SECTION 5.5
Solving Systems of Equations by Using Determinants

OBJECTIVE A
To evaluate a determinant

A matrix is a rectangular array of numbers. Each number in a matrix is called an element of the matrix. The matrix at the right, with three rows and four columns, is called a \(3 \times 4\) (read "3 by 4") matrix.

A matrix of \(m\) rows and \(n\) columns is said to be of order \(m \times n\). The matrix above has order \(3 \times 4\). The notation refers to the element of a matrix in the \(i\)th row and the \(j\)th column. For matrix \(A\), \(a_{31} = -3\), \(a_{11} = 6\), and \(a_{13} = 2\).

A square matrix is one that has the same number of rows as columns. A matrix and a matrix are shown at the right. Associated with every square matrix is a number called its determinant.

**Determinant of a \(2 \times 2\) Matrix**

The determinant of a \(2 \times 2\) matrix \[
\begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix}
\]

is written \[
\begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix}
\]

The value of this determinant is given by the formula

\[
\det(AB) = a_{11}a_{22} - a_{12}a_{21}
\]

**HOW TO • 1** Find the value of the determinant \[
\begin{vmatrix}
3 & 4 \\
-1 & 2
\end{vmatrix}
\]

\[
\begin{vmatrix}
3 & 4 \\
-1 & 2
\end{vmatrix} = 3 \cdot 2 - 4(-1) = 6 + 4 = 10
\]

The value of the determinant is \(10\).

For a square matrix whose order is \(3 \times 3\) or greater, the value of the determinant is found by using \(2 \times 2\) determinants.

The minor of an element in a \(3 \times 3\) determinant is the \(2 \times 2\) determinant that is obtained by eliminating the row and column that contain that element.

**HOW TO • 2** Find the minor of \(-3\) for the determinant

\[
\begin{vmatrix}
2 & -3 & 4 \\
0 & 4 & 8 \\
-1 & 3 & 6
\end{vmatrix}
\]

The minor of \(-3\) is the \(2 \times 2\) determinant created by eliminating the row and column that contain \(-3\).

\[
\begin{vmatrix}
2 & -3 & 4 \\
0 & 4 & 8 \\
-1 & 3 & 6
\end{vmatrix}
\]

Eliminate the row and column as shown:

\[
\begin{vmatrix}
-2 & 4 \\
0 & 8 \\
-1 & 6
\end{vmatrix}
\]

The minor of \(-3\) is \[
\begin{vmatrix}
0 & 8 \\
-1 & 6
\end{vmatrix}
\]

The word matrix was first used in a mathematical context in 1850. The root of this word is the Latin mater, meaning mother. A matrix was thought of as an object from which something else originates. The idea was that a determinant, discussed below, originated (was born) from a matrix. Today, matrices are one of the most widely used tools of applied mathematics.
Cofactor of an Element of a Matrix

The cofactor of an element of a matrix is \((-1)^{i+j}\) times the minor of that element, where \(i\) is the row number of the element and \(j\) is the column number of the element.

**HOW TO 3**

For the determinant \[
\begin{vmatrix}
3 & -2 & 1 \\
2 & -5 & -4 \\
0 & 3 & 1
\end{vmatrix}
\]
find the cofactor of \(-2\) and of \(-5\).

Because \(-2\) is in the first row and the second column, \(i = 1\) and \(j = 2\). Thus \((-1)^{i+j} = (-1)^{1+2} = (-1)^3 = -1\). The cofactor of \(-2\) is \((-1)^{i+j} \cdot M_{ij}\) where \(M_{ij}\) is the minor of \(-2\).

Because \(-5\) is in the second row and the second column, \(i = 2\) and \(j = 2\). Thus \((-1)^{i+j} = (-1)^{2+2} = (-1)^4 = 1\). The cofactor of \(-5\) is \((-1)^{i+j} \cdot M_{ij}\) where \(M_{ij}\) is the minor of \(-5\).

Note from this example that the cofactor of an element is \(-1\) times the minor of that element or \(1\) times the minor of that element, depending on whether the sum \(i + j\) is an odd or an even integer.

The value of a \(3 \times 3\) or larger determinant can be found by expanding by cofactors of any row or any column.

