Appendix C  Further Concepts in Geometry

C.1  Exploring Congruence and Similarity

- Identifying Congruent Figures
- Identifying Similar Figures
- Reading and Using Definitions
- Congruent Triangles
- Classifying Triangles

Identifying Congruent Figures

Two figures are congruent if they have the same shape and the same size. Each of the triangles in Figure C.1 is congruent to each of the other triangles. The triangles in Figure C.2 are not congruent to each other.

<table>
<thead>
<tr>
<th>Congruent</th>
<th>Not Congruent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure C.1</td>
<td>Figure C.2</td>
</tr>
</tbody>
</table>

Notice that two figures can be congruent without having the same orientation. If two figures are congruent, then either one can be moved (and turned or flipped if necessary) so that it coincides with the other figure.

EXAMPLE 1  Dividing Regions into Congruent Parts

Divide the region into two congruent parts.

[Diagram of a region divided into two congruent parts]

SOLUTION

There are many solutions to this problem. Some of the solutions are shown in Figure C.3. Can you think of others?
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Two of the figures are similar. Which two are they?

a.    b.    c.

Solution

The first figure has five sides and the other two figures have four sides. Because similar figures must have the same shape, the first figure is not similar to either of the others. Because you are told that two figures are similar, it follows that the second and third figures are similar.

Determining Similarity

You wrote an essay on Euclid, the Greek mathematician who is famous for writing a geometry book titled Elements of Geometry. You are making a copy of the essay using a photocopier that is set at 75% reduction. Is each image on the copied pages similar to its original?

Solution

Every image on a copied page is similar to its original. The copied pages are smaller, but that doesn’t matter because similar figures do not have to be the same size.
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**EXAMPLE 4**  Drawing an Object to Scale

You are drawing a floor plan of a building. You choose a scale of \( \frac{1}{8} \) inch to 1 foot. That is, \( \frac{1}{8} \) inch of the floor plan represents 1 foot of the actual building. What dimensions should you draw for a room that is 12 feet wide and 18 feet long?

**SOLUTION**

Because each foot is represented as \( \frac{1}{8} \) inch, the width of the room should be

\[
12 \left( \frac{1}{8} \right) = \frac{12}{8} = \frac{3}{2} = 1\frac{1}{2}
\]

and the length of the room should be

\[
18 \left( \frac{1}{8} \right) = \frac{18}{8} = \frac{9}{4} = 2\frac{1}{4}.
\]

The scale dimensions of the room are 1\( \frac{1}{2} \) inches by 2\( \frac{1}{4} \) inches. See Figure C.6.

**Reading and Using Definitions**

A definition uses known words to describe a new word. If no words were known, then no new words could be defined. So, some words such as point, line, and plane must be commonly understood without being defined. Some statements such as “a point lies on a line” and “point C lies between points A and B” are also not defined. See Figure C.7.

**Segments and Rays**

Consider the line \( \overline{AB} \) that contains the points A and B. (In geometry, the word line means a straight line.)

The line segment (or simply segment) \( \overline{AB} \) consists of the endpoints A and B and all points on the line \( \overline{AB} \) that lie between A and B.

The ray \( \overrightarrow{AB} \) consists of the initial point A and all points on the line \( \overline{AB} \) that lie on the same side of A as B lies. If C is between A and B, then \( \overrightarrow{CA} \) and \( \overrightarrow{CB} \) are opposite rays.

Points, segments, or rays that lie on the same line are collinear.
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It follows that $\overline{AB}$ and $\overline{BA}$ denote the same segment, but $\overline{AB}$ and $\overline{BA}$ do not denote the same ray. No length is given to lines or rays because each is infinitely long. The length of the line segment $\overline{AB}$ is denoted by $AB$.

**Angles**

An angle consists of two different rays that have the same initial point. The rays are the sides of the angle. The angle that consists of the rays $\overline{AB}$ and $\overline{AC}$ is denoted by $\angle BAC$, $\angle CAB$, or $\angle A$. The point $A$ is the vertex of the angle. See Figure C.9.

The measure of $\angle A$ is denoted by $m\angle A$. Angles are classified as acute, right, obtuse, and straight.

