15 Graphs of Functions as Models

15.1 An Arms Race

You might ask, Why study the arms race? One reason is that almost all modern wars are preceded by unstable arms races. Strong evidence suggests that an unstable arms race between great powers, characterized by a sharp acceleration in military capability, is an early warning indicator of war. In a 1979 article, Michael Wallace of the University of British Columbia studied 99 international disputes during the 1816–1965 period.\(^1\) He found that disputes preceded by an unstable arms race escalated to war 23 out of 28 times, whereas disputes not preceded by an arms race resulted in war only 3 out of 71 times. Wallace calculated an arms race index for the two nations involved in each dispute that correctly predicted war or no war in 91 out of the 99 cases studied. His findings do not mean that an arms race between the powers necessarily results in war or that there is a causal link between arms races and conflict escalation. They do establish, however, that rapid competitive military growth is strongly associated with the propensity to war. Thus, by studying the arms race, we have the potential for predicting war. If we can predict war, then there is hope that we can learn to avoid it.

There is another reason for studying the arms race. If the arms race can be approximated by a mathematical model, then it can be understood more concretely. You will see that the answers to such questions as, Will civil defense dampen the arms race? and Will the introduction of mobile missile launching pads help to reduce the arms requirements? are not simply matters of political opinion. There is an objective reality to the arms race that the mathematical model intends to capture.

The former Soviet Union and the United States were engaged in a nuclear arms race during the Cold War. At that time political and military strategists asked how the United States should react to changes in numbers and sophistication of the Soviet nuclear arsenal. To answer the difficult question, How many weapons are enough?, several factors had to be considered, including American objectives, Soviet objectives, and weapon technology. A former chairman of the Joint Chiefs of Staff, General Maxwell D. Taylor, suggested the following nuclear deterrence objectives for the American strategic forces:

The strategic forces, having the single capability of inflicting massive destruction, should have the single task of deterring the Soviet Union from resorting to any form of strategic warfare. To

maximize their deterrent effectiveness they must be able to survive a massive first strike and still be able to destroy sufficient enemy targets to eliminate the Soviet Union as a viable government, society, and economy, responsive to the national leaders who determine peace or war.\textsuperscript{2}

Note especially that Taylor’s deterrence strategy assumes the worst possible case: the Soviet’s launching a preemptive first attack to destroy America’s nuclear force.

How many weapons would be necessary to accomplish the objectives General Taylor suggests? After describing an appropriate system of Soviet targets (generally population and industrial centers), he states the following:

The number of weapons we shall need will be those required to destroy the specific targets within this system of which few will be hardened silos calling for the accuracy and short flight time of ICBMs. As a safety factor, we should add extra weapons to compensate for losses that may be suffered in a first strike and for uncertainties in weapon performance. The total weapons requirement should be substantially less than the numbers available to us in our present arsenal.\textsuperscript{3}

Thus, a minimum number of missiles would be required to destroy specific enemy targets (generally population and industrial centers) chosen to inflict unacceptable damage on the enemy. Additional missiles would be required to compensate for losses incurred in the Soviet’s presumed first strike. Implicitly, the number of such additional missiles depends on the size and effectiveness of the Soviet missile forces. Taylor concluded that meeting these objectives would allow for a reduction in America’s nuclear arsenal.

In response to a question on expenditures for national defense, Admiral Hyman G. Rickover testified before a congressional committee as follows:

For example, take the number of nuclear submarines; I’ll hit right close to home. I see no reason why we have to have just as many as the Russians do. At a certain point you get where it’s sufficient. What’s the difference whether we have 100 nuclear submarines or 200? I don’t see what difference it makes. You can sink everything on the ocean several times over with the number we have and so can they. That’s the point I’m making.\textsuperscript{4}

Again, Admiral Rickover concluded that a reduction in arms would be possible.

On July 14, 2001, in Genoa, Italy, President George W. Bush and Russian President Vladimir Putin agreed to seek cuts in their nuclear arsenals. Putin said he would accept a U.S. antimissile shield if it were linked to deep cuts in offensive nuclear weapons. He suggested that both nations could reduce their nuclear arsenals to approximately 1500 strategic weapons. (At that time the United States had approximately 7000 strategic weapons, and Russia had approximately 6500.)

We are going to develop a graphical model of the nuclear arms race based on the preceding remarks. The model will help answer the question, How many weapons are enough? Although the model applies to any kind of arms race, for purposes of discussion and illustration we focus on nuclear weapons delivered by long-range intercontinental ballistic missiles (ICBMs).

\textsuperscript{3}Ibid.
Developing the Graphical Model

Suppose that two countries, Country X and Country Y, are engaged in a nuclear arms race and that each country adopts the following strategies.

Friendly Strategy: To survive a massive first strike and inflict unacceptable damage on the enemy.

Enemy Strategy: To conduct a massive first strike to destroy the friendly missile force.

That is, each country follows the friendly strategy when determining its own missile force and presumes the enemy strategy for the opposing country. Note especially that the friendly strategy implies targeting population and industrial centers, whereas the enemy strategy implies targeting missile sites. This was the policy of nuclear deterrence advocated during the Cold War.

Now let’s define the following variables:

\[ x = \text{the number of missiles possessed by Country } X \]
\[ y = \text{the number of missiles possessed by Country } Y \]

Next, let \( y = f(x) \) denote the function representing the minimum number of missiles required by Country Y to accomplish its strategies when Country X has \( x \) missiles. Similarly, let \( x = g(y) \) represent the minimum number of missiles required by Country X to accomplish its objectives. When Country Y determines the required size of its missile force, it assumes that it has the friendly strategy and that Country X is following the enemy strategy. On the other hand, Country X has the friendly strategy when determining the size of its missile force and presumes Country Y is following the enemy strategy.

We begin by investigating the nature of the curve \( y = f(x) \). Because a certain number of missiles \( y_0 \) are required by Country Y to destroy the selected population and industrial centers of Country X, \( y_0 \) is the intercept when \( x = 0 \). That is, Country Y considers that it needs \( y_0 \) missiles even if Country X has none (basically, a psychological defense in the sense that Y fears attack or invasion by X). As Country X increases its missile force, Country Y must then add additional missiles because it assumes Country X is following the enemy strategy and targeting its missile force. Let’s assume that the weapons technology is such that Country X can destroy no more than one of Country Y’s missiles with each missile fired. Then the number of additional missiles Country Y needs for each missile added by Country X depends on the effectiveness of Country X’s missiles. Convince yourself that the curve \( y = f(x) \) must lie between the limiting lines shown in Figure 15.1. Line A, having slope 0, represents a state of absolute invulnerability of Country Y’s missiles to any attack. At the other extreme, line B, having slope 1, indicates that Country Y must add one new missile for each missile added by Country X.

