ column. Because negative numbers can’t be pivots, the only possible choice for the pivot is the $5$. (We need not compute the test ratios because there would only be one from which to choose.) We now clear this column, remembering to take care of the departing and entering variables.

\[
\begin{align*}
5R_2 + 3R_1 \\
R_3 + 13R_1
\end{align*}
\]
Notice how the value of \( p \) keeps climbing: It started at 0 in the first tableau, went up to \( \frac{180}{4} = 45 \) in the second, and is currently at \( \frac{232}{4} = 58 \). Because there are no more negative numbers in the bottom row, we are done and can write down the solution: \( p \) has a maximum value of \( \frac{232}{4} = 58 \), and this occurs when
\[
\begin{align*}
  x &= 0 \\
  y &= \frac{72}{20} = \frac{18}{5} \quad \text{and} \\
  z &= \frac{4}{5}.
\end{align*}
\]
The slack variables are both zero.

As a partial check on our answer, we can substitute these values into the objective function and the constraints:
\[
\begin{align*}
  58 &= 12(0) + 15\left(\frac{18}{5}\right) + 5\left(\frac{4}{5}\right) \quad \checkmark \\
  2(0) + 2\left(\frac{18}{5}\right) + \left(\frac{4}{5}\right) &= 8 \leq 8 \quad \checkmark \\
  0 + 4\left(\frac{18}{5}\right) - 3\left(\frac{4}{5}\right) &= 12 \leq 12. \quad \checkmark
\end{align*}
\]
We say that this is only a partial check, because it shows only that our solution is feasible, and that we have correctly calculated \( p \). It does not show that we have the optimal solution. This check will usually catch any arithmetic mistakes we make, but it is not foolproof.

**APPLICATIONS**

In the next example (further exploits of Acme Baby Foods—compare Example 2 in Section 2) we show how the simplex method relates to the graphical method.

**EXAMPLE 3 Resource Allocation**

Acme Baby Foods makes two puddings, vanilla and chocolate. Each serving of vanilla pudding requires 2 teaspoons of sugar and 25 fluid ounces of water, and each serving of chocolate pudding requires 3 teaspoons of sugar and 15 fluid ounces of water. Acme has available each day 3,600 teaspoons of sugar and 22,500 fluid ounces of water. Acme makes no more than 600 servings of vanilla pudding because that is all that it can sell each day. If Acme makes a profit of 10¢ on each serving of vanilla pudding and 7¢ on each serving of chocolate, how many servings of each should it make to maximize its profit?

**Solution** We first identify the unknowns. Let
\[
\begin{align*}
  x &= \text{the number of servings of vanilla pudding} \\
  y &= \text{the number of servings of chocolate pudding}.
\end{align*}
\]
The objective function is the profit \( p = 10x + 7y \), which we need to maximize. For the constraints, we start with the fact that Acme will make no more than 600 servings of vanilla: \( x \leq 600 \). We can put the remaining data in a table as follows:

<table>
<thead>
<tr>
<th></th>
<th>Vanilla</th>
<th>Chocolate</th>
<th>Total Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sugar (teaspoons)</td>
<td>2</td>
<td>3</td>
<td>3,600</td>
</tr>
<tr>
<td>Water (ounces)</td>
<td>25</td>
<td>15</td>
<td>22,500</td>
</tr>
</tbody>
</table>

Because Acme can use no more sugar and water than is available, we get the two constraints:

\[
2x + 3y \leq 3,600 \\
25x + 15y \leq 22,500.
\]

Note that all the terms are divisible by 5.

Thus our linear programming problem is this:

Maximize \( p = 10x + 7y \)
subject to
\[
\begin{align*}
x &\leq 600 \\
2x + 3y &\leq 3,600 \\
5x + 3y &\leq 4,500 \\
x &\geq 0, \ y &\geq 0.
\end{align*}
\]

Next, we introduce the slack variables and set up the initial tableau.

\[
\begin{align*}
x + s &= 600 \\
2x + 3y + t &= 3,600 \\
5x + 3y + u &= 4,500 \\
-10x - 7y + p &= 0
\end{align*}
\]

Note that we have had to introduce a third slack variable, \( u \). There need to be as many slack variables as there are constraints (other than those of the \( x \geq 0 \) variety).

**Q** What do the slack variables say about Acme puddings?

**A** The first slack variable, \( s \), represents the number you must add to the number of servings of vanilla pudding actually made to obtain the maximum of 600 servings. The second slack variable, \( t \), represents the amount of sugar that is left over once the puddings are made, and \( u \) represents the amount of water left over.

We now use the simplex method to solve the problem:

\[
\begin{array}{ccccccc}
 x & y & s & t & u & p \\
\hline
 s & \boxed{1} & 0 & 1 & 0 & 0 & 0 & 600 \\
 t & 2 & 3 & 0 & 1 & 0 & 0 & 3,600 \quad R_2 - 2R_1 \\
 u & 5 & 3 & 0 & 0 & 1 & 0 & 4,500 \quad R_3 - 5R_1 \\
 p & -10 & -7 & 0 & 0 & 0 & 1 & 0 \quad R_4 + 10R_1 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
 x & y & s & t & u & p \\
\hline
 x & 1 & 0 & 1 & 0 & 0 & 0 & 600 \\
 t & 0 & 3 & -2 & 1 & 0 & 0 & 2,400 \quad R_2 - R_3 \\
 u & 0 & \boxed{4} & -5 & 0 & 1 & 0 & 1,500 \quad 3R_4 + 7R_3 \\
 p & 0 & -7 & 10 & 0 & 0 & 1 & 6,000 \\
\end{array}
\]
Before we go on... Because the problem in Example 3 had only two variables, we could have solved it graphically. It is interesting to think about the relationship between the two methods. Figure 22 shows the feasible region. Each tableau in the simplex method corresponds to a corner of the feasible region, given by the corresponding basic solution. In this example, the sequence of basic solutions is 

\[(x, y) = (0, 0), (600, 0), (600, 500), (300, 1,000)\].

This is the sequence of corners shown in Figure 23. In general, we can think of the simplex method as walking from corner to corner of the feasible region, until it locates the optimal solution. In problems with many variables and many constraints, the simplex method usually visits only a small fraction of the total number of corners.

We can also explain again, in a different way, the reason we use the test ratios when choosing the pivot. For example, when choosing the first pivot we had to choose among the test ratios 600, 1,800, and 900 (look at the first tableau). In Figure 22, you can see that those are the three x-intercepts of the lines that bound the feasible region. If we had chosen 1,800 or 900, we would have jumped along the x-axis to a point outside of the feasible region, which we do not want to do. In general, the test ratios measure the distance from the current corner to the constraint lines, and we must choose the smallest such distance to avoid crossing any of them into the unfeasible region.

It is also interesting in an application like this to think about the values of the slack variables. We said above that \(s\) is the difference between the maximum 600 servings of vanilla that might be made and the number that is actually made. In the optimal solution, \(s = 300\), which says that 300 fewer servings of vanilla were made than the maximum possible. Similarly, \(t\) was the amount of sugar left over. In the optimal solution, \(t = 0\), which tells us that all of the available sugar is used. Finally, \(u = 0\), so all of the available water is used as well.
Summary: The Simplex Method for Standard Maximization Problems

To solve a standard maximization problem using the simplex method, we take the following steps:

1. Convert to a system of equations by introducing slack variables to turn the constraints into equations and by rewriting the objective function in standard form.

2. Write down the initial tableau.

3. Select the pivot column: Choose the negative number with the largest magnitude in the left-hand side of the bottom row. Its column is the pivot column. (If there are two or more candidates, choose any one.) If all the numbers in the left-hand side of the bottom row are zero or positive, then we are finished, and the basic solution maximizes the objective function. (See below for the basic solution.)

4. Select the pivot in the pivot column: The pivot must always be a positive number. For each positive entry $b$ in the pivot column, compute the ratio $a/b$, where $a$ is the number in the last column in that row. Of these test ratios, choose the smallest one. (If there are two or more candidates, choose any one.) The corresponding number $b$ is the pivot.

5. Use the pivot to clear the column in the normal manner (taking care to follow the exact prescription for formulating the row operations described in Chapter 2) and then relabel the pivot row with the label from the pivot column. The variable originally labeling the pivot row is the departing, or exiting, variable, and the variable labeling the column is the entering variable.

6. Go to step 3.

To get the basic solution corresponding to any tableau in the simplex method, set to zero all variables that do not appear as row labels. The value of a variable that does appear as a row label (an active variable) is the number in the rightmost column in that row divided by the number in that row in the column labeled by the same variable.

Troubleshooting the Simplex Method

Q: What if there is no candidate for the pivot in the pivot column? For example, what do we do with a tableau like the following?

<table>
<thead>
<tr>
<th></th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
<th>$s$</th>
<th>$t$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y$</td>
<td>-8</td>
<td>20</td>
<td>0</td>
<td>6</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$p$</td>
<td>-20</td>
<td>0</td>
<td>0</td>
<td>26</td>
<td>15</td>
<td>4</td>
</tr>
</tbody>
</table>

A: Here, the pivot column is the $x$ column, but there is no suitable entry for a pivot (because zeros and negative numbers can’t be pivots). This happens when the feasible region is unbounded and there is also no optimal solution. In other words, $p$ can be made as large as we like without violating the constraints.
A negative number will not appear above the bottom row in the rightmost column if we follow the procedure correctly. (The bottom right entry is allowed to be negative if the objective takes on negative values as in a negative profit, or loss.) Following are the most likely errors leading to this situation:

- The pivot was chosen incorrectly. (Don’t forget to choose the smallest test ratio.) When this mistake is made, one or more of the variables will be negative in the corresponding basic solution.
- The row operation instruction was written backwards or performed backwards (for example, instead of \( R_2 - R_1 \), it was \( R_1 - R_2 \)). This mistake can be corrected by multiplying the row by \(-1\).
- An arithmetic error occurred. (We all make those annoying errors from time to time.)

Zeros are permissible in the rightmost column. For example, the constraint \( x - y \leq 0 \) will lead to a zero in the rightmost column.

What happens if we choose a pivot column other than the one with the most negative number in the bottom row?

There is no harm in doing this as long as we choose the pivot in that column using the smallest test ratio. All it might do is slow the whole calculation by adding extra steps.

One last suggestion: If it is possible to do a simplification step (dividing a row by a positive number) at any stage, we should do so. As we saw in Chapter 2, this can help prevent the numbers from getting out of hand.

### 4.3 Exercises

#### More advanced

1. Maximize \( p = 2x + y \)
   subject to \( x + 2y \leq 6 \)
   \( x + y \leq 4 \)
   \( x, y \geq 0 \).

   **HINT** [See Examples 1 and 2]

2. Maximize \( p = x \)
   subject to \( x - y \leq 4 \)
   \( -x + 3y \leq 4 \)
   \( x, y \geq 0 \).

   **HINT** [See Examples 1 and 2]

3. Maximize \( p = x - y \)
   subject to \( 5x - 5y \leq 20 \)
   \( 2x - 10y \leq 40 \)
   \( x, y \geq 0 \).

4. Maximize \( p = 2x + 3y \)
   subject to \( 3x + 8y \leq 24 \)
   \( 6x + 4y \leq 30 \)
   \( x, y \geq 0 \).

5. Maximize \( p = 5x - 4y + 3z \)
   subject to \( 5x + 5z \leq 100 \)
   \( 5y - 5z \leq 50 \)
   \( 5x - 5y \leq 50 \)
   \( x, y, z \geq 0 \).

6. Maximize \( p = 6x + y + 3z \)
   subject to \( 3x + y \leq 15 \)
   \( 2x + 2y + 2z \leq 20 \)
   \( x, y, z \geq 0 \).

7. Maximize \( p = 7x + 5y + 6z \)
   subject to \( x + y - z \leq 3 \)
   \( x + 2y + z \leq 8 \)
   \( x, y, z \geq 0 \).
Maximize 

\[ p = 3x + 4y + 2z \]

subject to

\[ 3x + y + z \leq 5 \]
\[ x + 2y + z \leq 5 \]
\[ x + y + z \leq 4 \]
\[ x \geq 0, y \geq 0, z \geq 0. \]

Maximize 

\[ p = 3x + 7x_2 + 8x_3 \]

subject to

\[ 5x_1 - x_2 + x_3 \leq 1,500 \]
\[ 2x_1 + 2x_2 + x_3 \leq 2,500 \]
\[ 4x_1 + 2x_2 + x_3 \leq 2,000 \]
\[ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \]

Maximize 

\[ p = x + y + z + w \]

subject to

\[ x + y + z + w \leq 3 \]
\[ y + z + w \leq 4 \]
\[ x + z + w \leq 5 \]
\[ x + y + w \leq 6 \]
\[ x \geq 0, y \geq 0, z \geq 0, w \geq 0. \]

Maximize 

\[ p = x - y + z + w \]

subject to

\[ x + y + z \leq 3 \]
\[ y + z + w \leq 3 \]
\[ x + z + w \leq 4 \]
\[ x + y + w \leq 4 \]
\[ x \geq 0, y \geq 0, z \geq 0, w \geq 0. \]

Maximize 

\[ p = x + y + z + w + v \]

subject to

\[ x + y \leq 1 \]
\[ y + z \leq 2 \]
\[ z + w \leq 3 \]
\[ w + v \leq 4 \]
\[ x \geq 0, y \geq 0, z \geq 0, w \geq 0, v \geq 0. \]

Maximize 

\[ p = x + 2y + z + 2w + v \]

subject to

\[ x + y \leq 1 \]
\[ y + z \leq 2 \]
\[ z + w \leq 3 \]
\[ w + v \leq 4 \]
\[ x \geq 0, y \geq 0, z \geq 0, w \geq 0, v \geq 0. \]

\textbf{APPLICATIONS}

15. \textbf{Purchasing} You are in charge of purchases at the student-run used-book supply program at your college, and you must decide how many introductory calculus, history, and marketing texts should be purchased from students for resale. Due to budget limitations, you cannot purchase more than 650 of these textbooks each semester. There are also shelf-space limitations: Calculus texts occupy 2 units of shelf space each, history books 1 unit each, and marketing texts 3 units each, and you can spare at most 1,000 units of shelf space for the texts. If the used book program makes a profit of $10 on each calculus text, $4 on each history text, and $8 on each marketing text, how many of each type of text should you purchase to maximize profit? What is the maximum profit the program can make in a semester? \textbf{Hint} [See Example 3]?

16. \textbf{Sales} The Marketing Club at your college has decided to raise funds by selling three types of T-shirt: one with a single-color “ordinary” design, one with a two-color “fancy” design, and one with a three-color “very fancy” design. The club feels that
it can sell up to 300 T-shirts. “Ordinary” T-shirts will cost the club $6 each, “fancy” T-shirts $8 each, and “very fancy” T-shirts $10 each, and the club has a total purchasing budget of $3,000. It will sell “ordinary” T-shirts at a profit of $4 each, “fancy” T-shirts at a profit of $5 each, and “very fancy” T-shirts at a profit of $4 each. How many of each kind of T-shirt should the club order to maximize profit? What is the maximum profit the club can make? HINT [See Example 3.]

23. Resource Allocation Arctic Juice Company makes three juice blends: PineOrange, using 2 portions of pineapple juice and 2 portions of orange juice per gallon; PineKiwi, using 3 portions of pineapple juice and 1 portion of kiwi juice per gallon; and OrangeKiwi, using 3 portions of orange juice and 1 portion of kiwi juice per gallon. Each day the company has 800 portions of pineapple juice, 650 portions of orange juice, and 350 portions of kiwi juice available. Its profit on PineOrange is $1 per gallon, its profit on PineKiwi is $2 per gallon, and its profit on OrangeKiwi is $1 per gallon. How many gallons of each blend should it make each day to maximize profit? What is the largest possible profit the company can make?

24. Purchasing Trans Global Tractor Trailers has decided to spend up to $1,500,000 on a fleet of new trucks, and it is considering three models: the Gigahaul, which has a capacity of 6,000 cubic feet and is priced at $60,000; the Megahaul, with a capacity of 5,000 cubic feet and priced at $50,000; and the Picohaul, with a capacity of 2,000 cubic feet, priced at $40,000. The anticipated annual revenues are $500,000 for each new truck purchased (regardless of size). Trans Global would like a total capacity of up to 130,000 cubic feet, and feels that it cannot provide drivers and maintenance for more than 30 trucks. How many of each should it purchase to maximize annual revenue? What is the largest possible revenue it can make?

25. Resource Allocation The Enormous State University History Department offers three courses, Ancient, Medieval, and Modern History, and the department chairperson is trying to decide how many sections of each to offer this semester. They may offer up to 45 sections total, up to 5,000 students would like to take a course, and there are 60 professors to teach them (no student will take more than one course, and no professor will teach more than one section). Sections of Ancient History have 100 students each, sections of Medieval History have 50 students each, and sections of Modern History have 200 students each. Modern History sections are taught by a team of two professors, while Ancient and Medieval History need only one professor per section. Ancient History nets the university $10,000 per section, Medieval nets $20,000, and Modern History nets $30,000 per section. How many sections of each course should the department offer in order to generate the largest profit? What is the largest possible profit? Will there be any unused time slots, any students who did not get into classes, or any professors without anything to teach?

26. Resource Allocation You manage an ice cream factory that makes three flavors: Creamy Vanilla, Continental Mocha, and Succulent Strawberry. Into each batch of Creamy Vanilla go 2 eggs, 1 cup of milk, and 2 cups of cream. Into each batch of Continental Mocha go 1 egg, 1 cup of milk, and 2 cups of cream. Into each batch of Succulent Strawberry go 1 egg, 2 cups of milk, and 2 cups of cream. You have in stock 200 eggs, 120 cups of milk, and 200 cups of cream. You make a profit of $3 on each batch of Creamy Vanilla, $2 on each batch of Continental Mocha and $4 on each batch of Succulent Strawberry.

a. How many batches of each flavor should you make to maximize your profit?
b. In your answer to part (a), have you used all the ingredients?
c. Due to the poor strawberry harvest this year, you cannot make more than 10 batches of Succulent Strawberry. Does this affect your maximum profit?