- **Acute** $0^\circ < m\angle A < 90^\circ$
- **Right** $m\angle A = 90^\circ$
- **Obtuse** $90^\circ < m\angle A < 180^\circ$
- **Straight** $m\angle A = 180^\circ$

In geometry, unless specifically stated otherwise, angles are assumed to have a measure that is greater than 0° and less than or equal to 180°.

Every nonstraight angle has an interior and an exterior. A point $D$ is in the interior of $\angle A$ if it is between points that lie on each side of the angle. Two angles (such as $\angle ROS$ and $\angle SOP$ shown in Figure C.9) are adjacent if they share a common vertex and side, but have no common interior points. In Figure C.10, $\angle 1$ and $\angle 3$ share a vertex, but not a common side, so $\angle 1$ and $\angle 3$ are not adjacent.

**Segment and Angle Congruence**

Two segments are congruent, $\overline{AB} \equiv \overline{CD}$, if they have the same length. Two angles are congruent, $\angle P \equiv \angle Q$, if they have the same measure.

Definitions can always be interpreted “forward” and “backward.” For instance, the definition of congruent segments means (1) if two segments have the same measure, then they are congruent, and (2) if two segments are congruent, then they have the same measure. You learned that two figures are congruent if they have the same shape and size.
Section C.1  Exploring Congruence and Similarity

Congruent Triangles

If \( \triangle ABC \) is congruent to \( \triangle PQR \), then there is a correspondence between their angles and sides such that corresponding angles are congruent and corresponding sides are congruent. The notation \( \triangle ABC \cong \triangle PQR \) indicates the congruence and the correspondence, as shown in Figure C.11. If two triangles are congruent, then you know that they share many properties.

\[
\triangle ABC \cong \triangle PQR
\]

Corresponding angles are
\[
\angle A \cong \angle P, \quad \angle B \cong \angle Q, \quad \angle C \cong \angle R
\]

Corresponding sides are
\[
AB \cong PQ, \quad BC \cong QR, \quad CA \cong RP
\]

![Figure C.11](image)

**EXAMPLE 5  Naming Congruent Parts**

You and a friend have identical drafting triangles, as shown in Figure C.12. Name all congruent parts.

![Figure C.12](image)

**SOLUTION**

Given that \( \triangle DEF \cong \triangle RST \), the congruent angles and sides are as follows.

Angles:
\[
\angle D \cong \angle R, \quad \angle E \cong \angle S, \quad \angle F \cong \angle T
\]

Sides:
\[
DE \cong RS, \quad EF \cong ST, \quad FD \cong TR
\]
Classifying Triangles
A triangle can be classified by relationships among its sides or among its angles, as shown in the following definitions.

**Classification by Sides**
1. An **equilateral triangle** has three congruent sides.
2. An **isosceles triangle** has at least two congruent sides.
3. A **scalene triangle** has no sides congruent.

**Classification by Angles**
1. An **acute triangle** has three acute angles. If these angles are all congruent, then the triangle is also **equiangular**.
2. A **right triangle** has exactly one right angle.
3. An **obtuse triangle** has exactly one obtuse angle.

In \( \triangle ABC \), each of the points \( A \), \( B \), and \( C \) is a **vertex** of the triangle. (The plural of vertex is vertices.) The side \( BC \) is the side **opposite** \( \angle A \). Two sides that share a common vertex are **adjacent sides** (see Figure C.13).
The sides of right triangles and isosceles triangles are given special names. In a right triangle, the sides adjacent to the right angle are the legs of the triangle. The side opposite the right angle is the hypotenuse of the triangle (see Figure C.14).

An isosceles triangle can have three congruent sides. If it has only two, then the two congruent sides are the legs of the triangle. The third side is the base of the triangle (see Figure C.15).

**C.1 Exercises**

In Exercises 1 and 2, copy the region on a piece of dot paper. Then divide the region into two congruent parts. How many different ways can you do this? See Example 1.

1. ![Image of region to be divided]

2. ![Image of region to be divided]

3. Two of the figures are congruent. Which are they?
   (a) ![Circle]
   (b) ![Smaller circle]
   (c) ![Larger circle]

4. Two of the figures are similar. Which are they? See Example 2.
   (a) ![Shape A]
   (b) ![Shape B]
   (c) ![Shape C]

In Exercises 5 and 6, copy the region on a piece of paper. Then divide the region into four congruent parts.