To determine more precisely the shape of the graph of \( y = f(x) \), we will analyze what happens for various cases relating the relative sizes of the two missile forces. To determine the cases, we subdivide the region between lines A and B into smaller subregions defined by the lines \( x = y \), \( x = 2y \), \( x = 3y \), and so forth, as shown in Figure 15.2. We then approximate \( y = f(x) \) in each of these subregions. Remember that when Country Y determines the number of missiles it needs to deter Country X for the graph of \( y = f(x) \), Country Y is presumed to follow the friendly strategy, whereas Country X follows the enemy strategy.
Case 1: \( x < y \)  If Country \( X \) attacks in this situation, it fires all of its \( x \) missiles at the same number of Country \( Y \)'s missiles. Because the number \((y - x)\) of Country \( Y \)'s missiles could not be attacked, at least that many would survive. Of the number \( x \) of Country \( Y \)'s missiles that were fired on, a percentage \( s \) would survive, where \( 0 < s < 1 \). Thus, the total number of missiles surviving the attack is \( y - x + sx \). Now Country \( Y \) must have \( y_0 \) missiles survive to inflict unacceptable damage on Country \( X \). Hence,

\[
y_0 = y - x + sx \quad \text{for} \quad 0 < s < 1
\]

or, solving for \( y \),

\[
y = y_0 + (1 - s)x
\]  

(15.1)

Equation (15.1) gives the minimum number of missiles Country \( Y \) must have to be confident that \( y_0 \) missiles will survive an attack by Country \( X \).

Case 2: \( y = x \)  In firing all of its missiles, Country \( X \) fires exactly one of its missiles at each of Country \( Y \)'s missiles. Assuming the percentage \( s \) will survive the attack, the number \( sx = sy \) survive, in which case Country \( Y \) needs

\[
y = \frac{y_0}{s}
\]  

(15.2)

missiles to inflict unacceptable damage on Country \( X \).
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**Case 3: \( y < x < 2y \)** Country X, in firing all its missiles, targets each of Country Y’s missiles once and a portion of them twice, as illustrated in Figure 15.3.

**Figure 15.3**
An example of \( y < x < 2y \)

Convince yourself that \( x - y \) of Country Y’s missiles would be targeted twice and \( y - (x - y) = 2y - x \) would be targeted once. Of those targeted once, a percentage \( s(2y - x) \) will survive as before. Of those targeted twice, the percentage \( s(x - y) \) will survive the first round. Of those that survive the first round, the percentage \( s [s(x - y)] = s^2(x - y) \) will survive the second round. Hence, Country Y must have

\[
y_0 = s^2(x - y) + s(2y - x)
\]

missiles survive, or, solving for \( y \),

\[
y = \frac{y_0 + x(s - s^2)}{2s - s^2}
\]

is the minimum number of missiles required by Country Y.

**Case 4: \( x = 2y \)** Country X will fire exactly two missiles at each of Country Y’s missiles. If we reason as in Case 2, the number \( s^2y \) survive, so

\[
y = \frac{y_0}{s^2}
\]

is the minimum number of missiles required by Country Y.

Now let’s combine all of the preceding scenarios into a single graph. For convenience, we are going to assume that the discrete situation just discussed giving the minimum number of missiles can be represented by a continuous model (giving rise to fractions of missiles). First, observe that Equations (15.1) and (15.3) both represent straight-line segments: the first segment for \( x < y \) and the second segment for \( y < x < 2y \). In Case 1, when \( x < y \), we obtained the equation

\[
y_0 = y - x + sx
\]

As \( x \) approaches \( y \), this last equation becomes (in the limit) \( y_0 = sy \). In Case 3, when \( y < x < 2y \), we obtained the equation

\[
y_0 = s^2(x - y) + s(2y - x)
\]

Again, as \( x \) approaches \( y \), the equation becomes \( y_0 = sy \). Thus, the two line segments meet at \( x = y \) with the common value \( y = y_0/s \). Finally, as \( x \) approaches \( 2y \), Equation (15.5) becomes \( y_0 = s^2y \).

These observations mean that the two line segments defined by Equations (15.1) and (15.3) form a continuous curve meeting the lines \( y = x \) and \( 2y = x \). Moreover, the slope \( \frac{1-s}{2s} \) for the line segment represented by Equation (15.3) is less than the slope \( 1 - s \) of the line...
segment represented by Equation (15.1) because $2 - s > 1$. Thus, the curve is \textit{piecewise linear with decreasing slopes}. The graphical model is depicted in Figure 15.4. Note the graph lies within the cone-shaped region between lines $A$ and $B$ as discussed previously.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15.4}
\caption{A graphical model relating the number of missiles for Country $Y$ to the number of missiles for Country $X$ when $0 \leq x \leq 2y$.}
\end{figure}

We could continue to analyze additional cases, such as what happens when $2y < x < 3y$. Because we are interested only in qualitative information, however, let’s see if we can determine the general shape of the curves more simply.

To simplify the analysis, let’s replace our piecewise linear approximation by a single continuous smooth curve (one without corners) that passes through each of the points $(0, y_0), (x, x), (x, \frac{x}{2}), \ldots$ shown in Figure 15.4. (These are the points where the piecewise linear approximation crosses the $y$ axis and the lines $y = x$ and $2y = x$). We want a curve given by a single equation rather than one represented by a different equation in each subregion. Generalizing from our analysis in Cases 2 and 4, one such curve is given by the following model:

$$y = \frac{y_0}{s^{x/y}} \quad \text{for} \quad 0 < s < 1$$

(15.6)

An inspection of Equation (15.6) reveals that for every ratio $x/y$ we can find $y$. Thus, the curve $y = f(x)$ crosses each line $x = y, x = 2y, \ldots, x = ny$, as illustrated in Figure 15.5, at the same points as did our piecewise linear approximation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure15.5}
\caption{The curve $y = f(x)$ must cross every line $x = ny$.}
\end{figure}
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The situation for Country X is entirely symmetrical. (In determining its curve, Country X is assumed to have the friendly strategy to deter Country Y, and Country Y is presumed to have the enemy strategy.) Its minimum number of missiles is represented by a continuous curve \( x = g(y) \) that crosses every line \( y = x, y = 2x, \ldots, y = nx \). Thus, the two curves must intersect.

The preceding discussion leads us to consider two idealized continuously differentiable curves such as those drawn in Figure 15.6. Because the curve \( y = f(x) \) represents the minimum number of missiles required by Country Y, the region above the curve represents missile levels satisfactory to Country Y. Likewise, the region to the right of the curve \( x = g(y) \) represents missile levels satisfactory to Country X. Thus, the darkest region in Figure 15.6 represents missile levels satisfactory to both countries.