27. Agriculture Your small farm encompasses 100 acres, and you are planning to grow tomatoes, lettuce, and carrots in the coming planting season. Fertilizer costs per acre are: $5 for tomatoes, $4 for lettuce, and $2 for carrots. Based on past experience, you estimate that each acre of tomatoes will require an average of 4 hours of labor per week, while tending to lettuce and carrots will each require an average of 2 hours per week. You estimate a profit of $2,000 for each acre of tomatoes, $1,500 for each acre of lettuce, and $500 for each acre of carrots. You can afford to spend no more than $400 on fertilizer, and your farm laborers can supply up to 500 hours per week. How many acres of each crop should you plant to maximize total profits? In this event, will you be using all 100 acres of your farm?

28. Agriculture Your farm encompasses 500 acres, and you are planning to grow soybeans, corn, and wheat in the coming planting season. Fertilizer costs per acre are: $5 for soybeans, $2 for corn, and $1 for wheat. You estimate that each acre of soybeans will require an average of 5 hours of labor per week, while tending to corn and wheat will each require an average of 2 hours per week. Based on past yields and current market prices, you estimate a profit of $3,000 for each acre of soybeans, $2,000 for each acre of corn, and $1,000 for each acre of wheat. You can afford to spend no more than $3,000 on fertilizer, and your farm laborers can supply 3,000 hours per week. How many acres of each crop should you plant to maximize total profits? In this event, will you be using all the available labor?

29. Resource Allocation (Note that the following exercise is very similar to Exercise 36 on page 169, except for one important detail. Refer back to your solution of that problem—if you did it—and then attempt this one.) The Enormous State University Choral Society is planning its annual Song Festival, when it will serve three kinds of delicacies: granola treats, nutty granola treats, and nuttiest granola treats. The following table shows some of the ingredients required (in ounces) for a single serving of each delicacy, as well as the total amount of each ingredient available.
The society makes a profit of $6 on each serving of granola, $8 on each serving of nutty granola, and $3 on each serving of nuttiest granola. Assuming that the Choral Society can sell all that they make, how many servings of each will maximize profits? How much of each ingredient will be left over?

30. Resource Allocation
Repeat the preceding exercise, but this time assume that the Choral Society makes a $3 profit on each of its delicacies.

31. Recycling
Safety-Kleen operates the world’s largest oil refinery at Elgin, Illinois. You have been hired by the company to determine how to allocate its intake of up to 50 million gallons of used oil to its three refinery processes: A, B, and C. You are told that electricity costs for process A amount to $150,000 per million gallons treated, while for processes B and C, the costs are respectively $100,000 and $50,000 per million gallons treated. Process A can recover 60 percent of the used oil, process B can recover 55 percent, and process C can recover only 50 percent. Assuming a revenue of $4 million per million gallons of recovered oil and an annual electrical budget of $3 million, how much used oil would you allocate to each process in order to maximize total revenues?22

32. Recycling
Repeat the preceding exercise, but this time assume that process C can handle only up to 20 million gallons per year.

Creatine Supplements
Exercises 33 and 34 are based on the following data on four popular bodybuilding supplements. (Figures shown correspond to a single serving.)23

<table>
<thead>
<tr>
<th>Creatine (g)</th>
<th>Carbohydrates (g)</th>
<th>Taurine (g)</th>
<th>Alpha Lipoic Acid (mg)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cell-Tech® (MuscleTech)</strong></td>
<td>10</td>
<td>75</td>
<td>2</td>
</tr>
<tr>
<td><strong>RiboForce HP® (EAS)</strong></td>
<td>5</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td><strong>Creatine Transport® (Kaizen)</strong></td>
<td>5</td>
<td>35</td>
<td>1</td>
</tr>
<tr>
<td><strong>Pre-Load Creatine (Optimum)</strong></td>
<td>6</td>
<td>35</td>
<td>1</td>
</tr>
</tbody>
</table>

33. You are thinking of combining the first three supplements in the previous table to obtain a 10-day supply that gives you the maximum possible amount of creatine, but no more than 1,000 milligrams of alpha lipoic acid and 225 grams of carbohydrates. How many servings of each supplement should you combine to meet your specifications, and how much creatine will you get?

34. Repeat Exercise 33, but use the last three supplements in the table instead.

Investing
Exercises 35 and 36 are based on the following data on three stocks.24

<table>
<thead>
<tr>
<th>Price</th>
<th>Dividend Yield</th>
<th>Earnings per Share</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MON (Monsanto)</strong></td>
<td>$80</td>
<td>1%</td>
</tr>
<tr>
<td><strong>SNE (Sony Corp.)</strong></td>
<td>$20</td>
<td>2%</td>
</tr>
<tr>
<td><strong>IMCL (ImClone)</strong></td>
<td>$70</td>
<td>0%</td>
</tr>
</tbody>
</table>

35. You are planning to invest up to $10,000 in MON, SNE, and IMCL shares. You desire to maximize your share of the companies’ earnings but, for tax reasons, want to earn no more than $200 in dividends. Your broker suggests that because IMCL stock pays no dividends, you should invest only in IMCL. Is she right?

36. Repeat Exercise 35 under the assumption that MON stock has climbed to $120 on speculation, but its dividend yield and EPS (earnings per share) are unchanged.

37. Loan Planning
Enormous State University’s employee credit union has $5 million available for loans in the coming year. As VP in charge of finances, you must decide how much capital to allocate to each of four different kinds of loans, as shown in the following table.

<table>
<thead>
<tr>
<th>Type of Loan</th>
<th>Annual Rate of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Automobile</strong></td>
<td>8%</td>
</tr>
<tr>
<td><strong>Furniture</strong></td>
<td>10%</td>
</tr>
<tr>
<td><strong>Signature</strong></td>
<td>12%</td>
</tr>
<tr>
<td><strong>Other secured</strong></td>
<td>10%</td>
</tr>
</tbody>
</table>

State laws and credit union policies impose the following restrictions:
- Signature loans may not exceed 10 percent of the total investment of funds.

22 These figures are realistic: Safety-Kleen’s actual 1993 capacity was 50 million gallons, its recycled oil sold for approximately $4 per gallon, its recycling process could recover approximately 55 percent of the used oil, and its electrical bill was $3 million. Source: (Oil Recycler Greases Rusty City’s Economy), Chicago Tribune, May 30, 1993, Section 7, p.1.
23 Source: Nutritional information supplied by the manufacturers (www.netrition.com). Cost per serving is approximate.
• Furniture loans plus other secured loans may not exceed automobile loans.
• Other secured loans may not exceed 200 percent of automobile loans.

How much should you allocate to each type of loan to maximize the annual return?

38. Investments You have $100,000 which you are considering investing in three dividend-yielding stocks: Banco Santander, ConAgra Foods, and Dillard’s. You have the following data:26

<table>
<thead>
<tr>
<th>Company</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>STD (Banco Santander)</td>
<td>7%</td>
</tr>
<tr>
<td>CAG (ConAgra Foods)</td>
<td>5%</td>
</tr>
<tr>
<td>DDS (Dillard’s)</td>
<td>4%</td>
</tr>
</tbody>
</table>

Your broker has made the following suggestions:
• At least 50 percent of your total investment should be in DDS.
• No more than 10 percent of your total investment should be in STD.

How much should you invest in each stock to maximize your anticipated dividends while following your broker’s advice?

39. Portfolio Management If $x$ dollars are invested in a company that controls, say, 30 percent of the market with five brand-names, then 0.30$x$ is a measure of market exposure and 5$x$ is a measure of brand-name exposure. Now suppose you are a broker at a large securities firm, and one of your clients would like to invest up to $100,000 in recording industry stocks. You decide to recommend a combination of stocks in four of the world’s largest companies: Warner Music, Universal Music, Sony, and EMI. (See the table.)27

<table>
<thead>
<tr>
<th>Company</th>
<th>Warner Music</th>
<th>Universal Music</th>
<th>Sony</th>
<th>EMI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market Share</td>
<td>12%</td>
<td>20%</td>
<td>20%</td>
<td>15%</td>
</tr>
<tr>
<td>Number of Labels (Brands)</td>
<td>8</td>
<td>20</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

You would like your client’s brand-name exposure to be as large as possible but his total market exposure to be $15,000 or less. (This would reflect an average of 15 percent.) Furthermore, you would like at least 20 percent of the investment to be in Universal because you feel that its control of the DGG and Phillips labels is advantageous for its classical music operations. How much should you advise your client to invest in each company?

40. Portfolio Management Referring to Exercise 39, suppose instead that you wanted your client to maximize his total market exposure but limit his brand-name exposure to 1.5 million or less (representing an average of 15 labels or fewer per company), and still invest at least 20 percent of the total in Universal. How much should you advise your client to invest in each company?

41. Transportation Scheduling (This exercise is almost identical to Exercise 26 in Section 2.3 but is more realistic; one cannot always expect to fill all orders exactly, and keep all plants operating at 100 percent capacity.) The Tubular Ride Boogie Board Company has manufacturing plants in Tucson, Arizona, and Toronto, Ontario. You have been given the job of coordinating distribution of the latest model, the Gladiator, to their outlets in Honolulu and Venice Beach. The Tucson plant, when operating at full capacity, can manufacture 620 Gladiator boards per week, while the Toronto plant, beset by labor disputes, can produce only 410 boards per week. The outlet in Honolulu orders 500 Gladiator boards per week, while Venice Beach orders 530 boards per week. Transportation costs are as follows: Tucson to Honolulu: $10 per board; Tucson to Venice Beach: $5 per board; Toronto to Honolulu: $20 per board; Toronto to Venice Beach: $10 per board. Your manager has informed you that the company’s total transportation budget is $6,550. You realize that it may not be possible to fill all the orders, but you would like the total number of boogie boards shipped to be as large as possible. Given this, how many Gladiator boards should you order shipped from each manufacturing plant to each distribution outlet?

42. Transportation Scheduling Repeat the preceding exercise, but use a transportation budget of $3,050.

43. Transportation Scheduling Your publishing company is about to start a promotional blitz for its new book, Physics for the Liberal Arts. You have 20 salespeople stationed in Chicago and 10 in Denver. You would like to fly at most 10 into Los Angeles and at most 15 into New York. A round-trip plane flight from Chicago to LA costs $195;28 from Chicago to NY costs $182; from Denver to LA costs $395; and from Denver to NY costs $166. You want to spend at most $4,520 on plane tickets. How many salespeople should you fly from each of Chicago and Denver to each of LA and NY to have the most salespeople on the road?

44. Transportation Scheduling Repeat the preceding exercise, but this time, spend at most $5,320.

COMMUNICATION AND REASONING EXERCISES

45. Can the following linear programming problem be stated as a standard maximization problem? If so, do it; if not, explain why.

Maximize $p = 3x - 2y$
subject to $x - y + z \geq 0$
$x - y - z \leq 6$
$x \geq 0, y \geq 0, z \geq 0.$

46. Can the following linear programming problem be stated as a standard maximization problem? If so, do it; if not, explain why.

Maximize \( p = -3x - 2y \)

subject to

\begin{align*}
x - y + z &\geq 0 \\
x - y - z &\geq -6 \\
x \geq 0, y \geq 0, z \geq 0.
\end{align*}

47. Why is the simplex method useful? (After all, we do have the graphical method for solving LP problems.)

48. Are there any types of linear programming problems that cannot be solved with the methods of this section but that can be solved using the methods of the preceding section? Explain.

49. Your friend Janet is telling everyone that if there are only two constraints in a linear programming problem, then, in any optimal basic solution, at most two unknowns (other than the objective) will be nonzero. Is she correct? Explain.

50. Your other friend Jason is telling everyone that if there is only one constraint in a standard linear programming problem, then you will have to pivot at most once to obtain an optimal solution. Is he correct? Explain.

51. What is a “basic solution”? How might one find a basic solution of a given system of linear equations?

52. In a typical simplex method tableau, there are more unknowns than equations, and we know from the chapter on systems of linear equations that this typically implies the existence of infinitely many solutions. How are the following types of solutions interpreted in the simplex method?
   a. Solutions in which all the variables are positive.
   b. Solutions in which some variables are negative.
   c. Solutions in which the inactive variables are zero.

53. Can the value of the objective function decrease in passing from one tableau to the next? Explain.

54. Can the value of the objective function remain unchanged in passing from one tableau to the next? Explain.

4.4 The Simplex Method: Solving General Linear Programming Problems

As we saw in Section 4.2, not all LP problems are standard maximization problems. We might have constraints like \( 2x + 3y \geq 4 \) or perhaps \( 2x + 3y = 4 \). Or, you might have to minimize, rather than maximize, the objective function. General problems like this are almost as easy to deal with as the standard kind: There is a modification of the simplex method that works very nicely. The best way to illustrate it is by means of examples. First, we discuss nonstandard maximization problems.

Nonstandard Maximization Problems

**EXAMPLE 1 Maximizing with Mixed Constraints**

Maximize \( p = 4x + 12y + 6z \)

subject to

\begin{align*}
x + y + z &\leq 100 \\
4x + 10y + 7z &\leq 480 \\
x + y + z &\geq 60 \\
x \geq 0, y \geq 0, z \geq 0.
\end{align*}

**Solution** We begin by turning the first two inequalities into equations as usual because they have the standard form. We get

\begin{align*}
x + y + z + s &= 100 \\
4x + 10y + 7z + t &= 480.
\end{align*}

We are tempted to use a slack variable for the third inequality, \( x + y + z \geq 60 \), but adding something positive to the left-hand side will not make it equal to the right: It will get even bigger. To make it equal to 60, we must subtract some non-negative number.
We put a star next to the third row because the basic solution corresponding to this table is
\[ x = y = z = 0, \quad s = 100, \quad t = 480, \quad u = 60/(−1) = −60. \]

Several things are wrong here. First, the values \( x = y = z = 0 \) do not satisfy the third inequality \( x + y + z \geq 60 \). Thus, this basic solution is not feasible. Second—and this is really the same problem—the surplus variable \( u \) is negative, whereas we said that it should be non-negative. The star next to the row labeled \( u \) alerts us to the fact that the present basic solution is not feasible and that the problem is located in the starred row, where the active variable \( u \) is negative.

Whenever an active variable is negative, we star the corresponding row.

In setting up the initial tableau, we star those rows coming from \( \geq \) inequalities.

The simplex method as described in the preceding section assumed that we began in the feasible region, but now we do not. Our first task is to get ourselves into the feasible region. In practice, we can think of this as getting rid of the stars on the rows. Once we get into the feasible region, we go back to the method of the preceding section.

There are several ways to get into the feasible region. The method we have chosen is one of the simplest to state and carry out. (We will see why this method works at the end of the example.)

### The Simplex Method for General Linear Programming Problems

Star all rows that give a negative value for the associated active variable (except for the objective variable, which is allowed to be negative). If there are starred rows, you will need to begin with Phase I.

#### Phase I: Getting into the Feasible Region (Getting Rid of the Stars)

In the first starred row, find the largest positive number. Use test ratios as in Section 4.3 to find the pivot in that column (exclude the bottom row), and then pivot on that entry. (If the lowest ratio occurs both in a starred row and an un-starred row, pivot in a starred row rather than the un-starred one.) Check to see
Because there is a starred row, we need to use Phase I. The largest positive number in the starred row is 1, which occurs three times. Arbitrarily select the first, which is in the first column. In that column, the smallest test ratio happens to be given by the 1 in the \(u\) row, so this is our first pivot.

**Phase II: Use the Simplex Method for Standard Maximization Problems**

If there are any negative entries on the left side of the bottom row after Phase I, use the method described in the preceding section.

Because there is a starred row, we need to use Phase I. The largest positive number in the starred row is 1, which occurs three times. Arbitrarily select the first, which is in the first column. In that column, the smallest test ratio happens to be given by the 1 in the \(u\) row, so this is our first pivot.

**Pivot column**

<table>
<thead>
<tr>
<th></th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(s)</th>
<th>(t)</th>
<th>(u)</th>
<th>(p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(t)</td>
<td>4</td>
<td>10</td>
<td>7</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(u)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>(p)</td>
<td>-4</td>
<td>-12</td>
<td>-6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

This gives

<table>
<thead>
<tr>
<th></th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(s)</th>
<th>(t)</th>
<th>(u)</th>
<th>(p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(t)</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>(x)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(p)</td>
<td>0</td>
<td>-8</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>-4</td>
<td>1</td>
</tr>
</tbody>
</table>

Notice that we removed the star from row 3. To see why, look at the basic solution given by this tableau:

\[ x = 60,\ y = 0,\ z = 0,\ s = 40,\ t = 240,\ u = 0. \]

None of the variables is negative anymore, so there are no rows to star. The basic solution is therefore feasible—it satisfies all the constraints.

Now that there are no more stars, we have completed Phase I, so we proceed to Phase II, which is just the method of the preceding section.

<table>
<thead>
<tr>
<th></th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(s)</th>
<th>(t)</th>
<th>(u)</th>
<th>(p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(s)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(t)</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>(x)</td>
<td>6</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>(p)</td>
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<td>0</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
Before we go on... We owe you an explanation of why this method works. When we perform a pivot in Phase I, one of two things will happen. As in Example 1, we may pivot in a starred row. In that case, the negative active variable in that row will become inactive (hence zero) and some other variable will be made active with a positive value because we are pivoting on a positive entry. Thus, at least one star will be eliminated. (We will not introduce any new stars because pivoting on the entry with the smallest test ratio will keep all non-negative variables non-negative.)