**HOW TO 4**

Find the value of the determinant \[
\begin{vmatrix}
2 & -3 & 2 \\
1 & 3 & -1 \\
0 & -2 & 2
\end{vmatrix}
\]

We will expand by cofactors of the first row. Any row or column would work.

\[
\begin{vmatrix}
2 & -3 & 2 \\
1 & 3 & -1 \\
0 & -2 & 2
\end{vmatrix} = 2(-1)^{1+2} \begin{vmatrix}
-2 & 1 \\
-2 & 2
\end{vmatrix} + 3(-1)(-1)^{1+2} \begin{vmatrix}
-2 & 1 \\
-2 & 2
\end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix}
-2 & 1 \\
-2 & 2
\end{vmatrix}
\]

\[
= 2(1) \begin{vmatrix}
3 & -1 \\
-2 & 2
\end{vmatrix} + 3(-1)(-1) \begin{vmatrix}
1 & -1 \\
0 & 2
\end{vmatrix} + 2(1) \begin{vmatrix}
1 & 3 \\
0 & -2
\end{vmatrix}
\]

\[
= 2(6 - 2) + 3(2 - 0) + 2(-2) - 0
\]

\[
= 2(4) + 3(2) + 2(-2) = 8 + 6 - 4 = 10
\]

To illustrate that any row or column can be chosen when expanding by cofactors, we will now show the evaluation of the same determinant by expanding by cofactors of the second column.

\[
\begin{vmatrix}
2 & -3 & 2 \\
1 & 3 & -1 \\
0 & -2 & 2
\end{vmatrix} = -3(-1)^{1+2} \begin{vmatrix}
1 & -1 \\
0 & 2
\end{vmatrix} + 3(-1)^{2+2} \begin{vmatrix}
2 & 2 \\
0 & 2
\end{vmatrix} + (-2)(-1)^{1+2} \begin{vmatrix}
2 & 2 \\
1 & -1
\end{vmatrix}
\]

\[
= -3(-1) \begin{vmatrix}
1 & -1 \\
0 & 2
\end{vmatrix} + 3(1) \begin{vmatrix}
2 & 2 \\
0 & 2
\end{vmatrix} + (-2)(-1) \begin{vmatrix}
2 & 2 \\
1 & -1
\end{vmatrix}
\]

\[
= 3(2 - 0) + 3(4 - 0) + 2(-2) - 2
\]

\[
= 3(2) + 3(4) + 2(-2) = 6 + 12 + (-8) = 10
\]
Note that the value of the determinant is the same whether the first row or the second column is used to expand by cofactors. Any row or column can be used to evaluate a determinant by expanding by cofactors.

**EXAMPLE • 1**

Find the value of $\begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix}$.

**Solution**

$$\begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = 3(-4) - (-2)(6) = -12 + 12 = 0$$

The value of the determinant is 0.

**YOU TRY IT • 1**

Find the value of $\begin{vmatrix} -1 & -4 \\ 3 & -5 \end{vmatrix}$.

**EXAMPLE • 2**

Find the value of $\begin{vmatrix} -2 & 3 & 1 \\ 4 & -2 & 0 \\ 1 & -2 & 3 \end{vmatrix}$.

**Solution**

Expand by cofactors of the first row.

$$\begin{vmatrix} -2 & 3 & 1 \\ 4 & -2 & 0 \\ 1 & -2 & 3 \end{vmatrix} = -2 \begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= -2(-6 - 0) + 3(-4 + 0) + 1(-4 - 2)$$

$$= -12 - 36 - 6$$

$$= -30$$

The value of the determinant is −30.

**YOU TRY IT • 2**

Find the value of $\begin{vmatrix} 1 & 4 & -2 \\ 3 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix}$.

**EXAMPLE • 3**

Find the value of $\begin{vmatrix} 0 & -2 & 1 \\ 1 & 4 & 1 \\ 2 & -3 & 4 \end{vmatrix}$.

**Solution**

$$\begin{vmatrix} 0 & -2 & 1 \\ 1 & 4 & 1 \\ 2 & -3 & 4 \end{vmatrix} = 0 \begin{vmatrix} 4 & 1 \\ 2 & -3 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} + 1 \begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix}$$

$$= 0 - (-2)(4 - 2) + 1(-3 + 8)$$

$$= 2(2) + 1(-11)$$

$$= 4 - 11$$

$$= -7$$

The value of the determinant is −7.