5. ![Image of region to be divided]

6. ![Image of region to be divided]

In Exercises 7–10, use the triangular grid shown in the figure. In the grid, each small triangle has sides of one unit.

7. How many congruent triangles with one-unit sides are in the grid?

8. How many congruent triangles with two-unit sides are in the grid?

9. How many congruent triangles with three-unit sides are in the grid?

10. Does the grid contain triangles that are not similar to each other?
11. If two figures are congruent, are they similar? Explain.
12. If two figures are similar, are they congruent? Explain.
13. Can a triangle be similar to a square? Explain.

In Exercises 15–18, match the description with its correct notation.
(a) \( \overline{PQ} \)  
(b) \( PQ \)  
(c) \( \overrightarrow{PQ} \)  
(d) \( \overrightarrow{PQ} \)

15. The line through \( P \) and \( Q \)
16. The ray from \( P \) through \( Q \)
17. The segment between \( P \) and \( Q \)
18. The length of the segment between \( P \) and \( Q \)

In Exercises 20–22, use the figure shown.

20. The figure shows three angles whose vertex is \( X \). Write two names for each angle. Which two angles are adjacent?
21. Is \( Y \) in the interior or exterior of \( \angle WXY \)?
22. Which is the best estimate for \( \angle WXY \)?  
   (a) 15°  
   (b) 30°  
   (c) 45°

In Exercises 23–28, match the triangle with its name.
(a) Equilateral  
(b) Scalene  
(c) Obtuse  
(d) Equiangular  
(e) Isosceles  
(f) Right

23. Side lengths: 2 inches, 3 inches, 4 inches
24. Angle measures: 60°, 60°, 60°
25. Side lengths: 3 meters, 2 meters, 3 meters
26. Angle measures: 30°, 60°, 90°
27. Side lengths: 4 feet, 4 feet, 4 feet
28. Angle measures: 20°, 145°, 15°

In Exercises 29 and 30, use the figure, in which \( \triangle LMP \cong \triangle ONQ \). See Example 5.

29. Name three pairs of congruent angles. Name three pairs of congruent sides.
30. If \( \triangle LMP \) is isosceles, explain why \( \triangle ONQ \) must be isosceles.

In Exercises 31–34, use the definition of congruence to complete the statement.
31. If \( \triangle ABC \cong \triangle TUV \), then \( \angle C = \)
32. If \( \triangle PQR \cong \triangle XYZ \), then \( \angle P = \)
33. If \( \triangle LMN \cong \triangle TUV \), then \( \overline{LN} = \)
34. If \( \triangle DEF \cong \triangle NOP \), then \( DE = \)

35. Copy and complete the table. Write Yes if it is possible to sketch a triangle with both characteristics. Write No if it is not possible. Illustrate your results with sketches. (The first is done for you.)

<table>
<thead>
<tr>
<th></th>
<th>Scalene</th>
<th>Isosceles</th>
<th>Equilateral</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Acute</strong></td>
<td>Yes</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td><strong>Obtuse</strong></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td><strong>Right</strong></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Acute and Scalene
In Exercises 36–39, \(\triangle ABC\) is isosceles with \(\overline{AC} \cong \overline{BC}\). Solve for \(x\). Then decide whether the triangle is equilateral. (The figures are not necessarily drawn to scale.)

36. \[
\begin{align*}
A & \quad C \\
6 & \quad 2x + 2 \\
B & \quad x + 1
\end{align*}
\]

37. \[
\begin{align*}
A & \quad C \\
2x + 6 & \quad 12 \\
B & \quad x + 1
\end{align*}
\]

38. \[
\begin{align*}
C & \quad B \\
x + 1 & \quad 2x - 1 \\
A & \quad 4x - 2
\end{align*}
\]

39. \[
\begin{align*}
C & \quad B \\
4x - 6 & \quad 2x \\
A & \quad x + 3
\end{align*}
\]

40. **Landscape Design** You are designing a patio. Your plans use a scale of \(\frac{1}{8}\) inch to 1 foot. The patio is 24 feet by 36 feet. What are its dimensions on the plans? See Example 4.