The intersection point of the curves \( y = f(x) \) and \( x = g(y) \) represents the minimum level at which both sides are satisfied. To see that this is so, assume Country Y has \( y_0 \) missiles and observes that Country X has \( x_0 \) missiles. To meet its objectives, Country Y will have to add sufficient missiles to reach point 1 in Figure 15.6. In turn, Country X will have to add sufficient missiles to reach point 2. This process will continue until both sides are satisfied simultaneously. Notice that any point in the darkest region will suffice to satisfy both countries, and there are many points in the darkest region that are quite likely to occur. The intersection point \( (x_m, y_m) \) in Figure 15.6 represents the minimum force levels required of both countries to meet their objectives.

Uniqueness of the Intersection Point

We would like to know if the intersection point is unique. Note from Equation (15.6) that as the ratio \( x/y \) increases, \( y \) must increase; likewise, \( x = g(y) \) increases. Because both curves are increasing, it is tempting to conclude that the intersection point is unique. Consider Figure 15.7, however. In the figure both curves are steadily increasing: the curve \( y = f(x) \) crosses every line \( x = ny \), and \( x = g(y) \) crosses every line \( y = nx \). However, the curves have multiple intersection points. How can we ensure a unique intersection point? Notice that the slope of the curve \( y = f(x) \) in Figure 15.7 is steadily decreasing until the point \( x = x_1 \), when it begins to increase. Thus, the first derivative changes from a decreasing to an increasing function at \( x = x_1 \). That is, the tangent line changes from continuously turning in a clockwise direction to turning in a counterclockwise direction as \( x \) advances. In other words, the second derivative changes sign. If we can show such a sign change is
impossible, then we can conclude that the intersection point is unique. In fact, we will show that the second derivative of \( y = f(x) \) is always negative.

Taking the logarithm of Equation (15.6) yields

\[
\ln y = \ln y_0 - \frac{x}{y} \ln s
\]

Multiplying both sides of the equation by \( y \) and simplifying give

\[
y \ln y - y \ln y_0 = -x \ln s
\]

Differentiating implicitly with respect to \( x \) and simplifying yield

\[
y'(1 + \ln y - \ln y_0) = -\ln s
\]

or

\[
y' = \frac{-\ln s}{1 + \ln y - \ln y_0}
\]

Differentiating this last equation for the second derivative gives

\[
y'' = \frac{-(-\ln s)\frac{1}{y}y'}{(1 + \ln y - \ln y_0)^2}
\]

Next, we determine the sign of \( y' \). Rewrite \( y' \) as

\[
y' = \frac{\ln s}{1 + \ln \frac{y_0}{y}}
\]

Because \( 0 < s < 1 \), \( \ln s \) is negative.

Now, for the cases we are considering, \( y > y_0 \), which implies that \( \ln(y_0/y) < 0 \). Thus, \( y' > 0 \) everywhere, in which case \( y'' < 0 \) everywhere. Therefore, we can conclude that a unique intersection point does in fact exist. The model has the general shape shown in Figure 15.8.

**Graphical Behavior of \( y = f(x) \)**

The ways in which the graph of \( y = y_0/s^{x/y} \) behaves depends on three factors: the constant \( y_0 \), which is the minimum number of missiles required by Country \( Y \) after a preemptive first strike; the survivability percentage \( s \), which is determined by the technology and weapon effectiveness of Country \( X \)’s missiles as well as by how securely Country \( Y \)’s missiles
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Figure 15.8
A graphical model of the nuclear arms race

\[
x = g(y) \\
y = f(x)
\]

are protected; and the exchange ratio \( e = x/y \). If \( y_0 \) increases, then the curve \( y = f(x) \) shifts upward and also has a larger slope at each point than before (see Problem 5). If the survivability factor \( s \) increases, the curve rotates downward toward the horizontal line \( A: y = y_0 \) and has a smaller slope at each point than before.

If the exchange ratio \( e \) increases, then Country X can target Country Y’s missiles more than once, requiring Country Y to need more of them. This results in an increase in the slope and upward rotation of the curve toward the line \( B: y = y_0 + x \).

If you have access to a computer graphing package, you can see these effects by plotting the graph \( y = y_0/s^x/y \) for various values of the three factors.

Although the curve \( x = g(y) \) for Country X displays similar behavior, we note that its constant \( x_0 \), survivability, and exchange ratio factors are generally different from the values for Country Y. Let’s consider several situations in which we use these ideas and the graphical model to analyze the effects on the intersection point for different political and military strategies likely to be entertained by the two countries.

When analyzing the graphical effects of a particular strategy, we first consider how the strategy changes each of the factors \( y_0, s, \) and \( e \) for Country Y. Does the factor increase, decrease, or remain unchanged? Here we assume that Country Y has the friendly strategy, whereas Country X is following the enemy strategy. How each factor changes determines how the curve \( y = f(x) \) changes its position.

Next we consider how the strategy changes each of the factors \( x_0, s, \) and \( e \) for Country X. (Again, we stress that these factors are generally not the same as, and are independent of, their corresponding counterparts for Country Y.) Again, we ask if each factor increases, decreases, or remains unchanged due to the particular strategy. In answering this question, we assume that Country X has the friendly strategy, whereas Country Y follows the enemy strategy. How the factors change determines how the curve \( x = g(y) \) changes its position.

The combined changes of the two curves move the original intersection point \( (x_m, y_m) \) to a new position \( (x'_m, y'_m) \), where the shifted curves (resulting from our analysis of their factors) intersect.

Model Interpretation

EXAMPLE 1  Civil Defense

Suppose Country X decides to double its annual budget for civil defense. Presumably, Country Y will need more missiles to inflict an unacceptable level of damage on Country X’s population centers. Thus, \( y_0 \) increases. Because the effectiveness of Country X’s weapons
has not changed, nor has Country Y done anything to improve protection of its missiles in their silos, no change occurs in the survivability of Country Y’s missiles. Also, no change occurs in the exchange ratio for Country Y: one X missile can still destroy at most one Y missile. The net effect is that the curve \( y = f(x) \) shifts vertically upward with increasing slope at every \( x \).

For Country X, the factor \( x_0 \) does not change because increased protection of its population and industrial centers does not affect the minimum number of missiles it will need to retaliate against a preemptive first strike on the part of Country Y. Also, the survivability factor of Country X’s missiles does not change because civil defense does not improve the protection of its missiles typically located in silos in remote geographic regions, nor is it the case that the effectiveness of Country Y’s missiles has changed. Finally, one Y missile can still destroy at most one X missile, so the exchange ratio for Country X is unchanged. The net effect is that the curve \( x = g(y) \) does not change position at all.