**YOU TRY IT • 3**

Find the value of $\begin{vmatrix} 3 & -2 & 0 \\ 1 & 4 & 2 \\ -2 & 1 & 3 \end{vmatrix}$.

*Solutions on p. S1*
Cramer's Rule

The solution of the system of equations

\[ a_1x + b_1y = c_1 \]
\[ a_2x + b_2y = c_2 \]

is given by

\[ x = \frac{D_x}{D} \]
\[ y = \frac{D_y}{D} \]

where

\[ D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \]
\[ D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \]
\[ D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \]

and \( D \neq 0 \).

**HOW TO 5**

Solve by using Cramer’s Rule:

\[ 3x - 2y = 1 \]
\[ 2x + 5y = 3 \]

- Find the value of the coefficient determinant.
- Find the value of each of the numerator determinants.
- Use Cramer’s Rule to write the solution.

The solution is \( \left( \frac{11}{19}, \frac{7}{19} \right) \).

Cramer’s Rule is named after Gabriel Cramer, who used it in a book he published in 1750. However, this rule was also published in 1683 by the Japanese mathematician Seki Kowa. That publication occurred seven years before Cramer’s birth.

**Point of Interest**

The connection between determinants and systems of equations can be understood by solving a general system of linear equations.

Solve:

\[ (1) \quad a_1x + b_1y = c_1 \]
\[ (2) \quad a_2x + b_2y = c_2 \]

Eliminate \( y \). Multiply Equation (1) by and Equation (2) by \(-b_1\).

\[ a_1b_2x + b_1b_2y = c_1b_2 \]
\[ -a_2b_1x - b_1b_2y = -c_2b_1 \]

Add the equations.

\[ a_1b_2x - a_1b_1x = c_1b_2 - c_2b_1 \]
\[ (a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1 \]
\[ x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \]

The numerator \( c_1b_2 - c_2b_1 \) is the determinant obtained by replacing the first column in the coefficient determinant by the constants \( c_1 \) and \( c_2 \). This is called a numerator determinant.

The denominator \( a_1b_2 - a_2b_1 \) is the determinant of the coefficients of \( x \) and \( y \). This is called the coefficient determinant.

By following a similar procedure and eliminating \( x \), it is also possible to express the \( y \)-component of the solution in determinant form. These results are summarized in Cramer’s Rule.
A procedure similar to that followed for two equations in two variables can be used to extend Cramer’s Rule to three equations in three variables.

**Cramer’s Rule for a System of Three Equations in Three Variables**

The solution of the system of equations

\[ \begin{align*}
    a_1x + b_1y + c_1z &= d_1, \\
    a_2x + b_2y + c_2z &= d_2, \\
    a_3x + b_3y + c_3z &= d_3,
\end{align*} \]

is given by

\[ x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}, \]

where

\[ D = \begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3
\end{vmatrix}, \quad D_x = \begin{vmatrix}
    d_1 & b_1 & c_1 \\
    d_2 & b_2 & c_2 \\
    d_3 & b_3 & c_3
\end{vmatrix}, \quad D_y = \begin{vmatrix}
    a_1 & d_1 & c_1 \\
    a_2 & d_2 & c_2 \\
    a_3 & d_3 & c_3
\end{vmatrix}, \quad D_z = \begin{vmatrix}
    a_1 & b_1 & d_1 \\
    a_2 & b_2 & d_2 \\
    a_3 & b_3 & d_3
\end{vmatrix}, \quad \text{and } D \neq 0.
\]

**HOW TO 6**

Solve by using Cramer’s Rule:

\[ \begin{align*}
    2x - y + z &= 1, \\
    x + 3y - 2z &= -2, \\
    3x + y + 3z &= 4.
\end{align*} \]

Find the value of the coefficient determinant.

\[ D = \begin{vmatrix}
    2 & -1 & 1 \\
    1 & 3 & -2 \\
    3 & 1 & 3
\end{vmatrix} = 2(11) + 1(9) + 1(-8) = \frac{23}{3}. \]

Find the value of each of the numerator determinants.