The second possibility is that we may pivot on some row other than a starred row. Choosing the pivot via test ratios again guarantees that no new starred rows are created. A little bit of algebra shows that the value of the negative variable in the first starred row must increase toward zero. (Choosing the largest positive entry in the starred row will make it a little more likely that we will increase the value of that variable as much as possible; the rationale for choosing the largest entry is the same as that for choosing the most negative entry in the bottom row during Phase II.) Repeating this procedure as necessary, the value of the variable must eventually become zero or positive, assuming that there are feasible solutions to begin with.

So, one way or the other, we can eventually get rid of all of the stars.

Here is an example that begins with two starred rows.

**EXAMPLE 2 More Mixed Constraints**

Maximize \( p = 2x + y \)

subject to

\[
\begin{align*}
  x + y & \geq 35 \\
  x + 2y & \leq 60 \\
  2x + y & \geq 60 \\
  x & \leq 25 \\
  x & \geq 0, y \geq 0.
\end{align*}
\]

**Solution** We introduce slack and surplus variables, and write down the initial tableau:

\[
\begin{align*}
  x + y - s &= 35 \\
  x + 2y + t &= 60 \\
  2x + y - u &= 60 \\
  x + v &= 25 \\
  -2x - y + p &= 0.
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
<th>v</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>*s</td>
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<td>1</td>
<td>-1</td>
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<td>0</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*u</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>v</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

And we are finished. Thus the solution is

\[ p = 1,680/3 = 560, x = 120/6 = 20, y = 240/6 = 40, z = 0. \]

The slack and surplus variables are

\[ s = 40, t = 0, u = 0. \]
We locate the largest positive entry in the first starred row (row 1). There are two to choose from (both 1s); let’s choose the one in the $x$ column. The entry with the smallest test ratio in that column is the 1 in the $v$ row, so that is the entry we use as the pivot:

\[
\begin{array}{ccccccc}
\text{Pivot column} \\
& x & y & s & t & u & v & p \\
\text{*s} & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
t & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\
\text{*u} & 2 & 1 & 0 & 0 & -1 & 0 & 0 \\
v & 0 & 0 & 0 & 0 & 1 & 0 & 25 \\
p & -2 & -1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\text{R} & 2 & -2 & \text{R} & 1 \\
\text{R} & 3 & -\text{R} & 1 \\
\text{R} & 5 & +\text{R} & 1 \\
\end{array}
\]

Notice that both stars are still there because the basic solutions for $s$ and $u$ remain negative (but less so). The only positive entry in the first starred row is the 1 in the $y$ column, and that entry also has the smallest test ratio in its column. (Actually, it is tied with the 1 in the $u$ column, so we could choose either one.)

\[
\begin{array}{ccccccc}
\text{R} & 2 & -2 & \text{R} & 1 \\
\text{R} & 3 & -\text{R} & 1 \\
\text{R} & 5 & +\text{R} & 1 \\
\end{array}
\]

The basic solution is $x = 25$, $y = 10$, $s = 0$, $t = 15$, $u = 0/(−1) = 0$, and $v = 0$. Because there are no negative variables left (even $u$ has become 0), we are in the feasible region, so we can go on to Phase II, shown next. (Filling in the instructions for the row operations is an exercise.)
Before we go on... Because Example 2 had only two unknowns, we can picture the sequence of basic solutions on the graph of the feasible region. This is shown in Figure 24.

You can see that there was no way to jump from (0, 0) in the initial tableau directly into the feasible region because the first jump must be along an axis. (Why?) Also notice that the third jump did not move at all. To which step of the simplex method does this correspond?}

Minimization Problems

Now that we know how to deal with nonstandard constraints, we consider minimization problems, problems in which we have to minimize, rather than maximize, the objective function. The idea is to convert a minimization problem into a maximization problem, which we can then solve as usual.

Suppose, for instance, we want to minimize \( c = 10x - 30y \) subject to some constraints. The technique is as follows: Define a new variable \( p \) by taking \( p \) to be the negative of \( c \), so that \( p = -c \). Then, the larger we make \( p \), the smaller \( c \) becomes. For example, if we can make \( p \) increase from \(-10\) to \(-5\), then \( c \) will decrease from \(10\) to \(5\). So, if we are looking for the smallest value of \( c \), we might as well look for the largest value of \( p \) instead. More concisely,

\( \text{Minimizing } c \text{ is the same as maximizing } p = -c. \)

Now because \( c = 10x - 30y \), we have \( p = -10x + 30y \), and the requirement that we “minimize \( c = 10x - 30y \)” is now replaced by “maximize \( p = -10x + 30y \).”
From this table we get the following LP problem:

Minimize \( c = 2,000x + 3,000y \)
subject to \( 20x + 10y \leq 200 \)
\( 25x + 50y \geq 500 \)
\( 18x + 24y \geq 300 \)
\( x \geq 0, y \geq 0. \)

Before we start solving this problem, notice that all the inequalities may be simplified. The first is divisible by 10, the second by 25, and the third by 6. (However, this affects the meaning of the surplus variables; see Before we go on below.) Dividing gives the following simpler problem:

Minimize \( c = 2,000x + 3,000y \)
subject to \( 2x + y \geq 20 \)
\( x + 2y \geq 20 \)
\( 3x + 4y \geq 50 \)
\( x \geq 0, y \geq 0. \)
Following the discussion that preceded this example, we convert to a maximization problem:

Maximize \( p = -2,000x - 3,000y \)
subject to \( \begin{align*}
2x + y &\geq 20 \\
x + 2y &\geq 20 \\
3x + 4y &\geq 50 \\
x &\geq 0, \ y &\geq 0.
\end{align*} \)

We introduce surplus variables.

\[
\begin{align*}
2x + \quad y - s &= 20 \\
x + \quad 2y - t &= 20 \\
3x + \quad 4y - u &= 50 \\
2,000x + 3,000y + p &= 0.
\end{align*}
\]

The initial tableau is then

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>*s</td>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*t</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-1</td>
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<td>0</td>
</tr>
<tr>
<td>*u</td>
<td>3</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>2,000</td>
<td>3,000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The largest entry in the first starred row is the 2 in the upper left, which happens to give the smallest test ratio in its column.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
<th>p</th>
</tr>
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<tbody>
<tr>
<td>*s</td>
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<td>*t</td>
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<td>0</td>
<td>-1</td>
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<td>0</td>
</tr>
<tr>
<td>*u</td>
<td>3</td>
<td>4</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>p</td>
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<td>3,000</td>
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<td>1</td>
</tr>
</tbody>
</table>

\[ R_1 \rightarrow 3R_1 - R_2 \]

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
<th>u</th>
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<tbody>
<tr>
<td>*s</td>
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</tr>
<tr>
<td>*t</td>
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<td>2</td>
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<td>-1</td>
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</tr>
<tr>
<td>*u</td>
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<td>4</td>
<td>0</td>
<td>0</td>
<td>-1</td>
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</tr>
<tr>
<td>p</td>
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<td>1,000</td>
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</table>

\[ R_3 \rightarrow 3R_3 - 5R_2 \]

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>s</th>
<th>t</th>
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<th>p</th>
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<tbody>
<tr>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>*t</td>
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<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>*u</td>
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\[ 5R_1 - R_3 \]

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
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<th>t</th>
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<tr>
<td>*u</td>
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<td>4</td>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
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\[ 5R_2 + R_3 \]

<table>
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<th>t</th>
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<tr>
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<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
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\[ R_1/6 \]

<table>
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\[ R_2/3 \]

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<td>-1</td>
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<td>0</td>
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\[ R_3/2 \]

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<td>2,000</td>
<td>1,000</td>
<td>0</td>
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<td>1</td>
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\[ R_4/3 \]

<table>
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<tr>
<th></th>
<th>x</th>
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<td>1,000</td>
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<td>1</td>
</tr>
</tbody>
</table>
The optimal solution is
\[ x = \frac{50}{5} = 10, \\
y = \frac{50}{10} = 5, \\
p = -35,000, \]
so \( c = 35,000 \) \((s = 5, t = 0, u = 0)\).

You should buy 10 packages from Wall-to-Wall Furniture and 5 from Acme Furniture, for a minimum cost of $35,000.

**Before we go on...** The surplus variables in the preceding example represent pieces of furniture over and above the minimum requirements. The order you place will give you 50 extra tables \((s = 5, \text{ but } s \text{ was introduced after we divided the first inequality by 10, so the actual surplus is } 10 \times 5 = 50)\), the correct number of chairs \((t = 0)\), and the correct number of computer desks \((u = 0)\).

The preceding LP problem is an example of a standard minimization problem—in a sense the opposite of a standard maximization problem: We are minimizing an objective function, where all the constraints have the form \( Ax + By + Cz + \cdots \geq N \).

We will discuss standard minimization problems more fully in Section 4.5, as well as another method of solving them.

**FAQs**

**When to Switch to Phase II, Equality Constraints, and Troubleshooting**

**Q:** How do I know when to switch to Phase II?

**A:** After each step, check the basic solution for starred rows. You are not ready to proceed with Phase II until all the stars are gone.

**Q:** How do I deal with an equality constraint, such as \(2x + 7y - z = 90\)?

**A:** Although we haven’t given examples of equality constraints, they can be treated by the following trick: Replace an equality by two inequalities. For example, replace the equality \(2x + 7y - z = 90\) by the two inequalities \(2x + 7y - z \leq 90\) and \(2x + 7y - z \geq 90\). A little thought will convince you that these two inequalities amount to the same thing as the original equality!
4.4 EXERCISES

indicates exercises that should be solved using technology

1. Maximize $p = x + y$
   subject to $x + 2y \geq 6$
   $-x + y \leq 4$
   $2x + y \leq 8$
   $x \geq 0, y \geq 0$. HINT [See Examples 1 and 2.]

2. Maximize $p = 3x + 2y$
   subject to $x + 3y \geq 6$
   $-x + y \leq 4$
   $2x + y \leq 8$
   $x \geq 0, y \geq 0$. HINT [See Examples 1 and 2.]

3. Maximize $p = 12x + 10y$
   subject to $x + y \leq 25$
   $x \geq 10$
   $-x + 2y \geq 0$
   $x \geq 0, y \geq 0$.

4. Maximize $p = x + 2y$
   subject to $x + y \leq 25$
   $y \geq 10$
   $2x - y \geq 0$
   $x \geq 0, y \geq 0$.

5. Maximize $p = 2x + 5y + 3z$
   subject to $x + y + z \leq 150$
   $x + y + z \geq 100$
   $x \geq 0, y \geq 0, z \geq 0$.

6. Maximize $p = 3x + 2y + 2z$
   subject to $x + y + z \leq 38$
   $2x + y + z \geq 24$
   $x \geq 0, y \geq 0, z \geq 0$.

7. Maximize $p = 2x + 3y + z + 4w$
   subject to $x + y + z + w \leq 40$
   $2x + y - z - w \geq 10$
   $x + y + z + w \geq 10$
   $x \geq 0, y \geq 0, z \geq 0, w \geq 0$.

8. Maximize $p = 2x + 2y + z + 2w$
   subject to $x + y + z + w \leq 50$
   $2x + y - z - w \geq 10$
   $x + y + z + w \geq 20$
   $x \geq 0, y \geq 0, z \geq 0, w \geq 0$.

9. Minimize $c = 6x + 6y$
   subject to $x + 2y \geq 20$
   $2x + y \geq 20$
   $x \geq 0, y \geq 0$. HINT [See Example 3.]

10. Minimize $c = 3x + 2y$
    subject to $x + 2y \geq 20$
           $2x + y \geq 10$
           $x \geq 0, y \geq 0$. HINT [See Example 3.]

11. Minimize $c = 2x + y + 3z$
    subject to $x + y + z \geq 100$
           $2x + y \geq 50$
           $y + z \geq 50$
           $x \geq 0, y \geq 0, z \geq 0$.

12. Minimize $c = 2x + 2y + 3z$
    subject to $x + z \geq 100$
           $2x + y \geq 50$
           $y + z \geq 50$
           $x \geq 0, y \geq 0, z \geq 0$.

13. Minimize $c = 50x + 50y + 11z$
    subject to $2x + z \geq 3$
           $2x + y - z \geq 2$
           $3x + y - z \leq 3$
           $x \geq 0, y \geq 0, z \geq 0$.

14. Minimize $c = 50x + 11y + 50z$
    subject to $3x + z \geq 8$
           $3x + y - z \geq 6$
           $4x + y - z \leq 8$
           $x \geq 0, y \geq 0, z \geq 0$.

15. Minimize $c = x + y + z + w$
    subject to $5x - y + w \geq 1,000$
           $z + w \leq 2,000$
           $x + y \leq 500$
           $x \geq 0, y \geq 0, z \geq 0, w \geq 0$.

16. Minimize $c = 5x + y + z + w$
    subject to $5x - y + w \geq 1,000$
           $z + w \leq 2,000$
           $x + y \leq 500$
           $x \geq 0, y \geq 0, z \geq 0, w \geq 0$. **Q**: What happens if it is impossible to choose a pivot using the instructions in Phase I?

**A**: In that case, the LP problem has no solution. In fact, the feasible region is empty. If it is impossible to choose a pivot in phase II, then the feasible region is unbounded and there is no optimal solution.
1. In Exercises 17–22, we suggest the use of technology. Round all answers to two decimal places.

17. Maximize \( p = 2x + 3y + 1.1z + 4w \)
subject to \( 1.2x + y + z + w \leq 40.5 \)
\( 2.2x + y - z - w \geq 10 \)
\( 1.2x + y + z + 1.2w \geq 10.5 \)
\( x \geq 0, y \geq 0, z \geq 0, w \geq 0. \)

18. Maximize \( p = 2.2x + 2y + 1.1z + 2w \)
subject to \( x + 1.5y + 1.5z + w \leq 50.5 \)
\( 2x + 1.5y - z - w \geq 10 \)
\( x + 1.5y + z + 1.5w \geq 21 \)
\( x \geq 0, y \geq 0, z \geq 0, w \geq 0. \)

19. Minimize \( c = 2.2x + y + 3.3z \)
subject to \( x + 1.5y + 1.2z \geq 100 \)
\( 2x + 1.5y \geq 50 \)
\( 1.5y + 1.1z \geq 50 \)
\( x \geq 0, y \geq 0, z \geq 0. \)

20. Minimize \( c = 50.3x + 10.5y + 50.3z \)
subject to \( 3.1x + 1.1z \geq 28 \)
\( 3.1x + y + 1.1z \geq 23 \)
\( 4.2x + y + 1.1z \geq 28 \)
\( x \geq 0, y \geq 0, z \geq 0. \)

21. Minimize \( c = 1.1x + y + 1.5z - w \)
subject to \( 5.12x - y + w \leq 1,000 \)
\( z + w \geq 2,000 \)
\( 1.22x + y \leq 500 \)
\( x \geq 0, y \geq 0, z \geq 0, w \geq 0. \)

22. Minimize \( c = 5.45x + y + 1.5z + w \)
subject to \( 5.12x - y + w \geq 1,000 \)
\( z + w \geq 2,000 \)
\( 1.12x + y \leq 500 \)
\( x \geq 0, y \geq 0, z \geq 0, w \geq 0. \)

APPLICATIONS

23. Agriculture\(^{29}\) Your small farm encompasses 100 acres, and you are planning to grow tomatoes, lettuce, and carrots in the coming planting season. Fertilizer costs per acre are: $5 for tomatoes, $4 for lettuce, and $2 for carrots. Based on past experience, you estimate that each acre of tomatoes will require an average of 4 hours of labor per week, while tending to lettuce and carrots will each require an average of 2 hours per week. You estimate a profit of $2,000 for each acre of tomatoes, $1,500 for each acre of lettuce and $500 for each acre of carrots. You would like to spend at least $400 on fertilizer (your niece owns the company that manufactures it) and your farm laborers can supply up to 500 hours per week. How many acres of each crop should you plant to maximize total profits? In this event, will you be using all 100 acres of your farm? \textit{Hint:} (See Example 3.)

24. Agriculture\(^{30}\) Your farm encompasses 900 acres, and you are planning to grow soybeans, corn, and wheat in the coming planting season. Fertilizer costs per acre are: $5 for soybeans, $2 for corn, and $1 for wheat. You estimate that each acre of soybeans will require an average of 5 hours of labor per week, while tending to corn and wheat will each require an average of 2 hours per week. Based on past yields and current market prices, you estimate a profit of $3,000 for each acre of soybeans, $2,000 for each acre of corn, and $1,000 for each acre of wheat. You can afford to spend no more than $3,000 on fertilizer, but your labor union contract stipulates at least 2,000 hours per week of labor. How many acres of each crop should you plant to maximize total profits? In this event, will you be using more than 2,000 hours of labor?

25. Politics The political pollster Canter is preparing for a national election. It would like to poll at least 1,500 Democrats and 1,500 Republicans. Each mailing to the East Coast gets responses from 100 Democrats and 50 Republicans. Each mailing to the Midwest gets responses from 100 Democrats and 100 Republicans. And each mailing to the West Coast gets responses from 50 Democrats and 100 Republicans. Mailings to the East Coast cost $40 each to produce and mail, mailings to the Midwest cost $60 each, and mailings to the West Coast cost $50 each. How many mailings should Canter send to each area of the country to get the responses it needs at the least possible cost? What will it cost?

26. Purchasing Bingo’s Copy Center needs to buy white paper and yellow paper. Bingo’s can buy from three suppliers. Harvard Paper sells a package of 20 reams of white and 10 reams of yellow for $60, Yale Paper sells a package of 10 reams of white and 10 reams of yellow for $40, and Dartmouth Paper sells a package of 10 reams of white and 20 reams of yellow for $50. If Bingo’s needs 350 reams of white and 400 reams of yellow, how many packages should it buy from each supplier to minimize the cost? What is the least possible cost?