The overall effect of Country X increasing its civil defense budget is shown in Figure 15.9. The dashed curve is the new position of the function \( y = f(x) \) resulting from the civil defense of Country X. The point \((x'_m, y'_m)\) is the new intersection point. Note that although the course of action seemed fairly passive, the effect is to increase the minimum number of missiles required by both sides because \( x'_m > x_m \) and \( y'_m > y_m \).

**EXAMPLE 2 Mobile Launching Pads**

For this scenario, assume that Country X puts its missiles on mobile launching pads, which can be relocated during times of international crisis. The factor \( y_0 \) does not change because Country Y must still target the same number of population and industrial centers in retaliation for a first strike by Country X. Moreover, the fact that Country X’s missiles are launched from mobile pads does not alter the effectiveness of the missiles or improve the protection of Country Y’s silos. So the survivability of Country Y’s missiles is unchanged. No change in the exchange ratio for Country Y occurs because one X missile, though launched from a mobile launching pad, can still destroy at most one Y missile in a first strike. Thus, the curve \( y = f(x) \) does not change.

Regarding the factors for Country X, there is no change in \( x_0 \) because Country X still requires the same number of missiles to inflict unacceptable damage on Country Y’s
population and industrial centers. Country X’s missiles are less vulnerable than before because Country Y would not know their exact locations in executing a first strike. Thus, the survivability of Country X’s missiles is increased. Finally, the placement of a missile on a mobile pad does not alter the exchange ratio for Country X: one Y missile can still hit and destroy at most one X missile. The net effect of these changes is to flatten the curve $x = g(y)$ toward the y axis, as shown by the new dashed curve in Figure 15.10.

**Figure 15.10**
Country X uses mobile launching pads.

The overall effect of Country X placing its missiles on mobile launching pads is a movement of the intersection point so that $x_0^m < x_m$ and $y_0^m < y_m$, as depicted in Figure 15.10. Thus, the minimum number of missiles needed by both countries to deter the other side from engaging in hostile attack is reduced.

**EXAMPLE 3 Multiple Warheads**

Suppose now that Country X and Country Y both employ multiple warheads that can be targeted independently (MIRVs). In our development of the model we assumed each missile was armed with only one warhead, so counting missiles and counting warheads would be the same. With MIRVs this correspondence is no longer true. Let’s continue to count the numbers of missiles (not warheads) required by each country. Assume that each missile is armed with 16 smaller missiles, each possessing its own warhead. Because it still takes the same number of warheads to destroy the opponent’s population and industrial centers, it is reasonable to expect the number of larger missiles, $x_0$ and $y_0$, to be reduced by the factor 16.

Let’s consider the survivability factor $s$ for Country Y. If we assume a warhead released independently from an in-flight missile from Country X is just as effective in its destructive power as before, then there is no change in the survival possibility of the targeted missile in Country Y. That is, the factor $s$ is unchanged.

However, when one missile from Country X is headed for Country Y in a preemptive first strike, it carries 16 warheads, each of which can independently target a missile in Country Y. Thus, 1 X missile can destroy up to 16 Y missiles, and the exchange ratio factor for Country Y increases significantly. This means the curve $y = f(x)$ must rise...
more sharply than before to compensate for the increased destruction if it is to meet its friendly strategy objectives.

Because both countries have MIRVed, the same argument reveals that the survivability factor for Country $X$ remains unchanged but its exchange ratio factor is increased, causing a rise in the steepness of the curve $x = g(y)$ (away from the $y$ axis).

The new curves are represented in Figure 15.11. Because the reduction in values of the intercepts $x_0$ and $y_0$ and the changes in the slopes of the curves give different effects on the new location of the intersection point, it is difficult to determine from a graphical analysis whether the minimum number of missiles actually increases or decreases. This analysis demonstrates a limitation of graphical models. To determine the location of the equilibrium point $(x_0, y_0)$ would require a more detailed analysis and more exact information concerning the weapon effectiveness and technological capabilities of both countries (along with other factors, such as military intelligence).

**EXAMPLE 4  MIRVs Revisited: Counting Warheads**

Although we were unable to predict the effect of multiple warheads on the minimum number of missiles required by each side in Example 3, we can analyze the total number of warheads in this example’s strategy. Let $x$ and $y$ now represent the number of warheads possessed by Country $X$ and Country $Y$, respectively. The number of warheads needed by each country to inflict unacceptable damage on the opponent remain at the levels $x_0$ and $y_0$, as before. Also, because each warhead is located on a missile, its chance for survivability is the same as that of the missile, so the survivability factors are unchanged for each country.

Let’s examine what happens to the exchange ratio factor for Country $Y$. A single warhead released from an incoming missile from Country $X$ now has the capability of destroying 16 of Country $Y$’s warheads instead of just 1, because they are all clustered on a single missile targeted by the incoming warhead. This increase in the exchange ratio for Country $Y$ causes a sharp rise in the steepness of its curve $y = f(x)$. The same argument applies to the exchange ratio factor for Country $X$, so the curve $x = g(y)$ also increases in steepness. The new curves are displayed in Figure 15.12. Note that both countries require...
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Figure 15.12
Multiple warheads on each missile increase the total number of warheads required by each side.

more warheads if multiple warheads are introduced on each missile because \( x'_m > x_m \) and \( y'_m > y_m \) in Figure 15.12.

15.1 PROBLEMS

1. Analyze the effect on the arms race of each of the following strategies:
   a. Country X increases the accuracy of its missiles by using a better guidance system.
   b. Country X increases the payload (destructive power) of its missiles without sacrificing accuracy.
   c. Country X is able to retarget its missiles in flight so that it can aim for missiles that previous warheads have failed to destroy.
   e. Country Y adds long-range intercontinental bombers to its arsenal.
   f. Country X develops sophisticated jamming devices that dramatically increase the probability of neutralizing the guidance systems of Country Y’s missiles.

2. Discuss the appropriateness of the assumptions used in developing the nuclear arms race model. What is the effect on the number of missiles if each country believes the other country is also following the friendly strategy? Is disarmament possible?

3. Develop a graphical model based on the assumption that each side is following the enemy strategy. That is, each side desires a first-strike capability for destroying the missile force of the opposing side. What is the effect on the arms race if Country X now introduces antiballistic missiles?

4. Discuss how you might go about validating the nuclear arms race model. What data would you collect? Is it possible to obtain the data?

5. Use the polar coordinate substitution \( x = r \cos \theta \) and \( y = r \sin \theta \) in Equation (15.6) to show that a doubling of \( y_0 \) causes a doubling of \( r \) for every fixed \( \theta \). Show that if \( y_0 \) increases, then \( y = f(x) \) shifts upward with an increasing slope.
15.2 Modeling an Arms Race in Stages

Let’s again assume that two countries, Country X and Country Y, are engaged in an arms race. Each country follows a deterrent strategy that requires them to have a given number of weapons to deter the enemy (inflict unacceptable damage) even if the enemy has no weapons. Under this strategy, as the enemy adds weapons, the friendly force increases its arms inventory by some percentage of the number of attacking weapons that depends on how effective the friendly force perceives the enemy’s weapons to be.