\[ D_x = \begin{vmatrix}
    1 & -1 & 1 \\
    -2 & 3 & -2 \\
    4 & 1 & 3
\end{vmatrix} = 1(11) + 1(2) + 1(-14) = -1. \]

\[ D_y = \begin{vmatrix}
    2 & 1 & 1 \\
    1 & -2 & -2 \\
    3 & 4 & 3
\end{vmatrix} = 2(2) - 1(9) + 1(10) = 5. \]

\[ D_z = \begin{vmatrix}
    2 & -1 & 1 \\
    1 & 3 & -2 \\
    3 & 1 & 4
\end{vmatrix} = 2(14) + 1(10) + 1(-8) = 30. \]

Use Cramer’s Rule to write the solution.

\[ x = \frac{D_x}{D} = \frac{-1}{23}, \quad y = \frac{D_y}{D} = \frac{5}{23}, \quad z = \frac{D_z}{D} = \frac{30}{23}. \]

The solution is \((-\frac{1}{23}, \frac{5}{23}, \frac{30}{23})\).
EXAMPLE 4
Solve by using Cramer’s Rule.
6x − 9y = 5
4x − 6y = 4

Solution
\[
D = \begin{vmatrix} 6 & -9 \\ 4 & -6 \end{vmatrix} = 0
\]

Because \( D = 0 \), \( D_x \) is undefined. Therefore, the system is dependent or inconsistent.

YOU TRY IT 4
Solve by using Cramer’s Rule.
3x − y = 4
6x − 2y = 5

Your solution

EXAMPLE 5
Solve by using Cramer’s Rule.
3x − y + z = 5
x + 2y − 2z = −3
2x + 3y + z = 4

Solution
\[
D = \begin{vmatrix} 3 & -1 & 1 \\ 1 & 2 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 28,
\]
\[
D_x = \begin{vmatrix} 5 & -1 & 1 \\ -3 & 2 & -2 \\ 3 & 5 & 1 \end{vmatrix} = 28,
\]
\[
D_y = \begin{vmatrix} 3 & 5 & 1 \\ 1 & 2 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 0,
\]
\[
D_z = \begin{vmatrix} 3 & 1 & 5 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 56
\]
\[
x = \frac{D_x}{D} = \frac{28}{28} = 1
\]
\[
y = \frac{D_y}{D} = \frac{0}{28} = 0
\]
\[
z = \frac{D_z}{D} = \frac{56}{28} = 2
\]
The solution is (1, 0, 2).

YOU TRY IT 5
Solve by using Cramer’s Rule.
2x − y + z = −1
3x + 2y − z = 3
x + 3y + z = −2

Your solution

Solutions on p. S1
### Objective A: To evaluate a determinant

1. How do you find the value of the determinant associated with a $2 \times 2$ matrix?

2. What is the cofactor of a given element in a matrix?

For Exercises 3 to 14, evaluate the determinant.

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<td>14</td>
<td>4</td>
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</table>

15. What is the value of a determinant for which one row is all zeros?

16. What is the value of a determinant for which all the elements are the same number?

### Objective B: To solve a system of equations by using Cramer’s Rule

For Exercises 17 to 34, solve by using Cramer’s Rule.

17. $2x - 5y = 26$
   $5x + 3y = 3$

18. $3x + 7y = 15$
   $2x + 5y = 11$

19. $x - 4y = 8$
   $3x + 7y = 5$

20. $5x + 2y = -5$
   $3x + 4y = 11$

21. $2x + 3y = 4$
   $6x - 12y = -5$

22. $5x + 4y = 3$
   $15x - 8y = -21$

23. $2x + 5y = 6$
   $6x - 2y = 1$

24. $7x + 3y = 4$
   $5x - 4y = 9$
25. \(-2x + 3y = 7 \quad 26. \quad 9x + 6y = 7 \quad 27. \quad 2x - 5y = -2 \quad 28. \quad 8x + 7y = -3\)
\[
\begin{align*}
4x - 6y &= 9 \\
3x + 2y &= 4 \\
3x - 7y &= -3 \\
2x + 2y &= 5
\end{align*}
\]

29. \(2x - y + 3z = 9 \quad 30. \quad 3x - 2y + z = 2 \quad 31. \quad 3x - y + z = 11\)
\[
\begin{align*}
-x + 4y + 4z &= 5 \\
x + 3y + 2z &= -6 \\
x + 4y - 2z &= -12
\end{align*}
\]

32. \(x + 2y + 3z = 8 \quad 33. \quad 4x - 2y + 6z = 1 \quad 34. \quad x - 3y + 2z = 1\)
\[
\begin{align*}
2x - 3y + z &= 5 \\
3x + 4y + 2z &= 1 \\
2x - y + 3z &= 2
\end{align*}
\]

35. Can Cramer’s Rule be used to solve a dependent system of equations?

36. Suppose a system of linear equations in two variables has \(D_x \neq 0, D_y \neq 0,\) and \(D = 0.\) Is the system of equations independent, dependent, or inconsistent?