27. Resource Allocation Succulent Citrus produces orange juice and orange concentrate. This year the company anticipates a demand of at least 10,000 quarts of orange juice and 1,000 quarts of orange concentrate. Each quart of orange juice requires 10 oranges, and each quart of concentrate requires 50 oranges. The company also anticipates using at least 200,000 oranges for these products. Each quart of orange juice costs the company 50¢ to produce, and each quart of concentrate costs $2.00 to produce. How many quarts of each product should Succulent Citrus produce to meet the demand and minimize total costs?

28. Resource Allocation Fancy Pineapple produces pineapple juice and canned pineapple rings. This year the company anticipates a demand of at least 10,000 pint of pineapple juice and 1,000 cans of pineapple rings. Each pint of pineapple juice requires 2 pineapples, and each can of pineapple rings requires 1 pineapple. The company anticipates using at least 20,000 pineapples for these products. Each pint of pineapple juice

\(^{29}\) Compare Exercise 27 in Section 4.3.

\(^{30}\) Compare Exercise 28 in Section 4.3.
costs the company 20¢ to produce, and each can of pineapple rings costs 50¢ to produce. How many pints of pineapple juice and cans of pineapple rings should Fancy Pineapple produce to meet the demand and minimize total costs?

29. Music CD Sales In the year 2000, industry revenues from sales of recorded music amounted to $3.6 billion for rock music, $1.8 billion for rap music, and $0.4 billion for classical music.31 You would like the selection of music in your music store to reflect, in part, this national trend, so you have decided to stock at least twice as many rock music CDs as rap CDs. Your store has an estimated capacity of 20,000 CDs, and, as a classical music devotee, you would like to stock at least 5,000 classical CDs. Rock music CDs sell for $12 on average, rap CDs for $15, and classical CDs for $12. How many of each type of CD should you stock to get the maximum retail value?

30. Music CD Sales Your music store’s main competitor, Nuttal Hip Hop Classic Store, also wishes to stock at most 20,000 CDs, with at least half as many rap CDs as rock CDs and at least 2,000 classical CDs. It anticipates an average sale price of $15/rock CD, $10/rap CD and $10/classical CD. How many of each type of CD should you stock to get the maximum retail value, and what is the maximum retail value?

31. Nutrition Each serving of Gerber Mixed Cereal for Baby contains 60 calories and no vitamin C. Each serving of Gerber Mango Tropical Fruit Dessert contains 80 calories and 45 percent of the U.S. Recommended Daily Allowance (RDA) of vitamin C for infants. Each serving of Gerber Apple Banana Juice contains 60 calories and 120 percent of the RDA of vitamin C for infants.32 The cereal costs 10¢/serving, the dessert costs 53¢/serving, and the juice costs 27¢/serving. If you want to provide your child with at least 120 calories and at least 120 percent of the RDA of vitamin C, how can you do so at the least cost?

32. Nutrition Each serving of Gerber Mixed Cereal for Baby contains 60 calories, no vitamin C, and 11 grams of carbohydrates. Each serving of Gerber Mango Tropical Fruit Dessert contains 80 calories, 45 percent of the RDA of vitamin C for infants, and 21 grams of carbohydrates. Each serving of Gerber Apple Banana Juice contains 60 calories and 21 grams of the RDA of vitamin C for infants.33 Assume that the cereal costs 11¢/serving, the dessert costs 50¢/serving, and the juice costs 30¢/serving. If you want to provide your child with at least 120 calories, at least 120 percent of the RDA of vitamin C, and at least 37 grams of carbohydrates, how can you do so at the least cost?

33. Purchasing Cheapskate Electronics Store needs to update its inventory of stereos, TVs, and DVD players. There are three suppliers it can buy from: Nadir offers a bundle consisting of 5 stereos, 10 TVs, and 15 DVD players for $3,000. Blunt offers a bundle consisting of 10 stereos, 10 TVs, and 10 DVD players for $4,000. Sonny offers a bundle consisting of 15 stereos, 10 TVs, and 15 DVD players for $5,000. Cheapskate Electronics needs at least 150 stereos, 200 TVs, and 150 DVD players. How can it update its inventory at the least possible cost? What is the least possible cost?

34. Purchasing Federal Rent-a-Car is putting together a new fleet. It is considering package offers from three car manufacturers. Fred Motors is offering 5 small cars, 5 medium cars, and 10 large cars for $500,000. Admiral Motors is offering 5 small, 10 medium, and 5 large cars for $400,000. Chrysalis is offering 10 small, 5 medium, and 5 large cars for $300,000. Federal would like to buy at least 550 small cars, at least 500 medium cars, and at least 550 large cars. How many packages should it buy from each car maker to keep the total cost as small as possible? What will be the total cost?

35. (Compare Exercise 29 in Section 4.2.) You are thinking of combining Cell-Tech, RiboForce HP, and Creatine Transport to obtain a 10-day supply that provides at least 80 grams of creatine and at least 10 grams of taurine, but no more than 750 grams of carbohydrates and 1,000 milligrams of alpha lipoic acid. How many servings of each supplement should you combine to meet your specifications for the least cost?

36. (Compare Exercise 30 in Section 4.2.) You are thinking of combining RiboForce HP, Creatine Transport, and Pre-Load Creatine to obtain a 10-day supply that provides at least 80 grams of creatine and at least 10 grams of taurine, but no more than 600 grams of carbohydrates and 2,000 milligrams of alpha lipoic acid. How many servings of each supplement should you combine to meet your specifications for the least cost?

37. Subsidies The Miami Beach City Council has offered to subsidize hotel development in Miami Beach, and it is hoping for at least two hotels with a total capacity of at least 1,400.

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31“Ibid.” includes “Hip Hop.” Revenues are based on total manufacturers’ shipments at suggested retail prices and are rounded to the nearest $0.1 billion. Source: Recording Industry Association of America, www.riaa.com, March 2002.

32Source: Nutrition information supplied by Gerber.

33Ibid.

34Source: Nutritional information supplied by the manufacturers (www.netrition.com). Cost per serving is approximate.
Suppose that you are a developer interested in taking advantage of this offer by building a small group of hotels in Miami Beach. You are thinking of three prototypes: a convention-style hotel with 500 rooms costing $100 million, a vacation-style hotel with 200 rooms costing $20 million, and a small motel with 50 rooms costing $4 million. The City Council will approve your plans, provided you build at least one convention-style hotel and no more than two small motels.

a. How many of each type of hotel should you build to satisfy the city council’s wishes and stipulations while minimizing your total cost?

b. Now assume that the city council will give developers 20 percent of the cost of building new hotels in Miami Beach, up to $50 million. Will the city’s $50 million subsidy be sufficient to cover 20 percent of your total costs?

38. ▼ Subsidies Refer back to the preceding exercise. You are about to begin the financial arrangements for your new hotels when the city council informs you that it has changed its mind and now requires at least two vacation-style hotels and no more than four small motels.

a. How many of each type of hotel should you build to satisfy the city council’s wishes and stipulations while minimizing your total costs?

b. Will the city’s $50 million subsidy limit still be sufficient to cover 20 percent of your total costs?

39. ▼ Transportation Scheduling We return to your exploits coordinating distribution for the Tubular Ride Boogie Board Company. You will recall that the company has manufacturing plants in Tucson, Arizona and Toronto, Ontario, and you have been given the job of coordinating distribution of their latest model, the Gladiator, to their outlets in Honolulu and Venice Beach. The Tucson plant can manufacture up to 620 boards per week, while the Toronto plant, beset by labor disputes, can produce no more than 410 Gladiator boards per week. The outlet in Honolulu orders 500 Gladiator boards per week, while Venice Beach orders 530 boards per week. Transportation costs are as follows: Tucson to Honolulu: $10/board; Tucson to Venice Beach: $5/board; Toronto to Honolulu: $20/board; Toronto to Venice Beach: $10/board.

Your manager has said that you are to be sure to fill all orders and ship the boogie boards at a minimum total transportation cost. How will you do it?

40. ▼ Transportation Scheduling In the situation described in the preceding exercise, you have just been notified that workers at the Toronto boogie board plant have gone on strike, resulting in a total work stoppage. You are to come up with a revised delivery schedule by tomorrow with the understanding that the Tucson plant can push production to a maximum of 1,000 boards per week. What should you do?

41. ▼ Finance Senator Pork barrel habitually overdraws his three bank accounts, at the Congressional Integrity Bank, Citizens’ Trust, and Checks R Us. There are no penalties because the overdrafts are subsidized by the taxpayer. The Senate Ethics Committee tends to let slide irregular banking activities as long as they are not flagrant. At the moment (due to Congress’ preoccupation with a Supreme Court nominee), a total overdraft of up to $10,000 will be overlooked. Pork barrel’s conscience makes him hesitate to overdraw accounts at banks whose names include expressions like “integrity” and “citizens’ trust.” The effect is that his overdrafts at the first two banks combined amount to no more than one-quarter of the total. On the other hand, the financial officers at Integrity Bank, aware that Senator Pork barrel is a member of the Senate Banking Committee, “suggest” that he overdraw at least $2,500 from their bank. Find the amount he should overdraw from each bank in order to avoid investigation by the Ethics Committee and overdraw his account at Integrity by as much as his sense of guilt will allow.

42. ▼ Scheduling Because Joe Slim’s brother was recently elected to the State Senate, Joe’s financial advisement concern, Inside Information Inc., has been doing a booming trade, even though the financial counseling he offers is quite worthless. (None of his seasoned clients pays the slightest attention to his advice.) Slim charges different hourly rates to different categories of individuals: $5,000/hour for private citizens, $50,000/hour for corporate executives, and $10,000/hour for presidents of universities. Due to his taste for leisure, he feels that he can spend no more than 40 hours/week in consultation. On the other hand, Slim feels that it would be best for his intellect were he to devote at least 10 hours of consultation each week to university presidents. However, Slim always feels somewhat uncomfortable dealing with academics, so he would prefer to spend no more than half his consultation time with university presidents. Furthermore, he likes to think of himself as representing the interests of the common citizen, so he wishes to offer at least 2 more hours of his time each week to private citizens than to corporate executives and university presidents combined. Given all these restrictions, how many hours each week should he spend with each type of client in order to maximize his income?

43. ▼ Transportation Scheduling Your publishing company is about to start a promotional blitz for its new book, Physics for the Liberal Arts. You have 20 salespeople stationed in Chicago and 10 in Denver. You would like to fly at least 10 into Los Angeles and at least 15 into New York. A round-trip plane flight from Chicago to LA costs $195, from Chicago to NY costs $182; from Denver to LA costs $395; and from Denver to
NY costs $166. How many salespeople should you fly from each of Chicago and Denver to each of LA and NY to spend the least amount on plane flights?

44. **Transportation Scheduling** Repeat Exercise 43, but now suppose that you would like at least 15 salespeople in Los Angeles.

45. **Hospital Staffing** As the staff director of a new hospital, you are planning to hire cardiologists, rehabilitation specialists, and infectious disease specialists. According to recent data, each cardiology case averages $12,000 in revenue, each physical rehabilitation case $19,000, and each infectious disease case, $14,000. You judge that each specialist you employ will expand the hospital caseload by about 10 patients per week. You already have 3 cardiologists on staff, and the hospital is equipped to admit up to 200 patients per week. Based on past experience, each cardiologist and rehabilitation specialist brings in one government research grant per year, while each infectious disease specialist brings in three. Your board of directors would like to see a total of at least 30 grants per year and would like your weekly revenue to be as large as possible. How many of each kind of specialist should you hire?

46. **Hospital Staffing** Referring to Exercise 45, you completely misjudged the number of patients each type of specialist would bring to the hospital per week. It turned out that each cardiologist brought in 120 new patients per year, each rehabilitation specialist brought in 90 per year, and each infectious disease specialist brought in 70 per year. It also turned out that your hospital could deal with no more than 1,960 new patients per year. Repeat the preceding exercise in the light of this corrected data.

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**COMMUNICATION AND REASONING EXERCISES**

47. Explain the need for Phase I in a nonstandard LP problem.

48. Explain the need for Phase II in a nonstandard LP problem.

49. Explain briefly why we would need to use Phase I in solving a linear programming problem with the constraint $x + 2y - z \geq 3$.

50. Which rows do we star, and why?

51. Consider the following linear programming problem:

Maximize $p = x + y$

subject to $x - 2y \geq 0$

$2x + y \leq 10$

$x \geq 0, y \geq 0$.

This problem

(A) must be solved using the techniques of Section 4.4.

(B) must be solved using the techniques of Section 4.3.

(C) can be solved using the techniques of either section.

52. Consider the following linear programming problem:

Maximize $p = x + y$

subject to $x - 2y \geq 1$

$2x + y \leq 10$

$x \geq 0, y \geq 0$.

This problem

(A) must be solved using the techniques of Section 4.4.

(B) must be solved using the techniques of Section 4.3.

(C) can be solved using the techniques of either section.

53. Find a linear programming problem in three variables that requires one pivot in Phase I.

54. Find a linear programming problem in three variables that requires two pivots in Phase I.

55. Find a linear programming problem in two or three variables with no optimal solution, and show what happens when you try to solve it using the simplex method.

56. Find a linear programming problem in two or three variables with more than one optimal solution, and investigate which solution is found by the simplex method.

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38 These (rounded) figures are based on an Illinois survey of 1.3 million hospital admissions (Chicago Tribune, March 29, 1993, Section 4, p.1).

Source: Lutheran General Health System, Argus Associates, Inc.

39 These (rounded) figures were obtained from the survey referenced in the preceding exercise by dividing the average hospital revenue per physician by the revenue per case.

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**4.5 The Simplex Method and Duality**

We mentioned standard minimization problems in the last section. These problems have the following form.

**Standard Minimization Problem**

A standard minimization problem is an LP problem in which we are required to minimize (not maximize) a linear objective function

$$c = as + bt + cu + \cdots$$
We saw a way of solving minimization problems in Section 4.4, but a mathematically elegant relationship between maximization and minimization problems gives us another way of solving minimization problems that satisfy the non-negative objective condition. This relationship is called **duality**.

To describe duality, we must first represent an LP problem by a matrix. This matrix is not the first tableau but something simpler: Pretend you forgot all about slack variables and also forgot to change the signs of the objective function. As an example, consider the following two standard problems.

### Problem 1

**Maximize**  
\[ p = 20x + 20y + 50z \]

**subject to**  
\[ 2x + y + 3z \leq 2,000 \]
\[ x + 2y + 4z \leq 3,000 \]
\[ x \geq 0, y \geq 0, z \geq 0. \]

We saw a way of solving minimization problems in Section 4.4, but a mathematically elegant relationship between maximization and minimization problems gives us another way of solving minimization problems that satisfy the non-negative objective condition. This relationship is called **duality**.

To describe duality, we must first represent an LP problem by a matrix. This matrix is not the first tableau but something simpler: Pretend you forgot all about slack variables and also forgot to change the signs of the objective function. As an example, consider the following two standard problems.

### Problem 1

**Maximize**  
\[ p = 20x + 20y + 50z \]

**subject to**  
\[ 2x + y + 3z \leq 2,000 \]
\[ x + 2y + 4z \leq 3,000 \]
\[ x \geq 0, y \geq 0, z \geq 0. \]
We represent this problem by the matrix
\[
\begin{bmatrix}
2 & 1 & 3 & 2,000 \\
1 & 2 & 4 & 3,000 \\
20 & 20 & 50 & 0
\end{bmatrix}.
\]

Notice that the coefficients of the objective function go in the bottom row, and we place a zero in the bottom right corner.

**Problem 2** (from Example 3 in Section 4.4)

Minimize \( c = 2,000s + 3,000t \)
subject to \( 2s + t \geq 20 \)
\( s + 2t \geq 20 \)
\( 3s + 4t \geq 50 \)
\( s \geq 0, t \geq 0 \).

Problem 2 is represented by
\[
\begin{bmatrix}
2 & 1 & 20 \\
1 & 2 & 20 \\
3 & 4 & 50 \\
2,000 & 3,000 & 0
\end{bmatrix}.
\]

These two problems are related: The matrix for Problem 1 is the transpose of the matrix for Problem 2. (Recall that the transpose of a matrix is obtained by writing its rows as columns; see Section 3.1.) When we have a pair of LP problems related in this way, we say that the two are dual LP problems.

### Dual Linear Programming Problems

Two LP problems, one a maximization and one a minimization problem, are dual if the matrix that represents one is the transpose of the matrix that represents the other.

### Finding the Dual of a Given Problem

Given an LP problem, we find its dual as follows:

1. Represent the problem as a matrix (see above).
2. Take the transpose of the matrix.
3. Write down the dual, which is the LP problem corresponding to the new matrix. If the original problem was a maximization problem, its dual will be a minimization problem, and vice versa.

The original problem is called the primal problem, and its dual is referred to as the dual problem.

#### Quick Examples

**Primal problem**

Minimize \( c = s + 2t \)
subject to \( 5s + 2t \geq 60 \)
\( 3s + 4t \geq 80 \)
\( s + t \geq 20 \)
\( s \geq 0, t \geq 0 \).

\[
\begin{bmatrix}
5 & 2 \\
3 & 4 \\
1 & 1 \\
1 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
5 & 2 & 60 \\
3 & 4 & 80 \\
1 & 1 & 20 \\
1 & 2 & 0
\end{bmatrix}.
\]
The following theorem justifies what we have been doing, and says that solving the dual problem of an LP problem is equivalent to solving the original problem.