41. **Architecture** The length of each outer wall of the Pentagon, near Washington, D.C., is 921 feet. About how large would a \(\frac{1}{8}\)-inch to 1-foot scale drawing of the Pentagon be? Would such a scale be reasonable?

42. Find a location of \(S\) such that \(\triangle PQR \cong \triangle PQS\).

43. Find two locations of \(V\) such that \(\triangle PQR \cong \triangle TUV\).

44. **Logical Reasoning** Arrange 16 toothpicks as shown in the figure. What is the least number of toothpicks you must remove to create four congruent triangles? (Each toothpick must be the side of at least one triangle.) Sketch your result.

45. **Logical Reasoning** Show how you could arrange six toothpicks to form four congruent triangles. Each triangle has one toothpick for each side, and you cannot bend, break, or overlap the toothpicks. (Hint: The figure can be three-dimensional.)

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**C.2 Angles**

- Identifying Special Pairs of Angles
- Angles Formed by a Transversal
- Angles of a Triangle

**Identifying Special Pairs of Angles**

You have been introduced to several definitions concerning angles. For instance, you know that two angles are **adjacent** if they share a common vertex and side but have no common interior points. Some other definitions for pairs of angles are listed below.
Definitions for Pairs of Angles

Two angles are **vertical angles** if their sides form two pairs of opposite rays. (See Figure C.16.)

Two adjacent angles are a **linear pair** if their noncommon sides are opposite rays. (See Figure C.18.)

Two angles are **complementary** if the sum of their measures is $90^\circ$. Each angle is the **complement** of the other (See Figure C.17.)

Two angles are **supplementary** if the sum of their measures is $180^\circ$. Each angle is the **supplement** of the other. (See Figure C.18.)

In Example C.18, note that the linear pairs are also supplementary. This result is stated in the following postulate.

**Linear Pair Postulate**

If two angles form a linear pair, then they are supplementary — i.e., the sum of their measures is $180^\circ$.

**EXAMPLE 1**  Identifying Special Pairs of Angles

Use the terms defined above to describe relationships between the labeled angles in Figure C.19.

**SOLUTION**

a. $\angle 3$ and $\angle 5$ are vertical angles. So are $\angle 4$ and $\angle 6$.

b. There are four sets of linear pairs: $\angle 3$ and $\angle 4$, $\angle 4$ and $\angle 5$, $\angle 5$ and $\angle 6$, and $\angle 3$ and $\angle 6$. The angles in each of these pairs are also supplementary angles.

The relationship between vertical angles is stated in the following theorem.

**Vertical Angles Theorem**

If two angles are vertical angles, then they are congruent.
Section C.2  Angles

Angles Formed by a Transversal

A transversal is a line that intersects two or more coplanar lines at different points. The angles that are formed when the transversal intersects the lines have the following names.

### Angles Formed by a Transversal

In Figure C.20, the transversal $t$ intersects the lines $l$ and $m$.

Two angles are corresponding angles if they occupy corresponding positions, such as $\angle 1$ and $\angle 5$.

Two angles are alternate interior angles if they lie between $l$ and $m$ on opposite sides of $t$, such as $\angle 2$ and $\angle 8$.

Two angles are alternate exterior angles if they lie outside $l$ and $m$ on opposite sides of $t$, such as $\angle 1$ and $\angle 7$.

Two angles are consecutive interior angles if they lie between $l$ and $m$ on the same side of $t$, such as $\angle 2$ and $\angle 5$.

### Example 2  Naming Pairs of Angles

In Figure C.21, how is $\angle 9$ related to the other angles?

**Solution**

You can consider that $\angle 9$ is formed by the transversal $l$ as it intersects $m$ and $n$, or you can consider $\angle 9$ to be formed by the transversal $m$ as it intersects $l$ and $n$. Considering one or the other of these, you have the following.