Applying the Concepts

37. Determine whether the following statements are always true, sometimes true, or never true.
   a. The determinant of a matrix is a positive number.
   b. A determinant can be evaluated by expanding about any row or column of the matrix.
   c. Cramer’s Rule can be used to solve a system of linear equations in three variables.

38. Show that \[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}.
\]

39. Surveying

Surveyors use a formula to find the area of a plot of land. The surveyor’s area formula states that if the vertices \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) of a simple polygon are listed counterclockwise around the perimeter, then the area of the polygon is
\[
A = \frac{1}{2} \left| \begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array} \right| + \left| \begin{array}{cc} x_2 & x_3 \\ y_2 & y_3 \end{array} \right| + \left| \begin{array}{cc} x_3 & x_4 \\ y_3 & y_4 \end{array} \right| + \cdots + \left| \begin{array}{cc} x_n & x_1 \\ y_n & y_1 \end{array} \right|
\]
Use the surveyor’s area formula to find the area of the polygon with vertices \((9, -3), (26, 6), (18, 21), (16, 10),\) and \((1, 11).\) Measurements are given in feet.
SOLUTIONS TO CHAPTER 5 “YOU TRY IT”

SECTION 5.5

You Try It 1
\[
\begin{vmatrix}
0 & -1 & 4 \\
-3 & 0 & -5 \\
3 & 1 & 3 \\
\end{vmatrix}
= (-1)(-5) - (-4)(3)
= 5 + 12 = 17
\]
The value of the determinant is 17.

You Try It 2
Expand by cofactors of the first row.
\[
\begin{vmatrix}
1 & 4 & -2 \\
3 & 1 & 1 \\
0 & -2 & 2 \\
\end{vmatrix}
= 1 \left| \begin{array}{cc}
1 & 1 \\
0 & 2 \\
\end{array} \right| - 4 \left| \begin{array}{cc}
3 & 1 \\
0 & -2 \\
\end{array} \right| + (-2) \left| \begin{array}{cc}
3 & 1 \\
0 & 2 \\
\end{array} \right|
= 1(2 + 2) - 4(6 - 0) - 2(-6 - 0)
= 4 - 24 + 12
= -8
\]
The value of the determinant is -8.

You Try It 3
\[
\begin{vmatrix}
3 & -2 & 0 \\
1 & 4 & 2 \\
-2 & 1 & 3 \\
\end{vmatrix}
= 3 \left| \begin{array}{cc}
4 & 0 \\
1 & -2 \\
\end{array} \right| - (-2) \left| \begin{array}{cc}
2 & 1 \\
0 & -2 \\
\end{array} \right| + 0 \left| \begin{array}{cc}
2 & 1 \\
0 & -2 \\
\end{array} \right|
= 3(2 - 0) + 2(3 + 4) + 0
= 3(10) + 2(7)
= 30 + 14
= 44
\]
The value of the determinant is 44.

You Try It 4
\[
\begin{align*}
3x - y &= 4 \\
6x - 2y &= 5
\end{align*}
\]
\[
D = \begin{vmatrix}
3 & -1 \\
6 & -2 \\
\end{vmatrix}
= 0
\]

Because \(D = 0\), \(D_x\) is undefined.

Therefore, the system of equations is dependent or inconsistent. That is, the system is not independent.
ANSWERS TO CHAPTER 5 SELECTED EXERCISES

SECTION 5.5

3. 11 5. 18 7. 0 9. 15 11. -30 13. 0 15. 0 17. (3, -4) 19. (4, -1) 21. \( \left( \frac{11}{14}, \frac{17}{27} \right) \) 23. \( \left( \frac{1}{2}, 1 \right) \)
25. Not independent 27. (-1, 0) 29. (1, -1, 2) 31. (2, -2, 3) 33. Not independent 35. No 37. a. Sometimes true  b. Always true  c. Sometimes true 39. The area is 239 ft².