### Fundamental Theorem of Duality

a. If an LP problem has an optimal solution, then so does its dual. Moreover, the primal problem and the dual problem have the same optimal value for their objective functions.

b. Contained in the final tableau of the simplex method applied to an LP problem is the solution to its dual problem: It is given by the bottom entries in the columns associated with the slack variables, divided by the entry under the objective variable.

The theorem\(^{41}\) gives us an alternative way of solving minimization problems that satisfy the non-negative objective condition. Let’s illustrate by solving Problem 2 above.

#### EXAMPLE 1 Solving by Duality

Minimize \( c = 2,000s + 3,000t \)

subject to

\[
\begin{align*}
2s + t & \geq 20 \\
s + 2t & \geq 20 \\
3s + 4t & \geq 50 \\
s & \geq 0, t & \geq 0.
\end{align*}
\]

**Solution**

**Step 1 Find the dual problem.** Write the primal problem in matrix form and take the transpose:

\[
\begin{bmatrix}
\begin{array}{ccc}
2 & 1 & 20 \\
1 & 2 & 20 \\
3 & 4 & 50 \\
2,000 & 3,000 & 0
\end{array}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\begin{array}{ccc}
2 & 1 & 3 & 2,000 \\
1 & 2 & 4 & 3,000 \\
20 & 20 & 50 & 0
\end{array}
\end{bmatrix}.
\]

The dual problem is:

Maximize \( p = 20x + 20y + 50z \)

subject to

\[
\begin{align*}
2x + y + 3z & \leq 2,000 \\
x + 2y + 4z & \leq 3,000 \\
x & \geq 0, y & \geq 0, z & \geq 0.
\end{align*}
\]

\(^{41}\) The proof of the theorem is beyond the scope of this book but can be found in a textbook devoted to linear programming, like *Linear Programming* by Vašek Chvátal (San Francisco: W. H. Freeman and Co., 1983) which has a particularly well-motivated discussion.
Step 2 Use the simplex method to solve the dual problem. Because we have a standard maximization problem, we do not have to worry about Phase I but go straight to Phase II.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>-20</td>
<td>-20</td>
<td>-50</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t</td>
<td>-5</td>
<td>2</td>
<td>0</td>
<td>-4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>40</td>
<td>-10</td>
<td>0</td>
<td>50</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>9</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>y</td>
<td>-5</td>
<td>2</td>
<td>0</td>
<td>-4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>15</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>15</td>
<td>3</td>
</tr>
</tbody>
</table>

The solution to the primal problem is

\[ s = 30/3 = 10, \ t = 15/3 = 5, \ c = 105,000/3 = 35,000. \]

(Compare this with the method we used to solve Example 3 in the preceding section. Which method seems more efficient?)

Before we go on... Can you now see the reason for using the variable names \( s, t, u, \ldots \) in standard minimization problems? ■

Q: Is the theorem also useful for solving problems that do not satisfy the non-negative objective condition?

A: Consider a standard minimization problem that does not satisfy the non-negative objective condition, such as

Minimize \[ c = 2s - t \]
subject to \[ 2s + 3t \geq 2 \]
\[ s + 2t \geq 2 \]
\[ s \geq 0, t \geq 0. \]
Its dual would be

Maximize \( p = 2x + 2y \)
subject to
\[
2x + y \leq 2 \\
3x + 2y \leq -1 \\
x \geq 0, y \geq 0.
\]

This is not a standard maximization problem because the right-hand side of the second constraint is negative. In general, if a problem does not satisfy the non-negative condition, its dual is not standard. Therefore, to solve the dual by the simplex method will require using Phase I as well as Phase II, and we may as well just solve the primal problem that way to begin with. Thus, duality helps us solve problems only when the primal problem satisfies the non-negative objective condition.

In many economic applications, the solution to the dual problem also gives us useful information about the primal problem, as we will see in the following example.

**EXAMPLE 2 Shadow Costs**

You are trying to decide how many vitamin pills to take. SuperV brand vitamin pills each contain 2 milligrams of vitamin X, 1 milligram of vitamin Y, and 1 milligram of vitamin Z. Topper brand vitamin pills each contain 1 milligram of vitamin X, 1 milligram of vitamin Y, and 2 milligrams of vitamin Z. You want to take enough pills daily to get at least 12 milligrams of vitamin X, 10 milligrams of vitamin Y, and 12 milligrams of vitamin Z. However, SuperV pills cost 4¢ each and Toppers cost 3¢ each, and you would like to minimize the total cost of your daily dosage. How many of each brand of pill should you take? How would changing your daily vitamin requirements affect your minimum cost?

**Solution** This is a straightforward minimization problem. The unknowns are

\[
s = \text{number of SuperV brand pills} \\
t = \text{number of Topper brand pills}.
\]

The linear programming problem is

Minimize \( c = 4s + 3t \)
subject to
\[
2s + t \geq 12 \\
s + t \geq 10 \\
s + 2t \geq 12 \\
s \geq 0, t \geq 0.
\]

We solve this problem by using the simplex method on its dual, which is

Maximize \( p = 12x + 10y + 12z \)
subject to
\[
2x + y + z \leq 4 \\
x + y + 2z \leq 3 \\
x \geq 0, y \geq 0, z \geq 0.
\]

After pivoting three times, we arrive at the final tableau:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>s</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>6</td>
<td>0</td>
<td>-6</td>
<td>6</td>
<td>-6</td>
<td>0</td>
</tr>
<tr>
<td>y</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>
Therefore, the answer to the original problem is that you should take two SuperV vitamin pills and eight Toppers at a cost of 32¢ per day.

Now, the key to answering the last question, which asks you to determine how changing your daily vitamin requirements would affect your minimum cost, is to look at the solution to the dual problem. From the tableau we see that \( x = 1, \ y = 2, \) and \( z = 0. \) To see what \( x, \ y, \) and \( z \) might tell us about the original problem, let’s look at their units. In the inequality \( 2x + y + z \leq 4, \) the coefficient 2 of \( x \) has units “mg of vitamin X/SuperV pill,” and the 4 on the right-hand side has units “¢/SuperV pill.” For 2\( x \) to have the same units as the 4 on the right-hand side, \( x \) must have units “¢/mg of vitamin X.” Similarly, \( y \) must have units “¢/mg of vitamin Y” and \( z \) must have units “¢/mg of vitamin Z.” One can show (although we will not do it here) that \( x \) gives the amount that would be added to the minimum cost for each increase* of 1 milligram of vitamin X in our daily requirement. For example, if we were to increase our requirement from 12 milligrams to 14 milligrams, an increase of 2 milligrams, the minimum cost would change by \( 2x = 2\€, \) from 32¢ to 34¢. (Try it; you’ll end up taking four SuperV pills and six Toppers.) Similarly, each increase of 1 milligram of vitamin Y in the requirements would increase the cost by \( y = 2\€. \) These costs are called the **marginal costs** or the **shadow costs** of the vitamins.

What about \( z = 0? \) The shadow cost of vitamin Z is 0¢/mg, meaning that you can increase your requirement of vitamin Z without changing your cost. In fact, the solution \( s = 2 \) and \( t = 8 \) provides you with 18 milligrams of vitamin Z, so you can increase the required amount of vitamin Z up to 18 milligrams without changing the solution at all.

We can also interpret the shadow costs as the effective cost to you of each milligram of each vitamin in the optimal solution. You are paying 1¢/milligram of vitamin X, 2¢/milligram of vitamin Y, and getting the vitamin Z for free. This gives a total cost of \( 1 \times 12 + 2 \times 10 + 0 \times 12 = 32\€, \) as we know. Again, if you change your requirements slightly, these are the amounts you will pay per milligram of each vitamin.

* To be scrupulously correct, this works only for relatively small changes in the requirements, not necessarily for very large ones.

**Game Theory**

We return to a topic we discussed in Section 3.4: solving two-person zero-sum games. In that section, we described how to solve games that could be reduced to \( 2 \times 2 \) games or smaller. It turns out that we can solve larger games using linear programming and duality. We summarize the procedure, work through an example, and then discuss why it works.

**Solving a Matrix Game**

**Step 1** Reduce the payoff matrix by dominance.

**Step 2** Add a fixed number \( k \) to each of the entries so that they all become non-negative and no column is all zero.

**Step 3** Write 1s to the right of and below the matrix, and then write down the associated standard maximization problem. Solve this primal problem using the simplex method.
Step 4 Find the optimal strategies and the expected value as follows:

**Column Strategy**

1. Express the solution to the primal problem as a column vector.
2. Normalize by dividing each entry of the solution vector by \( p \) (which is also the sum of all the entries).
3. Insert zeros in positions corresponding to the columns deleted during reduction.

**Row Strategy**

1. Express the solution to the dual problem as a row vector.
2. Normalize by dividing each entry by \( p \), which will once again be the sum of all the entries.
3. Insert zeros in positions corresponding to the rows deleted during reduction.

**Value of the Game:**

\[ e = \frac{1}{p} - k \]

**EXAMPLE 3 Restaurant Inspector**

You manage two restaurants, Tender Steaks Inn (TSI) and Break for a Steak (BFS). Even though you run the establishments impeccably, the Department of Health has been sending inspectors to your restaurants on a daily basis and fining you for minor infractions. You’ve found that you can head off a fine if you’re present, but you can cover only one restaurant at a time. The Department of Health, on the other hand, has two inspectors, who sometimes visit the same restaurant and sometimes split up, one to each restaurant. The average fines you have been getting are shown in the following matrix.

<table>
<thead>
<tr>
<th>Health Inspectors</th>
<th>Both at BFS</th>
<th>Both at TSI</th>
<th>One at Each</th>
</tr>
</thead>
<tbody>
<tr>
<td>TSI</td>
<td>$8,000</td>
<td>$2,000</td>
<td></td>
</tr>
<tr>
<td>BFS</td>
<td>$0</td>
<td>$10,000</td>
<td>$4,000</td>
</tr>
</tbody>
</table>

How should you choose which restaurant to visit to minimize your expected fine?

**Solution** This matrix is not quite the payoff matrix because fines, being penalties, should be negative payoffs. Thus, the payoff matrix is the following:

\[
P = \begin{bmatrix}
-8,000 & 0 & -2,000 \\
0 & -10,000 & -4,000
\end{bmatrix}.
\]

We follow the steps above to solve the game using the simplex method.

**Step 1** There are no dominated rows or columns, so this game does not reduce.

**Step 2** We add \( k = 10,000 \) to each entry so that none are negative, getting the following new matrix (with no zero column):

\[
\begin{bmatrix}
2,000 & 10,000 & 8,000 \\
10,000 & 0 & 6,000
\end{bmatrix}.
\]
Step 3 We write 1s to the right and below this matrix:
\[
\begin{bmatrix}
2,000 & 10,000 & 8,000 & 1 \\
10,000 & 0 & 6,000 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}.
\]

The corresponding standard maximization problem is the following:

Maximize \( p = x + y + z \)
subject to \(2,000x + 10,000y + 8,000z \leq 1\)
\(10,000x + 6,000z \leq 1\)
\(x \geq 0, y \geq 0, z \geq 0.\)

Step 4 We use the simplex method to solve this problem. After pivoting twice, we arrive at the final tableau:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( s )</th>
<th>( t )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>50,000</td>
<td>34,000</td>
<td>5</td>
<td>-1</td>
</tr>
<tr>
<td>( x )</td>
<td>10,000</td>
<td>0</td>
<td>6,000</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( p )</td>
<td>0</td>
<td>0</td>
<td>14,000</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Column Strategy The solution to the primal problem is
\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
\frac{1}{10,000} \\
\frac{4}{50,000} \\
0
\end{bmatrix}.
\]

We divide each entry by \( p = 9/50,000 \), which is also the sum of the entries. This gives the optimal column strategy:
\[
C = \begin{bmatrix}
\frac{5}{9} \\
\frac{4}{9} \\
0
\end{bmatrix}.
\]

Thus, the inspectors’ optimal strategy is to stick together, visiting BFS with probability \( 5/9 \) and TSI with probability \( 4/9 \).

Row Strategy The solution to the dual problem is
\[
\begin{bmatrix}
s \\
t
\end{bmatrix} = \begin{bmatrix}
\frac{5}{50,000} \\
\frac{4}{50,000}
\end{bmatrix}.
\]

Once again, we divide by \( p = 9/50,000 \) to find the optimal row strategy:
\[
R = \begin{bmatrix}
\frac{5}{9} \\
\frac{4}{9}
\end{bmatrix}.
\]

Thus, you should visit TSI with probability \( 5/9 \) and BFS with probability \( 4/9 \).

Value of the Game Your expected average fine is
\[
e = \frac{1}{p} - k = \frac{50,000}{9} - 10,000 = -\frac{40,000}{9} \approx -4,444.
\]
Before we go on... We owe you an explanation of why the procedure we used in Example 3 works. The main point is to understand how we turn a game into a linear programming problem. It’s not hard to see that adding a fixed number $k$ to all the payoffs will change only the payoff, increasing it by $k$, and not change the optimal strategies. So let’s pick up Example 3 from the point where we were considering the following game:

$$P = \begin{bmatrix} 2,000 & 10,000 & 8,000 \\ 10,000 & 0 & 6,000 \end{bmatrix}.$$  

We are looking for the optimal strategies $R$ and $C$ for the row and column players, respectively; if $e$ is the value of the game, we will have $e = RPC$. Let’s concentrate first on the column player’s strategy $C = \begin{bmatrix} u \\ v \\ w \end{bmatrix}^T$, where $u$, $v$, and $w$ are the unknowns we want to find. Because $e$ is the value of the game, if the column player uses the optimal strategy $C$ and the row player uses any old strategy $S$, the expected value with these strategies has to be $e$ or better for the column player, so $SPC \leq e$. Let’s write that out for two particular choices of $S$. First, consider $S = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

$$2,000u + 10,000v + 8,000w \leq e.$$  

Next, do the same thing for $S = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$10,000u + 6,000w \leq e.$$  

It turns out that if these two inequalities are true, then $SPC \leq e$ for any $S$ at all, which is what the column player wants. These are starting to look like constraints in a linear programming problem, but the $e$ appearing on the right is in the way. We get around this by dividing by $e$, which we know to be positive because all of the payoffs are non-negative and no column is all zero (so the column player can’t force the value of the game to be 0; here is where we need these assumptions). We get the following inequalities:

$$2,000 \left( \frac{u}{e} \right) + 10,000 \left( \frac{v}{e} \right) + 8,000 \left( \frac{w}{e} \right) \leq 1 \quad 10,000 \frac{u}{e} + 6,000 \frac{w}{e} \leq 1.$$  

Now we’re getting somewhere. To make these look even more like linear constraints, we replace our unknowns $u$, $v$, and $w$ with new unknowns, $x = u/e$, $y = v/e$, and $z = w/e$. Our inequalities then become:

$$2,000x + 10,000y + 8,000z \leq 1 \quad 10,000x + 6,000z \leq 1.$$  

What about an objective function? From the point of view of the column player, the objective is to find a strategy that will minimize the expected value $e$. In order to write $e$ in terms of our new variables $x$, $y$, and $z$, we use the fact that our original variables, being the entries in the column strategy, have to add up to 1: $u + v + w = 1$. Dividing by $e$ gives

$$\frac{u}{e} + \frac{v}{e} + \frac{w}{e} = 1.$$  

The Simplex Method and Duality
or

\[ x + y + z = \frac{1}{e} . \]

Now we notice that, if we maximize \( p = x + y + z = 1/e \), it will have the effect of minimizing \( e \), which is what we want. So, we get the following linear programming problem:

Maximize  \[ p = x + y + z \]
subject to  \[ 2,000x + 10,000y + 8,000z \leq 1 \]
\[ 10,000x + 6,000z \leq 1 \]
\[ x \geq 0, y \geq 0, z \geq 0 . \]

Why can we say that \( x, y, \) and \( z \) should all be non-negative? Because the unknowns \( u, v, w, \) and \( e \) must all be non-negative.

So now, if we solve this linear programming problem to find \( x, y, z, \) and \( p \), we can find the column player’s optimal strategy by computing \( u = xe = x/p, v = y/p, \) and \( w = z/p \). Moreover, the value of the game is \( e = 1/p \). (If we added \( k \) to all the payoffs, we should now adjust by subtracting \( k \) again to find the correct value of the game.)

Turning now to the row player’s strategy, if we repeat the above type of argument from the row player’s viewpoint, we’ll end up with the following linear programming problem to solve:

Minimize  \[ c = s + t \]
subject to  \[ 2,000s + 10,000t \geq 1 \]
\[ 10,000s \geq 1 \]
\[ 8,000s + 6,000t \geq 1 \]
\[ s \geq 0, t \geq 0 . \]

This is, of course, the dual to the problem we solved to find the column player’s strategy, so we know that we can read its solution off of the same final tableau. The optimal value of \( c \) will be the same as the value of \( p \), so \( c = 1/e \) also. The entries in the optimal row strategy will be \( s/c \) and \( t/c \).

**FAQs**

**Q:** Given a minimization problem, when should I use duality, and when should I use the two-phase method in Section 4.4?

**A:** If the original problem satisfies the non-negative objective condition (none of the coefficients in the objective function are negative), then you can use duality to convert the problem to a standard maximization one, which can be solved with the one-phase method. If the original problem does not satisfy the non-negative objective condition, then dualizing results in a nonstandard LP problem, so dualizing may not be worthwhile.

**Q:** When is it absolutely necessary to use duality?

**A:** Never. Duality gives us an efficient but not necessary alternative for solving standard minimization problems.
4.5 Exercises

In Exercises 1–8, write down (without solving) the dual LP problem. HINT [See the Quick Example on page 322.]