Suppose Country Y believes it needs 120 weapons to deter the enemy. Furthermore, for every 2 weapons possessed by Country X, Country Y believes it needs to add 1 additional weapon (to ensure 120 weapons remain after a strike by Country X). Thus, the number of weapons needed by Country Y (y weapons) as a function of the number of weapons it believes Country X has (x weapons) is

\[ y = 120 + \frac{1}{2}x \]

Now suppose Country X is following a similar strategy, believing it needs 60 weapons even if Country Y has no weapons. Furthermore, for every 3 weapons that it believes Country Y possesses, X believes that it must add one weapon. Thus, the number x of weapons needed by Country X as a function of the number y of weapons it believes Country Y has is

\[ x = 60 + \frac{1}{3}y \]

How does the arms race proceed?

A Graphical Solution

Suppose that initially (stage \( n = 0 \)) Countries Y and X do not think the other side has arms. Then (stage \( n = 1 \)) they build 120 weapons and 60 weapons, respectively. Now assume each has perfect intelligence; that is, each knows the other has built weapons. In the next stage (stage \( n = 2 \)), Country Y increases its inventory to 150 weapons:

\[ y = 120 + \frac{1}{2}(60) = 150 \text{ weapons} \]

Similarly, Country X notes that Y had 120 weapons during the previous stage and increases its inventory to 100 weapons:

\[ x = 60 + \frac{1}{3}(120) = 100 \text{ weapons} \]

The arms race would proceed dynamically—that is, in successive stages. At each stage a country adjusts its inventory based on the strength of the enemy during the previous stage. In stage \( n = 3 \), Country Y realizes that Country X now has 100 weapons and reacts by increasing its inventory to \( y = 120 + \frac{1}{2}(100) = 170 \). Similarly, Country X increases its inventory to \( x = 60 + \frac{1}{3}(150) = 110 \). If we let \( n \) represent the stage of the arms race,
convince yourself that the following table represents the growth of the arms race under the assumptions we have made:

<table>
<thead>
<tr>
<th>Stage</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Country Y</td>
<td>0</td>
<td>120</td>
<td>150</td>
<td>170</td>
<td>175</td>
<td>178</td>
</tr>
<tr>
<td>Country X</td>
<td>0</td>
<td>60</td>
<td>100</td>
<td>110</td>
<td>117</td>
<td>118</td>
</tr>
</tbody>
</table>

Note that the growth in the arms race appears to be diminishing. The number of weapons needed by Country Y appears to be approaching approximately 180 weapons, whereas X appears to be approaching approximately 120 weapons (Figures 15.13 and 15.14). Does

![Figure 15.13](https://example.com/figure15_13.png)

**Figure 15.13**
Dynamics of an arms race

![Figure 15.14](https://example.com/figure15_14.png)

**Figure 15.14**
Arms race curves for each country
this model actually predict that an equilibrium value will be reached as suggested in the model developed earlier in this chapter? Is the equilibrium position stable in the sense that small changes in the number of weapons initially possessed by either side have little change on the final outcome? Is the outcome sensitive to changes in the coefficients of the model? Next, we build a dynamical system to answer these questions.

**Numerical Solution of an Arms Race as a Dynamical System Model**

Using the notation introduced in Section 15.1,

Let $n = \text{stage (years, decades, fiscal periods, etc.)}$

$x_n = \text{number of weapons possessed by X in stage n}$

$y_n = \text{number of weapons possessed by Y in stage n}$

Then our assumptions imply that at stage $n + 1$,

$$\begin{align*}
  y_{n+1} &= 120 + \frac{1}{2}x_n \\
  x_{n+1} &= 60 + \frac{1}{3}y_n
\end{align*}$$

(15.7)

with

$x_0 = 0$

$y_0 = 0$

Recall that the values $x_0$ and $y_0$ are called *initial values*. Along with the coefficients $\frac{1}{2}$ and $\frac{1}{3}$, they are *parameters* we ultimately would like to vary to determine the sensitivity of the predictions. In Table 15.1, we display a numerical solution for the model and initial conditions in Equations (15.7).

What happens if both countries start off with more than the minimum number of missiles? For example, what if Country X starts with 100 and Country Y with 200? Our model becomes

$$\begin{align*}
  y_{n+1} &= 120 + \frac{1}{2}x_n \\
  x_{n+1} &= 60 + \frac{1}{3}y_n
\end{align*}$$

(15.8)

with

$x_0 = 100$

$y_0 = 200$

Is the equilibrium reached, or is there uncontrolled growth? In the problems that follow this section, we ask you to explore the long-term behavior predicted by Equations (15.8) in terms of the stability of the arms race for different initial values and other parameters (the survival coefficients of Countries X and Y).
15.2 Modeling an Arms Race in Stages

Table 15.1 A numerical solution to Equations (15.7)

<table>
<thead>
<tr>
<th>n</th>
<th>x_n</th>
<th>y_n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>60</td>
<td>120</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>150</td>
</tr>
<tr>
<td>3</td>
<td>110</td>
<td>170</td>
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<td>4</td>
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<tr>
<td>5</td>
<td>118.3333</td>
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<td>119.4444</td>
<td>179.1667</td>
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<td>7</td>
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<td>179.7222</td>
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<td>8</td>
<td>119.9074</td>
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</tr>
<tr>
<td>9</td>
<td>119.9537</td>
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<td>179.9999</td>
</tr>
<tr>
<td>17</td>
<td>120</td>
<td>180</td>
</tr>
</tbody>
</table>

15.2 PROBLEMS

1. Build a numerical solution to Equations (15.8).
   a. Graph your results.
   b. Is an equilibrium value reached?
   c. Try other starting values. Do you think the equilibrium value is stable?
   d. Explore other values for the survival coefficients of Countries X and Y. Describe your results.

2. Recall from Section 15.1 that an equilibrium value for the arms race requires that \( x_{n+1} = x_n \) and \( y_{n+1} = y_n \) simultaneously. Is there an equilibrium value for Equations (15.7)? If so, find it.

15.2 PROJECTS

For Projects 1–4, complete the requirements in the referenced UMAP module (see enclosed CD), and prepare a short summary for classroom discussion.

1. “The Distribution of Resources,” by Harry M. Schey, UMAP 60–62 (one module). The author investigates a graphical model that can be used to measure the distribution of resources. The module provides an excellent review of the geometric interpretation of the derivative as applied to the economics of the distribution of a resource. Numerical calculation of the derivative and definite integral is also discussed.
2. “Nuclear Deterrence,” by Harvey A. Smith, UMAP 327. The author analyzes the stability of the arms race, assuming objectives similar to those suggested by General Taylor. The module develops analytic models using probabilistic arguments. An understanding of elementary probability is required.