**a.** $\angle 9$ and $\angle 10$ are a linear pair and are supplementary angles. So are $\angle 9$ and $\angle 12$.

**b.** $\angle 9$ and $\angle 11$ are vertical angles.

**c.** $\angle 9$ and $\angle 7$ are alternate exterior angles. So are $\angle 9$ and $\angle 3$.

**d.** $\angle 9$ and $\angle 5$ are corresponding angles. So are $\angle 9$ and $\angle 1$.

To help build understanding regarding angles formed by a transversal, consider relationships between two lines. Parallel lines are coplanar lines that do not intersect. (Recall that two nonvertical lines are parallel if and only if they have the same slope.) Intersecting lines are coplanar and have exactly one point in common. If intersecting lines meet at right angles, they are perpendicular; otherwise, they are oblique. See Figure C.22.
Many of the angles formed by a transversal that intersects parallel lines are congruent. The following postulate and theorems list useful results.

**Corresponding Angles Postulate**

If two parallel lines are cut by a transversal, then the pairs of corresponding angles are congruent.

Note that the hypothesis of this postulate states that the lines must be parallel, as shown in Figure C.23. If the lines are not parallel, then the corresponding angles are not congruent, as shown in Figure C.24.

**Angle Theorems**

**Alternate Interior Angles Theorem** If two parallel lines are cut by a transversal, then the pairs of alternate interior angles are congruent.

**Consecutive Interior Angles Theorem** If two parallel lines are cut by a transversal, then the pairs of consecutive interior angles are supplementary.

**Alternate Exterior Angles Theorem** If two parallel lines are cut by a transversal, then the pairs of alternate exterior angles are congruent.

**Perpendicular Transversal Theorem** If a transversal is perpendicular to one of two parallel lines, then it is perpendicular to the second.

**EXAMPLE 3** Using Properties of Parallel Lines

In Figure C.25, lines r and s are parallel lines cut by a transversal, l. Find the measure of each labeled angle.

**SOLUTION**

$\angle 1$ and the given angle are alternate exterior angles and are congruent. So, $m\angle 1 = 75^\circ$. $\angle 5$ and the given angle are vertical angles. Because vertical angles are congruent, they have the same measure. So, $m\angle 5 = 75^\circ$. Similarly, $\angle 1 \cong \angle 4$ and $m\angle 1 = m\angle 4 = 75^\circ$. There are several sets of linear pairs, including $\angle 1$ and $\angle 2$; $\angle 3$ and $\angle 4$; $\angle 5$ and $\angle 6$; $\angle 5$ and $\angle 7$.

The angles in each of these pairs are also supplementary angles; the sum of the measures of each pair of angles is $180^\circ$. Because one angle of each pair measures $75^\circ$, the supplements each measure $105^\circ$. So $\angle 2$, $\angle 3$, $\angle 6$, and $\angle 7$ each measure $105^\circ$. 
Angles of a Triangle

The word “triangle” means “three angles.” When the sides of a triangle are extended, however, other angles are formed. The original three angles of the triangle are the interior angles. The angles that are adjacent to the interior angles are the exterior angles of the triangle. Each vertex has a pair of exterior angles, as shown in Figure C.26.

Cut a triangle out of a piece of paper. Then tear off the three angles and place them adjacent to each other, as shown in Figure C.27. What do you observe? (You could perform this investigation by measuring with a protractor or using a computer drawing program.) You should arrive at the conclusion given in the following theorem.

**Triangle Sum Theorem**

The sum of the measures of the interior angles of a triangle is 180°.

**EXAMPLE 4** Using the Triangle Sum Theorem

Use the triangle shown in Figure C.28, to find \( m \angle 1 \), \( m \angle 2 \), and \( m \angle 3 \).

**SOLUTION**

To find the measure of \( \angle 3 \), use the Triangle Sum Theorem, as follows.

\[
m \angle 3 = 180^\circ - (51^\circ + 42^\circ) = 87^\circ
\]

Knowing the measure of \( \angle 3 \), you can use the Linear Pair Postulate to write

\[
m \angle 2 = 180^\circ - 87^\circ = 93^\circ.
\]

Using the Triangle Sum Theorem, you have

\[
m \angle 1 = 180^\circ - (28^\circ + 93^\circ) = 59^\circ.
\]
Appendix C  Further Concepts in Geometry

The next theorem is one that you might have anticipated from the investigation earlier. As shown in Figure C.29, if you had torn only two of the angles from the paper triangle, you could put them together to cover exactly one of the exterior angles.