1. Maximize \( p = 2x + y \)
   subject to \( x + 2y \leq 6 \)
   \(-x + y \leq 2 \)
   \( x \geq 0, y \geq 0 \).

2. Maximize \( p = x + 5y \)
   subject to \( x + y \leq 6 \)
   \(-x + 3y \leq 4 \)
   \( x \geq 0, y \geq 0 \).

3. Minimize \( c = 2s + t + 3u \)
   subject to \( s + t + u \geq 100 \)
   \( 2s + t \geq 50 \)
   \( s \geq 0, t \geq 0, u \geq 0 \).

4. Minimize \( c = 2s + 2t + 3u \)
   subject to \( s + u \geq 100 \)
   \( 2s + t \geq 50 \)
   \( s \geq 0, t \geq 0, u \geq 0 \).

5. Maximize \( p = x + y + z + w \)
   subject to \( x + y + z \leq 3 \)
   \( y + z + w \leq 4 \)
   \( x + z + w \leq 5 \)
   \( x + y + w \leq 6 \)
   \( x \geq 0, y \geq 0, z \geq 0, w \geq 0 \).

6. Maximize \( p = x + y + z + w \)
   subject to \( x + y + z \leq 3 \)
   \( y + z + w \leq 3 \)
   \( x + z + w \leq 4 \)
   \( x + y + w \leq 4 \)
   \( x \geq 0, y \geq 0, z \geq 0, w \geq 0 \).

7. Minimize \( c = s + 3t + u \)
   subject to \( 5s - t + v \geq 1,000 \)
   \( u - v \geq 2,000 \)
   \( s + t \geq 500 \)
   \( s \geq 0, t \geq 0, u \geq 0, v \geq 0 \).

8. Minimize \( c = 5s + 2u + v \)
   subject to \( s - t + 2u \geq 2,000 \)
   \( u + v \geq 3,000 \)
   \( s + t \geq 500 \)
   \( s \geq 0, t \geq 0, u \geq 0, v \geq 0 \).

In Exercises 9–22, solve the standard minimization problems using duality. (You may already have seen some of them in earlier sections, but now you will be solving them using a different method.) HINT [See Example 1.]

9. Minimize \( c = s + t \)
   subject to \( s + 2t \geq 6 \)
   \( 2s + t \geq 6 \)
   \( s \geq 0, t \geq 0 \).

10. Minimize \( c = s + 2t \)
   subject to \( s + 3t \geq 30 \)
   \( 2s + t \geq 30 \)
   \( s \geq 0, t \geq 0 \).

11. Minimize \( c = 6s + 6t \)
   subject to \( s + 2t \geq 20 \)
   \( 2s + t \geq 20 \)
   \( s \geq 0, t \geq 0 \).

12. Minimize \( c = 3s + 2t \)
   subject to \( s + 2t \geq 20 \)
   \( 2s + t \geq 10 \)
   \( s \geq 0, t \geq 0 \).

13. Minimize \( c = 0.2s + 0.3t \)
   subject to \( 0.2s + 0.1t \geq 1 \)
   \( 0.3s + 0.3t \geq 1.5 \)
   \( 10s + 10t \geq 80 \)
   \( s \geq 0, t \geq 0 \).

14. Minimize \( c = 0.4s + 0.1t \)
   subject to \( 30s + 20t \geq 600 \)
   \( 0.1s + 0.4t \geq 4 \)
   \( 0.2s + 0.3t \geq 4.5 \)
   \( s \geq 0, t \geq 0 \).

15. Minimize \( c = 2s + t \)
   subject to \( 3s + t \geq 30 \)
   \( s + t \geq 20 \)
   \( s + 3t \geq 30 \)
   \( s \geq 0, t \geq 0 \).

16. Minimize \( c = s + 2t \)
   subject to \( 4s + t \geq 100 \)
   \( 2s + t \geq 80 \)
   \( s + 3t \geq 150 \)
   \( s \geq 0, t \geq 0 \).

17. Minimize \( c = s + 2t + 3u \)
   subject to \( 3s + 2t + u \geq 60 \)
   \( 2s + t + 3u \geq 60 \)
   \( s \geq 0, t \geq 0, u \geq 0 \).
18. Minimize \( c = s + t + 2u \)
subject to \( s + 2t + 2u \geq 60 \)
\( 2s + t + 3u \geq 60 \)
\( s \geq 0, t \geq 0, u \geq 0 \).

19. Minimize \( c = 2s + t + 3u \)
subject to \( s + t + u \geq 100 \)
\( 2s + t \geq 50 \)
\( t + u \geq 50 \)
\( s \geq 0, t \geq 0, u \geq 0 \).

20. Minimize \( c = 2s + 2t + 3u \)
subject to \( s + u \geq 100 \)
\( 2s + t \geq 50 \)
\( t + u \geq 50 \)
\( s \geq 0, t \geq 0, u \geq 0 \).

21. Minimize \( c = s + t + u \)
subject to \( 3s + 2t + u \geq 60 \)
\( 2s + t + 3u \geq 60 \)
\( s + 3t + 2u \geq 60 \)
\( s \geq 0, t \geq 0, u \geq 0 \).

22. Minimize \( c = s + t + 2u \)
subject to \( s + 2t + 2u \geq 60 \)
\( 2s + t + 3u \geq 60 \)
\( s + 3t + 6u \geq 60 \)
\( s \geq 0, t \geq 0, u \geq 0 \).

In Exercises 23–28, solve the games with the given payoff matrices. HINT [See Example 3.]

23. \( P = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -2 \end{bmatrix} \)
24. \( P = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \)

25. \( P = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -2 \\ 1 & 2 & 0 \end{bmatrix} \)
26. \( P = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \)

27. \( P = \begin{bmatrix} -1 & 1 & 2 & -1 \\ 2 & -1 & -2 & -3 \\ 1 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \end{bmatrix} \)
28. \( P = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 2 \end{bmatrix} \)

**APPLICATIONS**

*The following applications are similar to ones in preceding exercise sets. Use duality to answer them.*

29. **Resource Allocation** Meow makes cat food out of fish and cornmeal. Fish has 8 grams of protein and 4 grams of fat per ounce, and cornmeal has 4 grams of protein and 8 grams of fat. A jumbo can of cat food must contain at least 48 grams of protein and 48 grams of fat. If fish and cornmeal both cost $5/ounce, how many ounces of each should Meow use in each can of cat food to minimize costs? What are the shadow costs of protein and of fat? HINT [See Example 2.]

30. **Resource Allocation** Oz makes lion food out of giraffe and gazelle meat. Giraffe meat has 18 grams of protein and 36 grams of fat per pound, while gazelle meat has 36 grams of protein and 18 grams of fat per pound. A batch of lion food must contain at least 36,000 grams of protein and 54,000 grams of fat. Giraffe meat costs $2/pound and gazelle meat costs $4/pound. How many pounds of each should go into each batch of lion food in order to minimize costs? What are the shadow costs of protein and fat?

31. **Nutrition** Ruff makes dog food out of chicken and grain. Chicken has 10 grams of protein and 5 grams of fat/ounce, and grain has 2 grams of protein and 2 grams of fat/ounce. A bag of dog food must contain at least 200 grams of protein and at least 150 grams of fat. If chicken costs 10¢/ounce and grain costs 1¢/ounce, how many ounces of each should Ruff use in each bag of dog food in order to minimize cost? What are the shadow costs of protein and fat?

32. **Purchasing** The Enormous State University’s Business School is buying computers. The school has two models to choose from, the Pomegranate and the iZac. Each Pomegranate comes with 400 MB of memory and 80 GB of disk space, while each iZac has 300 MB of memory and 100 GB of disk space. For reasons related to its accreditation, the school would like to be able to say that it has a total of at least 48,000 MB of memory and at least 12,800 GB of disk space. If both the Pomegranate and the iZac cost $2,000 each, how many of each should the school buy to keep the cost as low as possible? What are the shadow costs of memory and disk space?

33. **Nutrition** Each serving of Gerber Mixed Cereal for Baby contains 60 calories and no vitamin C. Each serving of Gerber Mango Tropical Fruit Dessert contains 80 calories and 45 percent of the U.S. Recommended Daily Allowance (RDA) of vitamin C for infants. Each serving of Gerber Apple Banana Juice contains 60 calories and 120 percent of the U.S. RDA of vitamin C for infants. The cereal costs 10¢/serving, the dessert costs 53¢/serving, and the juice costs 97¢/serving. If you want to provide your child with at least 120 calories and at least 120 percent of the U.S. RDA of vitamin C, how can you do so at the least cost? What are your shadow costs for calories and vitamin C?

34. **Nutrition** Each serving of Gerber Mixed Cereal for Baby contains 60 calories, no vitamin C, and 11 grams of carbohydrates. Each serving of Gerber Mango Tropical Fruit Dessert contains 80 calories, 45 percent of the U.S. Recommended Daily Allowance (RDA) of vitamin C for infants, and 21 grams of carbohydrates. Each serving of Gerber Apple Banana Juice contains 60 calories, 120 percent of the U.S. RDA of vitamin C for infants, and 15 grams of carbohydrates. Assume that the cereal costs 11¢/serving, the dessert

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42 Source: Nutrition information provided by Gerber.
43 Ibid.
costs 50¢/serving, and the juice costs 30¢/serving. If you want to provide your child with at least 180 calories, at least 120 percent of the U.S. RDA of vitamin C, and at least 37 grams of carbohydrates, how can you do so at the least cost? What are your shadow costs for calories, vitamin C, and carbohydrates?

35. Politics The political pollster Canter is preparing for a national election. It would like to poll at least 1,500 Democrats and 1,500 Republicans. Each mailing to the East Coast gets responses from 100 Democrats and 50 Republicans. Each mailing to the Midwest gets responses from 100 Democrats and 100 Republicans. And each mailing to the West Coast gets responses from 50 Democrats and 100 Republicans. Mailings to the East Coast cost $40 each to produce and mail, mailings to the Midwest cost $60 each, and mailings to the West Coast cost $50 each. How many mailings should Canter send to each area of the country to get the responses it needs at the least possible cost? What will it cost? What are the shadow costs of a Democratic response and a Republican response?

36. Purchasing Bingo’s Copy Center needs to buy white paper and yellow paper. Bingo’s can buy from three suppliers. Harvard Paper sells a package of 20 reams of white and 10 reams of yellow for $60; Yale Paper sells a package of 10 reams of white and 10 reams of yellow for $40, and Dartmouth Paper sells a package of 10 reams of white and 20 reams of yellow for $50. If Bingo’s needs 350 reams of white and 400 reams of yellow, how many packages should it buy from each supplier so as to minimize the cost? What is the lowest possible cost? What are the shadow costs of white paper and yellow paper?

37. Resource Allocation One day Gillian the Magician summoned the wisest of her women. “Devoted followers,” she began, “I have a quandary: As you well know, I possess great expertise in sleep spells and shock spells, but unfortunately, these are proving to be a drain on my aural energy resources; each sleep spell costs me 500 pico-shirleys of aural energy, while each shock spell requires 750 pico-shirleys. Clearly, I would like to hold my overall expenditure of aural energy to a minimum, and still meet my commitments in protecting the Sisterhood from the ever-present threat of trolls. Specifically, I have estimated that each sleep spell keeps us safe for an average of 2 minutes, while every shock spell protects us for about 3 minutes. We certainly require enough protection to last 24 hours of each day, and possibly more, just to be safe. At the same time, I have noticed that each of my sleep spells can immobilize 3 trolls at once, while one of my typical shock spells (having a narrower range) can immobilize only 2 trolls at once. We are faced, my sisters, with an onslaught of 1,200 trolls per day! Finally, as you are no doubt aware, the bylaws dictate that for a Magician of the Order to remain in good standing, the number of shock spells must be between one-quarter and one-third the number of shock and sleep spells combined. What do I do, oh Wise Ones?”

38. Risk Management The Grand Vizier of the Kingdom of Um is being blackmailed by numerous individuals and is having a very difficult time keeping his blackmailers from going public. He has been keeping them at bay with two kinds of payoff: gold bars from the Royal Treasury and political favors. Through bitter experience, he has learned that each payoff in gold gives him peace for an average of about one month, and each political favor seems to earn him about a month and a half of reprieve. To maintain his flawless reputation in the court, he feels he cannot afford any revelations about his tainted past to come to light within the next year. Thus, it is imperative that his blackmailers be kept at bay for 12 months. Furthermore, he would like to keep the number of gold payoffs at no more than one-quarter of the combined number of payoffs because the outward flow of gold bars might arouse suspicion on the part of the Royal Treasurer. The gold payoffs tend to deplete the Grand Vizier’s travel budget. (The treasury has been subsidizing his numerous trips to the Himalayas.) He estimates that each gold bar removed from the treasury will cost him four trips. On the other hand, because the administering of political favors tends to cost him valuable travel time, he suspects that each political favor will cost him about two trips. Now, he would obviously like to keep his blackmailers silenced and lose as few trips as possible. What is he to do? How many trips will he lose in the next year?

39. Game Theory—Politics Incumbent Tax N. Spend and challenger Trick L. Down are running for county executive, and polls show them to be in a dead heat. The election hinges on three cities: Littleville, Metropolis, and Urbantown. The candidates have decided to spend the last weeks before the election campaigning in those three cities; each day each candidate will decide in which city to spend the day. Pollsters have determined the following payoff matrix, where the payoff represents the number of votes gained or lost for each one-day campaign trip.

<table>
<thead>
<tr>
<th>City</th>
<th>T. N. Spend</th>
<th>T. L. Down</th>
</tr>
</thead>
<tbody>
<tr>
<td>Littleville</td>
<td>−200</td>
<td>−300</td>
</tr>
<tr>
<td>Metropolis</td>
<td>0</td>
<td>−100</td>
</tr>
<tr>
<td>Urbantown</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

What percentage of time should each candidate spend in each city in order to maximize votes gained? If both candidates use their optimal strategies, what is the expected vote?

40. Game Theory—Marketing Your company’s new portable phone/music player/PDA/bottle washer, the RunMan, will compete against the established market leader, the iNod, in a saturated market. (Thus, for each device you sell, one fewer iNod is sold.) You are planning to launch the RunMan with a traveling road show, concentrating on two cities, New York and Boston. The makers of the iNod will do the same to try to maintain their sales. If, on a given day, you both go to New York, you will lose 1,000 units in sales to the iNod. If you both go to Boston, you will lose 750 units in sales. On the other hand, if you go to New York and your competitor to Boston, you will gain 1,500 units in sales from them. If you
go to Boston and they to New York, you will gain 500 units in sales. What percentage of time should you spend in New York and what percentage in Boston, and how do you expect your sales to be affected?

41. **Game Theory—Morra Games** A three-finger Morra game is a game in which two players simultaneously show one, two, or three fingers at each round. The outcome depends on a predetermined set of rules. Here is an interesting example: If the number of fingers shown by A and B differ by 1, then A loses one point. If they differ by more than 1, the round is a draw. If they show the same number of fingers, A wins an amount equal to the sum of the fingers shown. Determine the optimal strategy for each player and the expected value of the game.

42. **Game Theory—Morra Games** Referring to the preceding exercise, consider the following rules for a three-finger Morra game: If the sum of the fingers shown is odd, then A wins an amount equal to that sum. If the sum is even, B wins the sum. Determine the optimal strategy for each player and the expected value of the game.

43. **Game Theory—Military Strategy** Colonel Blotto is a well-known game in military strategy. Here is a version of this game: Colonel Blotto has four regiments under his command, while his opponent, Captain Kije, has three. The armies are to try to occupy two locations, and each commander must decide how many regiments to send to each location. The army that sends more regiments to a location captures that location as well as the other army’s regiments. If both armies send the same number of regiments to a location, then there is a draw. The payoffs are one point for each location captured and one point for each regiment captured. Find the optimum strategy for each commander and also the value of the game.

44. **Game Theory—Military Strategy** Referring to the preceding exercise, consider the version of Colonel Blotto with the same payoffs given there except that Captain Kije earns two points for each location captured, while Colonel Blotto continues to earn only one point. Find the optimum strategy for each commander and also the value of the game.

**COMMUNICATION AND REASONING EXERCISES**

45. Give one possible advantage of using duality to solve a standard minimization problem.

46. To ensure that the dual of a minimization problem will result in a standard maximization problem,

(A) the primal problem should satisfy the non-negative objective condition.

(B) the primal problem should be a standard minimization problem.

(C) the primal problem should not satisfy the non-negative objective condition.

47. Give an example of a standard minimization problem whose dual is not a standard maximization problem. How would you go about solving your problem?

48. Give an example of a nonstandard minimization problem whose dual is a standard maximization problem.

49. Solve the following nonstandard minimization problem using duality. Recall from a footnote in the text that to find the dual you must first rewrite all of the constraints using “≥.” The Miami Beach City Council has offered to subsidize hotel development in Miami Beach, and is hoping for at least two hotels with a total capacity of at least 1,400. Suppose that you are a developer interested in taking advantage of this offer by building a small group of hotels in Miami Beach. You are thinking of three prototypes: a convention-style hotel with 500 rooms costing $100 million, a vacation-style hotel with 200 rooms costing $20 million, and a small motel with 50 rooms costing $4 million. The city council will approve your plans provided you build at least one convention-style hotel and no more than two small motels. How many of each type of hotel should you build to satisfy the city council’s wishes and stipulations while minimizing your total cost?

50. Given a minimization problem, when would you solve it by applying the simplex method to its dual, and when would you apply the simplex method to the minimization problem itself?

51. Create an interesting application that leads to a standard maximization problem. Solve it using the simplex method and note the solution to its dual problem. What does the solution to the dual tell you about your application?

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CHAPTER 4 REVIEW

KEY CONCEPTS

Web Site www.FiniteMath.org
Go to the student Web site at www.FiniteMath.org to find a comprehensive and interactive Web-based summary of Chapter 4.

