A P P E N D I X

Brief Review of Integration Techniques

**u-Substitution**

The basic idea underlying u-substitution is to perform a simple substitution that converts the integral into a recognizable form ready for immediate integration. For example, given

\[ \int \frac{\cos x}{1 + \sin x} \, dx \]

let \( u = 1 + \sin x \) and differentiate to find \( du = \cos x \, dx \). Substitution then yields

\[ \int \frac{\cos x}{1 + \sin x} \, dx = \int \frac{du}{u} = \ln |u| + C \]

Substituting for \( u \) again in this last expression gives

\[ \int \frac{\cos x}{1 + \sin x} \, dx = \ln |1 + \sin x| + C \]

**Integration by Parts**

Recall from calculus that

\[ \int u \, dv = uv - \int v \, du \]

In some cases it is necessary to apply the procedure several times before a form is obtained that can easily be integrated. In these and other situations, it is helpful to use the tabular method as follows:

<table>
<thead>
<tr>
<th>Sign</th>
<th>Derivatives</th>
<th>Integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>( u )</td>
<td>( dv )</td>
</tr>
<tr>
<td>-</td>
<td>( du )</td>
<td>( v )</td>
</tr>
</tbody>
</table>

Diagonal arrows in the table indicate terms to be multiplied (\( uv \) in this case). The bottom row in the table has horizontal arrows to indicate the final integral to be evaluated (\( \int v \, du \) in the above case). Finally, the sign column is associated with the differentiated term at each stage, beginning with a plus sign and alternating with the minus sign, as suggested by the table format.
Thus the table above would be read as follows:

\[
\int u \, dv = \left[ + uv \right] - \int v \, du
\]

To apply integration by parts successively, build the table by repeatedly differentiating the derivatives (middle) column and integrating the integrals (right) column, while the sign (left) column alternates. Terminate the table with a horizontal arrow between the middle and right column when you can readily integrate the product of the function in the last row or when the last row simply repeats the first row (up to a multiplicative constant). Let us consider several examples.

**EXAMPLE 1**

Find the integral \( \int xe^x \, dx \) by the tabular method.

**Solution** We set up the table as follows:

<table>
<thead>
<tr>
<th>Sign</th>
<th>Derivatives</th>
<th>Integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>( x )</td>
<td>( e^x )</td>
</tr>
<tr>
<td>−</td>
<td>1</td>
<td>( e^x )</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>( e^x )</td>
</tr>
</tbody>
</table>

Interpreting the table, we get

\[
\int xe^x \, dx = +xe^x - 1 \cdot e^x + \int 0 \cdot e^x \, dx + C = (x - 1)e^x + C
\]

**EXAMPLE 2**

Integrate \( \int x^2e^{2x} \, dx \) by the tabular method.

**Solution** We set up the table as before:

<table>
<thead>
<tr>
<th>Sign</th>
<th>Derivatives</th>
<th>Integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>( x^2 )</td>
<td>( e^{2x} )</td>
</tr>
<tr>
<td>−</td>
<td>2x</td>
<td>( \frac{e^{2x}}{2} )</td>
</tr>
<tr>
<td>+</td>
<td>2</td>
<td>( \frac{e^{2x}}{4} )</td>
</tr>
<tr>
<td>−</td>
<td>0</td>
<td>( \frac{e^{2x}}{8} )</td>
</tr>
</tbody>
</table>
Thus
\[ \int x^2 e^{2x} \, dx = \frac{x^2 e^{2x}}{2} - \frac{2x e^{2x}}{4} + \frac{2e^{2x}}{8} - \int \frac{0 \cdot e^{2x}}{8} \, dx + C \]
\[ = \frac{e^{2x}}{4} (2x^2 - 2x + 1) + C \]

**EXAMPLE 3**

Integrate \( \int e^x \sin x \, dx \).

**Solution** After filling in the table, we get

<table>
<thead>
<tr>
<th>Sign</th>
<th>Derivatives</th>
<th>Integrals</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>sin x</td>
<td>e^x</td>
</tr>
<tr>
<td>−</td>
<td>cos x</td>
<td>e^x</td>
</tr>
<tr>
<td>+</td>
<td>− sin x</td>
<td>e^x</td>
</tr>
</tbody>
</table>

Thus
\[ \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + \int (- \sin x)e^x \, dx + C \]

or
\[ \int e^x \sin x \, dx = \frac{e^x (\sin x - \cos x)}{2} + C_1 \]