![Figure C.29](image)

Exterior Angle Theorem

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote (nonadjacent) interior angles. (See Figure C.30.)

C.2 Exercises

In Exercises 1–6, sketch a pair of angles that fits the description. Label the angles as \( \angle 1 \) and \( \angle 2 \). See Example 1.

1. A linear pair of angles
2. Supplementary angles for which \( \angle 1 \) is acute
3. Acute vertical angles
4. Adjacent congruent complementary angles
5. Obtuse vertical angles
6. Adjacent congruent supplementary angles

In Exercises 7–12, use the figure to determine relationships between the given angles. See Example 2.

True or False? In Exercises 13-18, use the following information to decide whether the statement is true or false. Justify your answer. (Hint: Make a sketch.)

Vertical angles: \( \angle 1 \) and \( \angle 2 \);
Linear pairs: \( \angle 1 \) and \( \angle 3 \), \( \angle 1 \) and \( \angle 4 \).
13. If \( m\angle 3 = 30^\circ \), then \( m\angle 4 = 150^\circ \).
14. If \( m\angle 1 = 150^\circ \), then \( m\angle 4 = 30^\circ \).
15. \( \angle 2 \) and \( \angle 3 \) are congruent.
16. \( m\angle 3 + m\angle 1 = m\angle 4 + m\angle 2 \)
17. \( \angle 3 \equiv \angle 4 \)
18. \( m\angle 3 = 180^\circ - m\angle 2 \)

In Exercises 19–24, find the value of \( x \).

19. 
20. 

7. \( \angle AOC \) and \( \angle COD \)
8. \( \angle AOB \) and \( \angle BOC \)
9. \( \angle BOC \) and \( \angle COE \)
10. \( \angle AOB \) and \( \angle EOD \)
11. \( \angle BOC \) and \( \angle COF \)
12. \( \angle AOB \) and \( \angle AOE \)
21. 

22. 

23. 

24. 

25. In the figure, $P$, $S$, and $T$ are collinear. If $m\angle P = 40^\circ$ and $m\angle QST = 110^\circ$, what is $m\angle Q$? (Hint: $m\angle P + m\angle Q + m\angle PSQ = 180^\circ$)

(a) 40° (b) 55° (c) 70° (d) 110° (e) 140°

26. Use the figure below.

(a) Name two corresponding angles.
(b) Name two alternate interior angles.
(c) Name two alternate exterior angles.
(d) Name two consecutive interior angles.

27. In Exercises 27–30, $l_1 \parallel l_2$. Find the measures of $\angle 1$ and $\angle 2$. Explain your reasoning. See Example 3.

28. 

29. 

30. 

In Exercises 31–33, $m \parallel n$ and $k \parallel l$. Determine the values of $a$ and $b$.

31. 

32. 

33. 

34. Use the figure below.

(a) Name the interior angles of the triangle.
(b) Name the exterior angles of the triangle.
(c) Two angle measures are given in the figure. Find the measure of the eight labeled angles.
35. Use the figure below, in which \( \triangle ABC \cong \triangle DEF \). See Example 4.

\[
\begin{array}{c}
A \quad 105^\circ \quad B \\
C \quad 35^\circ \quad D \\
E \\
F
\end{array}
\]

(a) What is the measure of \( \angle D \)?
(b) What is the measure of \( \angle B \)?
(c) What is the measure of \( \angle C \)?
(d) What is the measure of \( \angle F \)?

36. Can a right triangle have an obtuse angle? Explain.

37. Is it true that a triangle that has two 60° angles must be equiangular? Explain.

38. If a right triangle has two congruent angles, does it have two 45° angles? Explain.

In Exercises 39 and 40, find the measure of each labeled angle.

39.

40.

In Exercises 41 and 42, draw and label a right triangle, \( \triangle ABC \), for which the right angle is \( \angle C \). What is \( m\angle B \)?

41. \( m\angle A = 13^\circ \)

42. \( m\angle A = 47^\circ \)

In Exercises 43 and 44, draw two noncongruent, isosceles triangles that have an exterior angle with the given measure.

43. 130°

44. 145°

In Exercises 45–48, find the measures of the interior angles.

45.

46.

47.

48.