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REVIEW EXERCISES

In each of Exercises 1–4, sketch the region corresponding to the given inequalities, say whether it is bounded, and give the coordinates of all corner points.

1. \(2x - 3y \leq 12\)  
2. \(x \leq 2y\)
3. \(x + 2y \leq 20\)  
\(3x + 2y \leq 30\)  
\(x \geq 0, y \geq 0\)

In each of Exercises 5–8, solve the given linear programming problem graphically.

5. Maximize \(p = 2x + y\)  
subject to  
\(3x + y \leq 30\)
\(x + y \leq 12\)
\(x + 3y \leq 30\)
\(x \geq 0, y \geq 0\).

6. Maximize \(p = 2x + 3y\)  
subject to  
\(2x + y \geq 10\)
\(2x + y \geq 12\)
\(x + y \leq 20\)
\(x \geq 0, y \geq 0\).

7. Minimize \(c = 2x + y\)  
subject to  
\(3x + y \geq 30\)
\(x + 2y \geq 20\)
\(2x - y \geq 0\)
\(x \geq 0, y \geq 0\).

8. Minimize \(c = 3x + y\)  
subject to  
\(3x + 2y \geq 6\)
\(2x - 3y \leq 0\)
\(3x - 2y \geq 0\)
\(x \geq 0, y \geq 0\).

In each of Exercises 9–16, solve the given linear programming problem using the simplex method. If no optimal solution exists, indicate whether the feasible region is empty or the objective function is unbounded.

9. Maximize \(p = x + y + 2z\)  
subject to  
\(x + 2y + 2z \leq 60\)
\(2x + y + 3z \leq 60\)
\(x \geq 0, y \geq 0, z \geq 0\).

10. Maximize \(p = x + y + 2z\)  
subject to  
\(x + 2y + 2z \leq 60\)
\(2x + y + 3z \leq 60\)
\(x + 3y + 6z \leq 60\)
\(x \geq 0, y \geq 0, z \geq 0\).

11. Maximize \(p = x + y + 3z\)  
subject to  
\(x + y + z \geq 100\)
\(y + z \leq 80\)
\(x \geq 0, y \geq 0, z \geq 0\).

12. Maximize \(p = 2x + y\)  
subject to  
\(x + 2y \geq 12\)
\(2x + y \leq 12\)
\(x + y \leq 5\)
\(x \geq 0, y \geq 0\).
13. Minimize subject to
\[ c = x + 2y + 3z \]
\[ 3x + 2y + z \geq 60 \]
\[ 2x + y + 3z \geq 60 \]
\[ x \geq 0, y \geq 0, z \geq 0. \]

14. Minimize subject to
\[ c = x + y - z \]
\[ 3x + 2y + z \geq 60 \]
\[ 2x + y + 3z \geq 60 \]
\[ x + 3y + 2z \geq 60 \]
\[ x \geq 0, y \geq 0, z \geq 0. \]

15. Minimize subject to
\[ c = x + y + z + w \]
\[ x + y \geq 30 \]
\[ x + z \geq 20 \]
\[ x + y - w \leq 10 \]
\[ y + z - w \leq 10 \]
\[ x \geq 0, y \geq 0, z \geq 0, w \geq 0. \]

16. Minimize subject to
\[ c = 4x + y + z + w \]
\[ x + y \geq 30 \]
\[ y - z \leq 20 \]
\[ z - w \leq 10 \]
\[ x \geq 0, y \geq 0, z \geq 0, w \geq 0. \]

In each of Exercises 17–20, solve the given linear programming problem using duality.

17. Minimize subject to
\[ c = 2x + y \]
\[ 3x + 2y \geq 60 \]
\[ 2x + y \geq 60 \]
\[ x + 3y \geq 60 \]
\[ x \geq 0, y \geq 0. \]

18. Minimize subject to
\[ c = 2x + y + 2z \]
\[ 3x + 2y + z \geq 100 \]
\[ 2x + y + 3z \geq 200 \]
\[ x \geq 0, y \geq 0, z \geq 0. \]

19. Minimize subject to
\[ c = 2x + y \]
\[ 3x + 2y \geq 10 \]
\[ 2x - y \leq 30 \]
\[ x + 3y \geq 60 \]
\[ x \geq 0, y \geq 0. \]

20. Minimize subject to
\[ c = 2x + y + 2z \]
\[ 3x - 2y + z \geq 100 \]
\[ 2x + y - 3z \leq 200 \]
\[ x \geq 0, y \geq 0, z \geq 0. \]

In each of Exercises 21–24, solve the game with the given payoff matrix.

21. \[ P = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 3 & -1 & 0 \end{bmatrix} \]

22. \[ P = \begin{bmatrix} -3 & 0 & 1 \\ -4 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix} \]

23. \[ P = \begin{bmatrix} -3 & -2 & 3 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{bmatrix} \]

24. \[ P = \begin{bmatrix} -4 & -2 & -3 \\ 1 & -3 & -2 \\ -3 & 1 & -4 \end{bmatrix} \]

Exercises 25–28 are adapted from the Actuarial Exam on Operations Research.

25. You are given the following linear programming problem:
\[ \text{Minimize } c = x + 2y \]
\[ \text{subject to } \begin{align*} -2x + y & \geq 1 \\ x - 2y & \geq 1 \\ x \geq 0, y \geq 0. \end{align*} \]

Which of the following is true?
(A) The problem has no feasible solutions.
(B) The objective function is unbounded.
(C) The problem has optimal solutions.

26. Repeat the preceding exercise with the following linear programming problem:
\[ \text{Maximize } p = x + y \]
\[ \text{subject to } \begin{align*} -2x + y & \leq 1 \\ x - 2y & \leq 2 \\ x \geq 0, y \geq 0. \end{align*} \]

27. Determine the optimal value of the objective function. You are given the following linear programming problem.
\[ \text{Maximize } Z = x_1 + 4x_2 + 2x_3 - 10 \]
\[ \text{subject to } \begin{align*} 4x_1 + x_2 + x_3 & \leq 45 \\ -x_1 + x_2 + 2x_3 & \leq 0 \\ x_1, x_2, x_3 & \geq 0. \end{align*} \]

28. Determine the optimal value of the objective function. You are given the following linear programming problem.
\[ \text{Maximize } Z = x_1 + 4x_2 + 2x_3 + x_4 + 40 \]
\[ \text{subject to } \begin{align*} 4x_1 + x_2 + x_3 & \leq 45 \\ -x_1 + 2x_2 + x_4 & \geq 40 \\ x_1, x_2, x_3 & \geq 0. \end{align*} \]

APPLICATIONS

Purchases In Exercises 29–32, you are the buyer for OHaganBooks.com and are considering increasing stocks of romance and horror novels at the new OHaganBooks.com warehouse in Texas. You have offers from several publishers: Duffin House, Higgins Press, McPhearson Imprints, and O’Conell Books. Duffin offers a package of 5 horror novels and 5 romance novels for $50, Higgins offers a package of 5 horror and 10 romance novels for $80, McPhearson offers a package of 10 horror novels and 5 romance novels for $80, and O’Conell offers a package of 10 horror novels and 10 romance novels for $90.

29. How many packages should you purchase from Duffin House and Higgins Press to obtain at least 4,000 horror novels and 6,000 romance novels at minimum cost? What is the minimum cost?

30. How many packages should you purchase from McPhearson Imprints and O’Conell Books to obtain at least 5,000 horror novels and 4,000 romance novels at minimum cost? What is the minimum cost?

31. Refer to the scenario in Exercise 29. As it turns out, John O’Hagan promised Marjory Duffin that OHaganBooks.com...
would buy at least 20 percent more packages from Duffin as from Higgins, but you still want to obtain at least 4,000 horror novels and 6,000 romance novels at minimum cost.

a. Without solving the problem, say which of the following statements are possible:

(A) The cost will stay the same.
(B) The cost will increase.
(C) The cost will decrease.
(D) It will be impossible to meet all the conditions.
(E) The cost will become unbounded.

b. If you wish to meet all the requirements in part (a) at minimum cost, how many packages should you purchase from each publisher? What is the minimum cost?

32. Refer to Exercise 30. You are about to place the order meeting the requirements of Exercise 30 when you are told that you can order no more than a total of 500 packages, and that at least half of the packages should be from McPhearson. Explain why this is impossible by referring to the feasible region for Exercise 30.

33. Investments Marjory Duffin’s portfolio manager has suggested two high-yielding stocks: European Emerald Emporium (EEE) and Royal Ruby Retailers (RRR). EEE shares cost $50, yield 4.5% in dividends, and have a risk index of 2.0 per share. RRR shares cost $55, yield 5% in dividends, and have a risk index of 3.0 per share. Marjory has up to $12,100 to invest and would like to earn at least $550 in dividends. How many shares of each stock should she purchase to meet her requirements and minimize the total risk index for her portfolio? What is the minimum total risk index?

34. Investments Marjory Duffin’s other portfolio manager has suggested another two high-yielding stocks: Countrynarrow Mortgages (CNM) and Scotland Subprime (SS). CNM shares cost $40, yield 5.5% in dividends, and have a risk index of 1.0 per share. SS shares cost $25, yield 7.5% in dividends, and have a risk index of 1.5 per share. Marjory will invest up to $30,000 in these stocks and would like to earn at least $1,650 in dividends. How many shares of each stock should she purchase in order to meet her requirements and minimize the total risk index for her portfolio?

35. Resource Allocation Billy-Sean O’Hagan has joined the Physics Society at Suburban State University, and the group is planning to raise money to support the dying space program by making and selling umbrellas. The society intends to make three models: the Sprinkle, the Storm, and the Hurricane. The amounts of cloth, metal, and wood used in making each model are given in this table:

<table>
<thead>
<tr>
<th>Model</th>
<th>Cloth (sq. yd)</th>
<th>Metal (lbs)</th>
<th>Wood (lbs)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sprinkle</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Storm</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Hurricane</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

The table also shows the amounts of each material available in a given day and the profits to be made from each model. How many of each model should the society make in order to maximize its profit?

36. Profit Duffin Press, which is now the largest publisher of books sold at the OHaganBooks.com site, prints three kinds of books: paperback, quality paperback, and hardcover. The amounts of paper, ink, and time on the presses required for each kind of book are given in this table:

<table>
<thead>
<tr>
<th></th>
<th>Paperback</th>
<th>Quality Paperback</th>
<th>Hardcover</th>
<th>Total Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paper (pounds)</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6,000</td>
</tr>
<tr>
<td>Ink (gallons)</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>6,000</td>
</tr>
<tr>
<td>Time (minutes)</td>
<td>10</td>
<td>10</td>
<td>3</td>
<td>22,000</td>
</tr>
<tr>
<td>Profit ($)</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

The table also lists the total amounts of paper, ink, and time available in a given day and the profits made on each kind of book. How many of each kind of book should Duffin print to maximize profit?

37. Purchases You are just about to place book orders from Duffin and Higgins (see Exercise 29) when everything changes: Duffin House informs you that, due to a global romance crisis, its packages now each will contain 5 horror novels but only 2 romance novels and still cost $50 per package. Packages from Higgins will now contain 10 of each type of novel, but now cost $150 per package. Ewing Books enters the fray and offers its own package of 5 horror and 5 romance novels for $100.

The sales manager now tells you that at least 50% of the packages must come from Higgins Press and, as before, you want to obtain at least 4,000 horror novels and 6,000 romance novels at minimum cost. Taking all of this into account, how many packages should you purchase from each publisher? What is the minimum cost?

38. Purchases You are about to place book orders from McPhearson and O’Conell (see Exercise 30) when you get an e-mail from McPhearson Imprints saying that, sorry, but they have stopped publishing romance novels due to the global romance crisis and can now offer only packages of 10 horror novels for $50. O’Conell is still offering packages of 10 horror novels and 10 romance novels for $90, and now the United States Treasury, in an attempt to bolster the floundering romance industry, is offering its own package of 20 romance novels for $120. Furthermore, Congress, in approving this measure, has passed legislation dictating that at least 2/3 of the packages in every order must come from the U.S. Treasury. As before, you wish to obtain at least 5,000 horror novels and 4,000 romance novels at minimum cost. Taking all of this into account, how many packages should you purchase from each supplier? What is the minimum cost?

39. Degree Requirements During his lunch break, John O’Hagan decides to devote some time to assisting his son Billy-Sean, who continues to have a terrible time planning his college
course schedule. The latest Bulletin of Suburban State University claims to have added new flexibility to its course requirements, but it remains as complicated as ever. It reads as follows:

All candidates for the degree of Bachelor of Arts at SSU must take at least 120 credits from the Sciences, Fine Arts, Liberal Arts and Mathematics combined, including at least as many Science credits as Fine Arts credits, and at most twice as many Mathematics credits as Science credits, but with Liberal Arts credits exceeding Mathematics credits by no more than one-third of the number of Fine Arts credits.

Science and fine arts credits cost $300 each, and liberal arts and mathematics credits cost $200 each. John would like to have Billy-Sean meet all the requirements at a minimum total cost.

a. Set up (without solving) the associated linear programming problem.

b. Use technology to determine how many of each type of credit Billy-Sean should take. What will the total cost be?

40. **Degree Requirements** No sooner had the “new and flexible” course requirement been released than the English Department again pressured the University Senate to include their vaunted “Verbal Expression” component in place of the fine arts requirement in all programs (including the sciences):

All candidates for the degree of Bachelor of Science at SSU must take at least 120 credits from the Liberal Arts, Sciences, Verbal Expression, and Mathematics, including at most as many Science credits as Liberal Arts credits, and at least twice as many Verbal Expression credits as Science credits and Liberal Arts credits combined, with Liberal Arts credits exceeding Mathematics credits by at least a quarter of the number of Verbal Expression credits.

Science credits cost $300 each, while each credit in the remaining subjects now costs $400. John would like to have Billy-Sean meet all the requirements at a minimum total cost.

a. Set up (without solving) the associated linear programming problem.

b. Use technology to determine how many of each type of credit Billy-Sean should take. What will the total cost be?

41. **Shipping** On the same day that the sales department at Duffin House received an order for 600 packages from the OHaganBooks.com Texas headquarters, it received an additional order for 200 packages from FantasyBooks.com, based in California. Duffin House has warehouses in New York and Illinois. The New York warehouse has 600 packages in stock, but the Illinois warehouse is closing down and has only 300 packages in stock. Shipping costs per package of books are as follows: New York to Texas: $20; New York to California: $50; Illinois to Texas: $30; Illinois to California: $40. What is the lowest total shipping cost for which Duffin House can fill the orders? How many packages should be sent from each warehouse to each online bookstore at a minimum shipping cost?

42. **Transportation Scheduling** Duffin Press is about to start a promotional blitz for its new book, *Physics for the Liberal Arts*. The company has 20 salespeople stationed in Chicago and 10 in Denver, and would like to fly at least 15 to sales fairs in each of Los Angeles and New York. A round-trip plane flight from Chicago to LA costs $200; from Chicago to NY costs $150; from Denver to LA costs $400; and from Denver to NY costs $200. How many salespeople should the company fly from each of Chicago and Denver to each of LA and NY for the lowest total cost in air fare?

43. **Marketing** Marjory Duffin, head of Duffin House, reveals to John O’Hagan that FantasyBooks.com is considering several promotional schemes: It may offer two books for the price of one, three books for the price of two, or possibly a free copy of *Brain Surgery for Klutzes* with each order. O’HaganBooks.com’s marketing advisers Floody and O’Lara seem to have different ideas as to how to respond. Floody suggests offering three books for the price of one, while O’Lara suggests instead offering a free copy of the *Finite Mathematics Student Solutions Manual* with every purchase. After a careful analysis, O’Hagan comes up with the following payoff matrix, where the payoffs represent the number of customers, in thousands, O’Hagan expects to gain from FantasyBooks.com.

<table>
<thead>
<tr>
<th></th>
<th>FantasyBooks.com</th>
<th>OHaganBooks.com</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Promo</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>2 for Price of 1</td>
<td>60</td>
<td>20</td>
</tr>
<tr>
<td>3 for Price of 2</td>
<td>100</td>
<td>15</td>
</tr>
<tr>
<td>Finite Math</td>
<td>200</td>
<td>0</td>
</tr>
</tbody>
</table>

Find the optimal strategies for both companies and the expected shift in customers.

44. **Study Techniques** Billy-Sean’s friend Pat from college has been spending all of his time in fraternity activities, and thus knows absolutely nothing about any of the three topics on tomorrow’s math test. He has turned to Billy-Sean for advice as to how to spend his “all-nighter.” The table below shows the scores Pat could expect to earn if the entire test were to be in a specific subject. (Because he knows no linear programming or matrix algebra, the table shows, for instance, that studying game theory all night will not be much use in preparing him for this topic.)

<table>
<thead>
<tr>
<th>Pat’s Strategies</th>
<th>Game Theory</th>
<th>Linear Programming</th>
<th>Matrix Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Study Game Theory</td>
<td>30</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>Study Linear Programming</td>
<td>0</td>
<td>70</td>
<td>0</td>
</tr>
<tr>
<td>Study Matrix Algebra</td>
<td>0</td>
<td>0</td>
<td>70</td>
</tr>
</tbody>
</table>

What percentage of the night should Pat spend on each topic, assuming the principles of game theory, and what score can he expect to get?
The Galaxy Nutrition health-food mega-store chain provides free online nutritional advice and support to its customers. As Web site technical consultant, you are planning to construct an interactive Web page to assist customers in preparing a diet tailored to their nutritional and budgetary requirements. Ideally, the customer would select foods to consider and specify nutritional and/or budgetary constraints, and the tool should return the optimal diet meeting those requirements. You would also like the Web page to allow the customer to decide whether, for instance, to find the cheapest possible diet meeting the requirements, the diet with the lowest number of calories, or the diet with the least total carbohydrates.