3. “The Geometry of the Arms Race,” by Steven J. Brams, Morton D. Davis, and Philip D. Straffin, Jr., UMAP 311. This module analyzes the possibilities of both parties disarming by introducing elementary game theory. Interesting conclusions are based on Country X’s ability to detect Country Y’s intentions, and vice versa.


15.2 Further Reading


15.3 Managing Nonrenewable Resources: The Energy Crisis

During the past century, the United States has shifted into nearly complete dependence on nonrenewable energy sources. Petroleum and natural gas now constitute about three-fourths of the nation’s fuel, and nearly half of our crude oil comes from foreign sources. The rise of the Organization of Petroleum-Exporting Countries cartel (OPEC) has caused some analysts to fear for supply security, especially during periods of political unrest when we are threatened by constraints on the supply of foreign oil, such as oil embargoes. Thus, there are significant attempts to conserve energy so as to reduce our long-term oil consumption. There is also interest in more drastic short-term reductions to survive a crisis situation.

Various solutions have been proposed to address these long- and short-term needs. One solution is gas rationing. Another is to place a surcharge tax on each gallon of gasoline sold at the local pump. Basically, the idea behind this solution is that gasoline companies will pass the tax on to the consumer by increasing the price per gallon by the amount of the tax. Accordingly, it is supposed the consumer will reduce consumption because of the higher price. Let’s study this proposal by constructing a graphical model and qualitatively addressing the following questions:

1. What is the effect of the surcharge tax on short- and long-term consumer demand?
2. Who actually pays the tax—the consumer or the oil companies?
3. Does the tax contribute to inflation?
15.3 Managing Nonrenewable Resources: The Energy Crisis

Stated more succinctly, the problem is to determine the effect of a surcharge tax on the market price of, and consumer demand for, gasoline. In the following analysis we are concerned with gaining a qualitative understanding of the principal factors involved with the problem. A graphical analysis is appropriate to gain this understanding, especially because precise data would be difficult to obtain. We begin by graphically analyzing some pertinent general economic principles. In the ensuing sections we interpret the conclusions of the graphical model as they apply to the oil situation.

**Constructing a Graphical Model**

Suppose a firm within a large competitive industry produces a single product. A question facing the firm is how many units to produce to maximize profits. Assume that the industry in question is so large that any particular firm’s production has no appreciable effect on the market price. Hence, the firm may assume the price of the product is constant and need consider only the difference between the price and the firm’s costs in producing the product. Individual firms encounter *fixed costs*, which are independent of the amount produced over a wide range of production levels. These costs include rent and utilities, equipment capitalization costs, and management costs. The *variable costs* depend on the quantity produced. Variable costs include the cost of raw materials, taxes, and labor. When the fixed costs are divided by the quantity produced, the share apportioned to each unit is obtained. This per-unit share is relatively high when production levels are low. However, as production levels increase, not only does the per-unit share of the fixed costs diminish but also economies of scale (such as buying raw materials in large quantities at reduced rates) often reduce some of the variable cost rates. Eventually, production levels are reached that strain the capabilities of the firm. At this point the firm is faced with hiring additional employees, paying overtime, or capitalizing additional machinery or similar costs. Because the per-unit costs tend to be relatively high when production levels are either very low or very high, one intuitively expects the existence of a production level $q^*$ that yields a maximum profit over the range of production levels being considered. This idea is illustrated in Figure 15.15. Next, consider the characteristics of $q^*$ mathematically.

![Figure 15.15](Image)

Profit is maximized at $q^*$.

At a given level of production $q$, total profit $TP(q)$ is the difference between total revenue $TR(q)$ and total cost $TC(q)$. That is,

$$TP(q) = TR(q) - TC(q)$$
A necessary condition for a relative maximum to exist is that the derivative of $TP$ with respect to $q$ must be 0:

$$TP' = TR' - TC' = 0$$

or, at the level $q^*$ of maximum profit,

$$TR'(q^*) = TC'(q^*) \quad (15.9)$$

Thus, at $q^*$ it is necessary that the slope of the total revenue curve equal the slope of the total cost curve. This condition is depicted in Figure 15.16.

Let's interpret economically the meaning of the derivatives $TR'$ and $TC'$. From the definition of the derivative,

$$TR'(q) \approx \frac{TR(q + \Delta q) - TR(q)}{\Delta q}$$

for $\Delta q$ small. Thus, if $\Delta q = 1$, you can see that $TR'(q)$ approximates $TR(q + 1) - TR(q)$, which is the revenue generated by the next unit sold, or the marginal revenue (MR) of the $q + 1$st unit. Because total revenue is the price per unit times the number of units, it follows that the marginal revenue of the $q + 1$st unit is the price of that unit less the revenue lost on previous units resulting from price reductions (see Problem 4). Similarly, $TC'(q)$ represents the marginal cost (MC) of the $q + 1$st unit; that is, the extra cost in changing output to include one additional unit. If Equation (15.9) is interpreted in these new terms, a necessary condition for maximum profit to occur at $q^*$ is that marginal revenue equal marginal cost:

$$MR(q^*) = MC(q^*) \quad (15.10)$$

For the critical point defined by Equations (15.8) to be a relative maximum, it is sufficient that the second derivative $TP''$ be negative. Because $TP' = MR - MC$, we have

$$TP''(q^*) = MR'(q^*) - MC'(q^*) < 0$$

or

$$MR'(q^*) < MC'(q^*) \quad (15.11)$$
15.3 Managing Nonrenewable Resources: The Energy Crisis

This means that at the level $q^*$ of maximum profit the slope of the marginal revenue curve is less than the slope of the marginal cost curve. The results (15.10) and (15.11) together imply that the marginal revenue and marginal cost curves intersect at $q^*$, with the marginal cost curve rising more rapidly. These results are illustrated in Figure 15.17.

**Interpreting the Graphical Model**

Now let’s interpret the graphical model represented by Figure 15.17. The MR curve represents the revenue generated by the next unit sold. The curve is drawn horizontally because in a large competitive industry, the amount one particular firm produces seldom influences the market price so there is no loss in revenue on previous units resulting from price reduction. Thus, the MR curve represents the (constant) price of the product. Given a market price determined by the entire industry and aggregate consumer demand, a firm attempting to maximize profits will continue to produce units until the cost of the next unit produced exceeds its market price. Verify that this situation is suggested by the graphical model in Figure 15.17.

**15.3 Problems**

1. Justify mathematically, and interpret economically, the graphical model for the theory of the firm given in Figure 15.18. What are the major assumptions on which the model is based?