Examples 1–3 illustrate the two basic strategies of integration by parts: (1) Choose a term to differentiate whose successive derivatives eventually become zero or repeat, and (2) continue to differentiate by parts until the integrand (up to a multiplicative constant) is repeated in the bottom row, as in Example 3. In choosing the term \( dv \) to integrate, you may find the following mnemonic “detail ladder” useful:

- dv
- exponential
- trigonometric
- algebraic
- inverse trigonometric
- logarithmic

To use the ladder, choose the term \( dv \) to integrate in order of priority from the top to the bottom. Conversely, the term \( u \) to differentiate is chosen from bottom to top. For example, when integrating
\[ \int x^2 e^x \, dx \]
which involves a polynomial and an exponential, integrate the exponential $dv = e^x \, dx$ and differentiate the polynomial $u = x^2$. The above mnemonic device is a rule of thumb only and may not work in some cases.

### Rational Functions

Given an algebraic fraction with a polynomial in both the numerator and the denominator (that is, a rational function), division may lead to a simpler form. If the highest power in the numerator is equal to or greater than the highest power in the denominator, first perform polynomial division and then integrate the result. For example,

$$\frac{y + 1}{y - 1} = 1 + \frac{2}{y - 1}$$

so

$$\int \frac{y + 1}{y - 1} \, dy = \int \left( 1 + \frac{2}{y - 1} \right) \, dy = y + 2 \ln |y - 1| + C$$

### Partial Fractions

In algebra you learned to sum fractional expressions by finding a common denominator. For example,

$$\frac{2}{x - 1} + \frac{4}{x + 3} = \frac{2(x + 3) + 4(x - 1)}{(x - 1)(x + 3)} = \frac{6x + 2}{x^2 + 2x - 3}$$

For purposes of integration we need to reverse this procedure. That is, given the integral

$$\int \frac{6x + 2}{x^2 + 2x - 3} \, dx$$

we use partial fraction decomposition to obtain a new expression that is readily integrable:

$$\int \left( \frac{2}{x - 1} + \frac{4}{x + 3} \right) \, dx = 2 \ln |x - 1| + 4 \ln |x + 3| + C$$

This process of splitting a fraction $f(x)/g(x)$ into a sum of fractions with linear or quadratic denominators is called partial fraction decomposition. For the method to work, the degree of the numerator $f(x)$ must be less than the degree of the denominator $g(x)$; otherwise, you must first perform polynomial long division. To use the method, the denominator must be factored into linear and quadratic factors. In Examples 4–6 we review three cases that may exist for the factored denominator:
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1. Distinct linear factors
2. Repeated linear factors
3. Quadratic factors

EXAMPLE 4  Distinct Linear Factors

Find the integral \( \int \frac{2x^2 - x + 1}{(x + 1)(x - 3)(x + 2)} \, dx \).

Solution  We must find constants \( A, B, \) and \( C \) such that

\[
\frac{2x^2 - x + 1}{(x + 1)(x - 3)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{x + 2} \tag{1}
\]

Algebraic Method  In this method you multiply through by the factored denominator to obtain

\[
2x^2 - x + 1 = A(x - 3)(x + 2) + B(x + 1)(x + 2) + C(x + 1)(x - 3)
\]

Then expand the right-hand side and combine like powers of \( x \):

\[
2x^2 - x + 1 = (A + B + C)x^2 + (-A + 3B - 2C)x + (-6A + 2B - 3C)
\]

Next equate the coefficients of like powers of \( x \) on both sides of this last equation. This procedure results in a system of linear algebraic equations involving our three unknowns:

\[
\begin{align*}
A + B + C &= 2 \\
-A + 3B - 2C &= -1 \\
-6A + 2B - 3C &= 1
\end{align*}
\]

Solution of this system by elimination or by the method of determinants yields

\[
A = -1, \quad B = \frac{4}{5}, \quad \text{and} \quad C = \frac{11}{5}
\]

Thus

\[
\int \frac{2x^2 - x + 1}{(x + 1)(x - 3)(x + 2)} \, dx = - \int \frac{dx}{x + 1} + \frac{4}{5} \int \frac{dx}{x - 3} + \frac{11}{5} \int \frac{dx}{x + 2}
\]

\[
= -\ln |x + 1| + \frac{4}{5} \ln |x - 3| + \frac{11}{5} \ln |x + 2| + C
\]