After doing a little research, you notice that the kind of problem you are trying to solve is quite well known and referred to as the diet problem, and that solving the diet problem is a famous example of linear programming. Indeed, there are already some online pages that solve versions of the problem that minimize total cost, so you have adequate information to assist you as you plan the page.*

You decide to start on a relatively small scale, starting with a program that uses a list of 10 foods, and minimizes either total caloric intake or total cost and satisfies a small list of requirements. Following is a small part of a table of nutritional information from the Argonne National Laboratory Web site (all the values shown are for a single serving) as well as approximate minimum daily requirements:

* NOTE See, for instance, the Diet Problem page at the Argonne National Laboratory Web site: http://www-neos.mcs.anl.gov/CaseStudies/dietpy/WebForms/index.html

<table>
<thead>
<tr>
<th></th>
<th>Price per Serving</th>
<th>Calories</th>
<th>Total Fat g</th>
<th>Carbs g</th>
<th>Dietary Fiber g</th>
<th>Protein g</th>
<th>Vit C IU</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tofu</td>
<td>$0.31</td>
<td>88.2</td>
<td>5.5</td>
<td>2.2</td>
<td>1.4</td>
<td>9.4</td>
<td>0.1</td>
</tr>
<tr>
<td>Roast Chicken</td>
<td>$0.84</td>
<td>277.4</td>
<td>10.8</td>
<td>0</td>
<td>0</td>
<td>42.2</td>
<td>0</td>
</tr>
<tr>
<td>Spaghetti w/Sauce</td>
<td>$0.78</td>
<td>358.2</td>
<td>12.3</td>
<td>58.3</td>
<td>11.6</td>
<td>8.2</td>
<td>27.9</td>
</tr>
<tr>
<td>Tomato</td>
<td>$0.27</td>
<td>25.8</td>
<td>0.4</td>
<td>5.7</td>
<td>1.4</td>
<td>1.0</td>
<td>23.5</td>
</tr>
<tr>
<td>Oranges</td>
<td>$0.15</td>
<td>61.6</td>
<td>0.2</td>
<td>15.4</td>
<td>3.1</td>
<td>1.2</td>
<td>69.7</td>
</tr>
<tr>
<td>Wheat Bread</td>
<td>$0.05</td>
<td>65.0</td>
<td>1.0</td>
<td>12.4</td>
<td>1.3</td>
<td>2.2</td>
<td>0</td>
</tr>
<tr>
<td>Cheddar Cheese</td>
<td>$0.25</td>
<td>112.7</td>
<td>9.3</td>
<td>0.4</td>
<td>0</td>
<td>7.0</td>
<td>0</td>
</tr>
<tr>
<td>Oatmeal</td>
<td>$0.82</td>
<td>145.1</td>
<td>2.3</td>
<td>25.3</td>
<td>4.0</td>
<td>6.1</td>
<td>0</td>
</tr>
<tr>
<td>Peanut Butter</td>
<td>$0.07</td>
<td>188.5</td>
<td>16.0</td>
<td>6.9</td>
<td>2.1</td>
<td>7.7</td>
<td>0</td>
</tr>
<tr>
<td>White Tuna in Water</td>
<td>$0.69</td>
<td>115.6</td>
<td>2.1</td>
<td>0</td>
<td>0</td>
<td>22.7</td>
<td>0</td>
</tr>
<tr>
<td>Minimum requirements</td>
<td>2,200</td>
<td>20</td>
<td>80</td>
<td>25</td>
<td>60</td>
<td>90</td>
<td></td>
</tr>
</tbody>
</table>

Source: www-neos.mcs.anl.gov/CaseStudies/dietpy/WebForms/complete_table.html

Now you get to work. As always, you start by identifying the unknowns. Since the output of the Web page will consist of a recommended diet, the unknowns should logically
be the number of servings of each item of food selected by the user. In your first trial run, you decide to include all the 10 food items listed, so you take

\[
\begin{align*}
  x_1 &= \text{Number of servings of tofu} \\
  x_2 &= \text{Number of servings of roast chicken} \\
  \vdots \\
  x_{10} &= \text{Number of servings of white tuna in water.}
\end{align*}
\]

You now set up a linear programming problem for two sample scenarios:

**Scenario 1 (Minimum Cost):** Satisfy all minimum nutritional requirements at a minimum cost. Here the linear programming problem is:

Minimize

\[
\begin{align*}
  c &= 0.31x_1 + 0.84x_2 + 0.78x_3 + 0.27x_4 + 0.15x_5 + 0.05x_6 + 0.25x_7 \\
  &\quad + 0.82x_8 + 0.07x_9 + 0.69x_{10}
\end{align*}
\]

Subject to

\[
\begin{align*}
  88.2x_1 + 277.4x_2 + 358.2x_3 + 25.8x_4 + 61.6x_5 + 65x_6 + 112.7x_7 \\
  &\quad + 145.1x_8 + 188.5x_9 + 115.6x_{10} \geq 2,200 \\
  5.5x_1 + 10.8x_2 + 12.3x_3 + 0.4x_4 + 0.2x_5 + 1x_6 + 9.3x_7 + 2.3x_8 \\
  &\quad + 16x_9 + 2.1x_{10} \geq 20 \\
  2.2x_1 + 58.3x_2 + 5.7x_3 + 15.4x_5 + 12.4x_6 + 0.4x_7 + 25.3x_8 + 6.9x_9 \geq 80 \\
  1.4x_1 + 11.6x_3 + 1.4x_4 + 3.1x_5 + 1.3x_6 + 4x_8 + 2.1x_9 \geq 25 \\
  9.4x_1 + 42.2x_2 + 8.2x_3 + 1x_4 + 1.2x_5 + 2.2x_6 + 7x_7 + 6.1x_8 + 7.7x_9 \\
  &\quad + 22.7x_{10} \geq 60 \\
  0.1x_1 + 27.9x_3 + 23.5x_4 + 69.7x_5 \geq 90.
\end{align*}
\]

This is clearly the kind of linear programming problem no one in their right mind would like to do by hand (solving it requires 16 tableaux!) so you decide to use the online simplex method tool at the Web site (www.FiniteMath.org → Student Web Site → Online Utilities → Simplex Method Tool).

Here is a picture of the input, entered almost exactly as written above (you need to enter each constraint on a new line, and “Minimize \( c = 0.31x_1 + \cdots \) Subject to” must be typed on a single line):

![Image of input]

Clicking “Solve” results in the following solution:

\[
\begin{align*}
  c &= 0.981126; \\
  x_1 &= 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 1.29125, x_6 = 0, x_7 = 0, \\
  x_8 = 0, x_9 &= 11.2491, x_{10} = 0.
\end{align*}
\]

This means that you can satisfy all the daily requirements for less than $1 on a diet of 1.3 servings of orange juice and 11.2 servings of peanut butter! Although you enjoy...
peanut butter, 11.2 servings seems a little over the top, so you modify the LP problem by adding a new constraint (which also suggests to you that some kind of flexibility needs to be built into the site to allow users to set limits on the number of servings of any one item):

$$x_9 \leq 3.$$  

This new constraint results in the following solution:

$$c = 1.59981; x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 1.29125, x_6 = 23.9224,$$
$$x_7 = 0, x_8 = 0, x_9 = 3, x_{10} = 0.$$

Because wheat bread is cheap and, in large enough quantities, supplies ample protein, the program has now substituted 23.9 servings of wheat bread for the missing peanut butter for a total cost of $1.60.

Unfettered, you now add

$$x_5 \leq 4$$

and obtain the following spaghetti, bread, and peanut butter diet for $3.40 per day:

$$c = 3.40305; x_1 = 0, x_2 = 0, x_3 = 3.83724, x_4 = 0, x_5 = 0, x_6 = 4, x_7 = 0,$$
$$x_8 = 0, x_9 = 3, x_{10} = 0.$$

**Scenario 2 (Minimum Calories):** Minimize total calories and satisfy all minimum nutritional requirements (except for caloric intake).

Here, the linear programming problem is

Minimize

$$c = 88.2x_1 + 277.4x_2 + 358.2x_3 + 25.8x_4 + 61.6x_5 + 65x_6 + 112.7x_7$$
$$+ 145.1x_8 + 188.5x_9 + 115.6x_{10}$$

Subject to

$$5.5x_1 + 10.8x_2 + 12.3x_3 + 0.4x_4 + 0.2x_5 + 1x_6 + 9.3x_7 + 2.3x_8 + 16x_9$$
$$+ 2.1x_{10} \geq 20$$
$$2.2x_1 + 58.3x_3 + 5.7x_4 + 15.4x_5 + 12.4x_6 + 0.4x_7 + 25.3x_8 + 6.9x_9 \geq 80$$
$$1.4x_1 + 11.6x_3 + 1.4x_4 + 3.1x_5 + 1.3x_6 + 4x_8 + 2.1x_9 \geq 25$$
$$9.4x_1 + 42.2x_2 + 8.2x_3 + 1x_4 + 1.2x_5 + 2.2x_6 + 7x_7 + 6.1x_8 + 7.7x_9$$
$$+ 22.7x_{10} \geq 60$$
$$0.1x_1 + 27.9x_3 + 23.5x_4 + 69.7x_5 \geq 90.$$

You obtain the following 716-calorie tofu, tomato, and tuna diet:

$$x_1 = 2.07232, x_2 = 0, x_3 = 0, x_4 = 15.7848, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = 0,$$
$$x_9 = 0, x_{10} = 1.08966.$$

As 16 servings of tomatoes seems a little over the top, you add the new constraint $$x_4 \leq 3$$

and obtain a 783-calorie tofu, tomato, orange, and tuna diet:

$$x_1 = 2.81682, x_2 = 0, x_3 = 0, x_4 = 3, x_5 = 5.43756, x_6 = 0, x_7 = 0, x_8 = 0,$$
$$x_9 = 0, x_{10} = 1.05713.$$

What the trial runs have shown you is that your Web site will need to allow the user to set reasonable upper bounds for the number of servings of each kind of food considered. You now get to work writing the algorithm, which appears here:

www.FiniteMath.org → Student Web Site → Online Utilities → Diet Problem Solver
EXERCISES

1. Briefly explain why roast chicken, which supplies protein more cheaply than either tofu or tuna, does not appear in the optimal solution in either scenario.

2. Consider the optimal solution obtained in Scenario 1 when peanut butter and bread were restricted. Experiment on the Simplex Method tool by increasing the protein requirement 10 grams at a time until chicken appears in the optimal diet. At what level of protein does the addition of chicken first become necessary?

3. What constraints would you add for a person who wants to eat at most two servings of chicken a day and is allergic to tomatoes and peanut butter? What is the resulting diet for Scenario 2?

4. What is the linear programming problem for someone who wants as much protein as possible at a cost of no more than $6 per day with no more than 50 g of carbohydrates per day assuming they want to satisfy the minimum requirements for all the remaining nutrients. What is the resulting diet?

5. Is it possible to obtain a diet with no bread or peanut butter in Scenario 1 costing less than $4 per day?
When we introduce slack variables, we get the following system of equations:
\[
\begin{align*}
    x + s &= 600 \\
    2x + 3y + t &= 3,600 \\
    5x + 3y + u &= 4,500 \\
    -10x - 7y + p &= 0 
\end{align*}
\]

We use the PIVOT program for the TI-83/84 Plus to help with the simplex method. This program is available at the Web site by following Everything for Finite Math → Math Tools for Chapter 4. Because the calculator handles decimals as easily as integers, there is no need to avoid them, except perhaps to save limited screen space. If we don’t need to avoid decimals, we can use the traditional Gauss-Jordan method (see the discussion at the end of Section 2.2): After selecting your pivot, and prior to clearing the pivot column, divide the pivot row by the value of the pivot, thereby turning the pivot into a 1.

The main drawback to using the TI-83/84 Plus is that we can’t label the rows and columns. We can mentally label them as we go, but we can do without labels entirely if we wish. We begin by entering the initial tableau as the matrix \( A \). (Another drawback to using the TI-83/84 Plus is that it can’t show the whole tableau at once. Here and below we show tableaux across several screens. Use the TI-83/84 Plus’s arrow keys to scroll a matrix left and right so you can see all of it.)

**Section 4.1**

Some calculators, including the TI-83/84 Plus, will shade one side of a graph, but you need to tell the calculator which side to shade. For instance, to obtain the solution set of \( 2x + 3y \leq 6 \) shown in Figure 4:

1. Solve the corresponding equation \( 2x + 3y = 6 \) for \( y \) and use the following input:

2. The icon to the left of “Y1” tells the calculator to shade above the line. You can cycle through the various shading options by positioning the cursor to the left of Y1 and pressing **ENTER** until you see the one you want. Here’s what the graph will look like:

**Section 4.3**

**Example 3 (page 298)** The Acme Baby Foods example in the text leads to the following linear programming problem:

Maximize \( p = 10x + 7y \)

subject to
\[
\begin{align*}
    x &\leq 600 \\
    2x + 3y &\leq 3,600 \\
    5x + 3y &\leq 4,500 \\
    x &\geq 0, \quad y &\geq 0. 
\end{align*}
\]

Solve it using technology.
After determining that the next pivot is in the third row and second column, we divide the third row by the pivot, 3, and then pivot:

The next pivot is the 3 in the second row, third column. We divide its row by 3 and pivot:

Afterward, we continue the process of finding the next pivot and performing the necessary operations to reach the optimal solution.
There are no negative numbers in the bottom row, so we’re finished. How do we read off the optimal solutions if we don’t have labels, though? Look at the columns containing one 1 and three 0s. They are the $x$ column, the $y$ column, the $s$ column, and the $p$ column. Think of the 1 that appears in each of these columns as a pivot whose column has been cleared. If we had labels, the row containing a pivot would have the same label as the column containing that pivot. We can now read off the solution as follows:

- **$x$ column**: The pivot is in the first row, so row 1 would have been labeled with $x$. We look at the rightmost column to read off the value $x = 300$.
- **$y$ column**: The pivot is in row 3, so we look at the rightmost column to read off the value $y = 1,000$.
- **$s$ column**: The pivot is in row 2, so we look at the rightmost column to read off the value $s = 300$.
- **$p$ column**: The pivot is in row 3, so we look at the rightmost column to read off the value $p = 10,000$.

Thus, the maximum value of $p$ is $10,000¢ = $100, which occurs when $x = 300$ and $y = 1,000$. The values of the slack variables are $s = 300$ and $t = u = 0$. (Look at the $t$ and $u$ columns to see that they must be inactive.)

---

**EXCEL Technology Guide**

**Section 4.1**

Excel is not a particularly good tool for graphing linear inequalities because it cannot easily shade one side of a line. One solution is to use the “error bar” feature to indicate which side of the line should be shaded. For example, here is how we might graph the inequality $2x + 3y \leq 6$.

1. Create a scatter graph using two points to construct a line segment (as in Chapter 1). (Notice that we had to solve the equation $2x + 3y = 6$ for $y$.)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>2</td>
<td>-10</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>2</td>
</tr>
</tbody>
</table>

2. Double-click on the line-segment, and use the “X-Error Bars” feature to obtain a diagram similar to the one below, where the error bars indicate the direction of shading.

Alternatively, you can use the Drawing Palette to create a polygon with a semi-transparent fill, as shown above.

**Section 4.3**

**Example 3 (page 298)** The Acme Baby Foods example in the text leads to the following linear programming problem:

Maximize $p = 10x + 7y$

subject to

\[
\begin{align*}
2x + 3y & \leq 3,600 \\
5x + 3y & \leq 4,500 \\
x & \geq 0, \quad y \geq 0.
\end{align*}
\]

Solve it using technology.
Solution with Technology

When we introduce slack variables, we get the following system of equations:

\[
\begin{align*}
    x + s &= 600 \\
    2x + 3y + t &= 3,600 \\
    5x + 3y + u &= 4,500 \\
    -10x - 7y + p &= 0.
\end{align*}
\]

We use the Excel Row Operation and Pivoting Utility to help with the simplex method. This spreadsheet is available at the Web site by following Everything for Finite Math → Math Tools for Chapter 4.

Because Excel handles decimals as easily as integers, there is no need to avoid them. If we don’t need to avoid decimals, we can use the traditional Gauss-Jordan method (see the discussion at the end of Section 2.2): After selecting your pivot, and prior to clearing the pivot column, divide the pivot row by the value of the pivot, thereby turning the pivot into a 1.

The main drawback to using a tool like Excel is that it will not automatically keep track of row and column labels. We could enter and modify labels by hand as we go, or we can do without them entirely if we wish. The following is the sequence of tableaux we get while using the simplex method with the help of the pivoting tool.

There are no negative numbers in the bottom row, so we’re finished. How do we read off the optimal solutions if we don’t have labels, though? Look at the columns containing one 1 and three 0s. They are the \( x \) column, the \( y \) column, the \( s \) column, and the \( p \) column. Think of the 1 that appears in each of these columns as a pivot whose column has been cleared. If we had labels, the row containing a pivot would have the same label as the column containing that pivot. We can now read off the solution as follows:

\( x \) column: The pivot is in the first row, so row 1 would have been labeled with \( x \). We look at the rightmost column to read off the value \( x = 300 \).

\( y \) column: The pivot is in row 3, so we look at the rightmost column to read off the value \( y = 1,000 \).

\( s \) column: The pivot is in row 2, so we look at the rightmost column to read off the value \( s = 300 \).

\( p \) column: The pivot is in row 3, so we look at the rightmost column to read off the value \( p = 10,000 \).

Thus, the maximum value of \( p \) is \( 10,000 \times \$1 = \$100 \), which occurs when \( x = 300 \) and \( y = 1,000 \). The values of the slack variables are \( s = 300 \) and \( t = u = 0 \). (Look at the \( t \) and \( u \) columns to see that they must be inactive.)