2. Show that for total profit to reach a relative minimum, $MR = MC$ and $MC' < MR'$.

3. Suppose the large competitive industry is the oil industry, and the firm within that industry is a gasoline station. How well does the model depicted in Figure 15.17 reflect the reality of that situation? How would you adjust the graphical model to make improvements?

4. Verify the result that the marginal revenue of the $q + 1$st unit equals the price of that unit minus the loss in revenue on previous units resulting from price reduction.
15.4 Effects of Taxation on the Energy Crisis

Let’s suppose a firm is currently maximizing its profits; that is, given a market price MR, it is producing $q^*$ units as suggested by Figure 15.17. Assume further that a tax is added to each unit sold. Because the firm must pay the government the amount of the tax for each unit sold, the marginal cost to the firm of each unit increases by the amount of the tax. Geometrically, that means the marginal cost curve shifts upward by the amount of the tax. Assume for the moment that the entire industry is able to increase the market price by simply adding on the amount of the tax to the price of each unit. Under this condition the MR curve also shifts upward by the amount of the tax. This situation is depicted in Figure 15.19. Note from the figure that the optimal production quantity is still $q^*$. Hence the model predicts no change in production as a result of the tax. Rather, the firm will produce the same amount...
15.4 Effects of Taxation on the Energy Crisis

Figure 15.19
Both the marginal revenue and marginal cost curves shift upward by the amount of the tax, leaving the optimal production at the same level $q^*$.

but charge a higher price, thereby contributing to inflation. Note too that it is the consumer who pays the full amount of the tax in the form of a price increase.

The shortcoming with the model in Figure 15.19 is that it does not reveal whether the entire industry can in fact continue to sell the same quantity at the higher price. To find out, we need to construct a model for the industry. Thus, for each firm in the industry, consider the intersection of the firm’s various marginal revenue curves with each MC curve. (Remember that each horizontal MR curve corresponds to a price of a unit.) This situation is depicted for one firm in Figure 15.20a.

For each price, calculate the total that all firms in the industry would optimally produce. This summing procedure yields a curve for the entire industry. Because this curve represents the amount the industry would supply at various price levels, it is called a supply curve, an example of which is depicted in Figure 15.20b. Qualitatively, as the market price increases, the industry is willing to produce greater quantities.

Figure 15.20
The industry’s supply curve is obtained by summing together the amounts the firms would produce at each price level.

Next, consider aggregate consumer demand for the product at various market price levels. From a consumer’s point of view, the quantity demanded is a function of the market price. However, it is traditional to plot market price as a function of quantity (Figure 15.21a). Conceptually, for each price level, individual consumer demands could be summed as in the procedure for obtaining the industry’s supply curve. This summation is depicted graphically in Figure 15.21. Qualitatively, as the price increases, we expect the aggregate demand for the product to decrease as consumers begin to use less or substitute cheaper alternative products (Figure 15.21b).

Finally, consider the industry’s supply and demand curves together. Suppose the two curves intersect at a unique point $(q^*, p^*)$ as depicted in Figure 15.22. If the industry supplies
Chapter 15  Graphs of Functions as Models

Figure 15.21
The industry’s demand curve \( D \) represents the aggregate demand for the product at various price levels and is obtained by summing individual consumer demands at those levels. Notice that we plot price versus quantity for demand curves rather than vice versa.

Figure 15.22
The intersection of the supply and demand curves gives a market price and a market quantity that satisfy both consumers and suppliers alike.

\( q^* \) and charges \( p^* \) (supply curve), then consumers are willing to buy the amount \( q^* \) at the price \( p^* \) (demand curve). Thus, there is equilibrium in the sense that no excess supply exists at that price and both consumers and suppliers are satisfied.

Obviously, an industry does not know the precise demand curve for its product. Therefore, it is important to determine what occurs if the industry supplies an amount other than \( q^* \). For example, suppose the industry supplies an amount \( q_1 \) greater than \( q^* \) (Figure 15.23a). Then consumers are willing to buy an amount as large as \( q_1 \) only if the price is as low as \( p_1 \), forcing a reduction in price. However, if the market price drops to \( p_1 \), the industry is willing to supply only \( q_2 \) units and will cut production back to that level. Then at \( q_2 \) the unsatisfied consumers would drive the price up to \( p_2 \). (Convince yourself that the process converges to \( (q^*, p^*) \) in Figure 15.23a, where the supply curve is steeper than the demand curve.) In that situation, market forces actually drive supply and demand to the equilibrium point.

On the other hand, consider Figure 15.23b, in which the supply curve is more horizontal than the demand curve. In this case, the equilibrium point \( (q^*, p^*) \) will not be achieved by the iterative process just described. Instead, there is likely to be wild fluctuation in the amount supplied and the market price as the industry and consumers search for the equilibrium point. (Convince yourself from Figure 15.23b that the equilibrium point is difficult to achieve when the supply curve is not as steep as the demand curve.)

The demand curve is steep at \( q^* \) when consumers cannot switch in the short run to an alternative product after the price \( p^* \) increases. Water and electricity are examples of such products essential to today’s consumers. The supply curve is steep at \( q^* \) when industry cannot supply more of the product unless it incurs significant additional cost, causing a sharp rise in \( p^* \). This situation occurs when the industry is operating at full supply capacity.
15.4 Effects of Taxation on the Energy Crisis

The ease with which the equilibrium point of supply and demand is achieved depends on the relative slopes of the supply and demand curves.

and demand for the product increases sharply. At this point, the industry must increase its costs (e.g., from capitalizing additional machinery or hiring additional employees), and these costs are passed on to the consumers in the form of price increases. The power crisis in California during 2000 and 2001 is such an example.

Now consider the effect of a tax on the supply and demand curves. Suppose that a particular industry is in an equilibrium market position \((q^*, p^*)\) when a tax is added to each unit sold. Because each firm has to pay the tax to the government, each marginal cost curve shifts upward by the amount of the tax (Figure 15.19). These individual shifts cause the aggregate supply curve for the industry to shift upward by the amount of the tax as well. This phenomenon is depicted in Figure 15.24. If there is no reason for a shift in the demand curve, the intersection of the demand curve with the new supply curve shifts upward toward the left to a new equilibrium point \((q_1, p_1)\), indicating an increase in the equilibrium market price with a corresponding decrease in market quantity. Furthermore,
notice from Figure 15.24 that the increase in price, \( p_1 - p^* \), is less than the tax. Thus, the model predicts that the consumer and industry share the tax. Study Figure 15.24 carefully and convince yourself that the proportion of the tax that the consumer pays and the relative reduction in the quantity supplied at equilibrium depend on the slopes of the supply and demand curves at the time the tax is imposed.