Heaviside Method  There is a shortcut method for finding the constants in the partial fraction decomposition of \( f(x)/g(x) \). First, write the rational function with \( g(x) \) completely factored into its linear terms:

\[
f(x) = f(x) = \frac{f(x)}{(x - r_1)(x - r_2) \cdots (x - r_n)} \tag{2}
\]
To find the constant \( A_i \) associated with the term

\[
\frac{A_i}{x - r_i}
\]

in the partial fraction decomposition, cover the factor \( x - r_i \) in the denominator of the right-hand side of Equation (2) and replace all the uncovered \( x \)'s with the number \( r_i \). For instance, to find the constant \( A \) in Equation (1), cover the factor \( x + 1 \) in the denominator and replace all the uncovered \( x \)'s with \( x = -1 \).

\[
A = \frac{2 - (-1) + 1}{(x + 1)(-1 - 3)(-1 + 2)} = \frac{4}{(-4)(1)} = -1
\]

Likewise, we find \( B \) by covering the factor \( x - 3 \) and replacing all the uncovered \( x \)'s with \( x = 3 \).

\[
B = \frac{2(9) - 3 + 1}{(3 + 1)(x - 3)(3 + 2)} = \frac{16}{4(5)} = \frac{4}{5}
\]

Finally, \( C \) is determined when \( x = -2 \).

\[
C = \frac{2(4) - (-2) + 1}{(-2 + 1)(-2 - 3)(x + 2)} = \frac{11}{(-1)(-5)} = \frac{11}{5}
\]

The integration is the same as before. We emphasize that the Heaviside method can be used only with distinct linear factors. In the next example, we present another method for finding the constants when the linear factors are repeated. Of course, you can always resort to the more tedious algebraic method.

**EXAMPLE 5 A Repeated Linear Factor**

Find the integral \( \int \frac{3P}{(P + 4)^2(P + 1)} \, dP \).

**Solution** We need to find constants \( A, B, \) and \( C \) such that

\[
\frac{3P}{(P + 4)^2(P + 1)} = \frac{A}{P + 4} + \frac{B}{(P + 4)^2} + \frac{C}{P + 1}
\]

or

\[
3P = A(P + 4)(P + 1) + B(P + 1) + C(P + 4)^2 \quad (3)
\]
Substitution Method  Since Equation (3) is an identity, it holds for every value of \( P \). Thus, to obtain three equations for finding the unknowns \( A \), \( B \), and \( C \), we simply substitute convenient values for \( P \):

\[
\begin{align*}
P = -4: & \quad -12 = -3B \\
P = -1: & \quad -3 = 9C \\
P = 0: & \quad 0 = 4A + B + 16C
\end{align*}
\]

to give the solutions \( A = \frac{1}{3}, B = 4, C = -\frac{1}{3} \). Thus

\[
\begin{align*}
\int \frac{3P}{(P + 4)^2(P + 1)} \, dP &= \int \left[ \frac{1}{3(P + 4)} + \frac{4}{(P + 4)^2} - \frac{1}{3(P + 1)} \right] \, dP \\
&= \frac{1}{3} \ln |P + 4| - \frac{4}{P + 4} - \frac{1}{3} \ln |P + 1| + C
\end{align*}
\]

EXAMPLE 6  A Quadratic Factor

Find the integral \( \int \frac{dP}{(P + 1)(P^2 + 1)} \).

Solution  We must find constants \( A \), \( B \), and \( C \) such that

\[
\frac{1}{(P + 1)(P^2 + 1)} = \frac{A}{P + 1} + \frac{BP + C}{P^2 + 1}
\]

Thus

\[
1 = A(P^2 + 1) + (BP + C)(P + 1)
\]

Since this expression is to hold for all \( P \), the coefficients of like powers of \( P \) on both sides of the equation must be equal. After collecting like powers of \( P \) on the right-hand side, we get

\[
0P^2 + 0P^1 + 1P^0 = (A + B)P^2 + (B + C)P + (A + C)
\]

which yields the linear system

\[
\begin{align*}
0 &= A + B \\
0 &= B + C \\
1 &= A + C
\end{align*}
\]

The solution is \( A = \frac{1}{2}, B = -\frac{1}{2}, \) and \( C = \frac{1}{2} \). Thus

\[
\begin{align*}
\int \frac{dP}{(P + 1)(P^2 + 1)} &= \int \left[ \frac{1}{2(P + 1)} + \frac{-\frac{P}{2} + \frac{1}{2}}{P^2 + 1} \right] \, dP \\
&= \frac{1}{2} \ln |P + 1| - \frac{1}{4} \ln |P^2 + 1| + \frac{1}{2} \tan^{-1} P + C
\end{align*}
\]