## 15.4 PROBLEMS

1. Show that when the demand curve is very steep, a tax added to each item sold will fall primarily on consumers. Now show that when the demand curve is more nearly horizontal, the tax is paid mostly by the industry. What if the supply curve is very steep? What if the supply curve is nearly horizontal?

2. Consider the oil industry. Discuss the conditions for which the demand curve will be steep near the equilibrium. What are the situations for which the demand curve will be more horizontal (or flat)?

3. Criticize the following quotation:

   The effect of a tax on a commodity might seem at first sight to be an advance in price to the consumer. But an advance in price will diminish the demand, and a reduced demand will send the price down again. It is not certain, therefore, after all, that the tax will really raise the price.\(^5\)

4. Suppose the government pays producers a subsidy for each unit produced instead of levying a tax. Discuss the effect on the equilibrium point of the supply and demand curves. What happens to the new price and the new quantity? Discuss how the proportion of the benefits to the consumer and to the industry depends on the slopes of the supply and demand curves at the time the subsidy is given (see Problem 1).

## 15.5 A Gasoline Shortage and Taxation

Now let’s consider the energy crisis. Suppose a shortage of oil imports exists at a time when the vast majority of the population depends on the automobile to get to work and that no alternative mass transportation system is immediately available. Assume also that in the short term, most people cannot switch to more fuel-efficient cars because they are not readily available or easily affordable. These assumptions suggest qualitatively a demand curve that is steep over a wide range of values because to get to work, consumers will suffer a high increase in price before significantly cutting back on demand. Of course, eventually price levels are reached at which it no longer pays for consumers to go to work. The demand curve is portrayed in Figure 15.25. Note that as \( q \) increases, consumers enter regions where the use of additional gasoline is for leisure. In such flat regions the consumer is most sensitive to price changes.

15.5 A Gasoline Shortage and Taxation

Next, consider the oil industry’s supply curve. If the shortage of foreign oil catches the industry by surprise, most likely it will seek to find and develop alternative sources of oil. Possibly, the industry will be forced to turn to more expensive sources to provide the same quantities as before the shortage. Furthermore, in the short term, the oil industry will be more sensitive to price because it will be very difficult for the industry to provide immediate increases in supply. These arguments suggest an upward shift of the supply curve, which may become more vertical as well. Study Figure 15.25 and convince yourself that if the demand curve is steep, a significant price increase may result, but that it will take an appreciable shift in the supply curve to reduce demand significantly. Decide whether the new equilibrium point can easily be attained in Figure 15.25.

Now consider the supply–demand curves depicted in Figure 15.26. Suppose the government is dissatisfied with the reduction in demand for oil resulting from the shift in the supply curve, so it imposes a tax on each gallon of gasoline to reduce demand further. As discussed in the previous section, the tax causes the supply curve to shift upward (Figure 15.26). If the consumers are less sensitive to price than the industry, the new equilibrium point will be difficult to achieve. Furthermore, the new equilibrium point suggests that consumers will pay most of the tax in the form of a price increase. In summary, the graphical model
suggests that large fluctuations in price are probable, only modest reductions in demand are achievable, and the consumer bears the large portion of the tax burden.

What about the long-term effects of the foreign oil shortage? After the crisis, the oil industry will again have the foreign oil as well as the new sources that were developed during the crisis. These two sources cause the supply curve to shift downward and perhaps become more nearly horizontal than during the foreign oil shortage. Meanwhile, a change in consumer demand has occurred as well. Carpoools have been formed, mass transportation systems are in place, and a larger proportion of the people have switched to fuel-efficient cars. These changes in supply and demand effectively transform the x axis for the consumers. That is, for the same amount of gasoline, the consumer is operating closer to the leisure range, where the demand curve is flat (Figure 15.27). The effect of these shifts in the supply and demand curves promises lower prices, but it is difficult to determine whether a significant reduction in demand will occur from the qualitative model depicted in Figure 15.27 (see Problem 1).

Finally, suppose the government is still dissatisfied with the level of demand and imposes a tax to reduce demand further. The supply curve shifts upward by the amount of the tax as before (Figure 15.28). Notice from Figure 15.28 that because the demand curve is

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**Figure 15.27**

After the crisis both the demand and supply curves shift.

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**Figure 15.28**

If the demand curve is relatively flat, the industry pays the larger portion of the tax, and the reduction in demand is more significant.
15.5 A Gasoline Shortage and Taxation

more horizontal than the supply curve at the original equilibrium, the increase in price due to the shift in the supply curve caused by the tax is small compared to the amount of the tax. In essence, the oil industry suffers the burden of the tax. Moreover, notice that the reduction in demand is more significant with this flatter demand curve. Finally, the new equilibrium position is more easily obtained.

15.5 PROBLEMS

1. Consider the graphical model in Figure 15.27. Argue that if the demand curve fails to shift significantly to the left, an increase in the equilibrium quantity could occur after the crisis.

2. Consider the situation in which demand is a fixed curve but there is an increase in supply, so the supply curve shifts downward. Discuss how the slope of the demand curve affects the change in price and the change in quantity: How does the price change, and when does it change the most? When does it change the least? Answer similar questions for the quantity.

3. Criticize the graphical model of the oil industry. Name some major factors that have been neglected. Which of the underlying assumptions are not satisfied by the crisis situation? Did the graphical model help you identify some of the key factors and their interactions? How could you adjust the model?

15.5 PROJECTS

For Projects 1–4, complete the requirements in the referenced UMAP module (see enclosed CD) and prepare a short summary for classroom discussion.

1. “Differentiation, Curve Sketching, and Cost Functions,” by Christopher H. Nevison, UMAP 376. In this module costs and revenue for a firm are discussed using elementary calculus. The author discusses several of the economic ideas presented in this chapter.

2. “Price Discrimination and Consumer Surplus: An Application of Calculus to Economics,” by Christopher H. Nevison, UMAP 294. The topics in this module are analyzed in a competitive market, and two-tier price discrimination is also discussed. The module examines several of the economic ideas presented in this chapter.

3. “Economic Equilibrium: Simple Linear Models,” by Philip M. Tuchinsky, UMAP 208. In this module linear supply and demand functions are constructed, and the equilibrium market position is analyzed for an industry producing one product. The result is then extended to \( n \) products. The author concludes by briefly considering nonlinear and discontinuous functions.

4. “I Will If You Will . . . A Critical Mass Model,” by Jo Anne S. Grownney, UMAP 539. A graphical model is presented to treat the problems of individual behavior in a group when the individual makes a choice dependent on his or her perception of the behavior of fellow group members. The model can provide insight into paradoxical situations.
in which members of a group prefer one type of behavior but actually engage in the opposite behavior (such as not cheating versus cheating in a class).

15.5 Further Reading

