Mathematical Systems

In this section, you will review the properties of real numbers. These properties make up the third component of what is called a mathematical system. These three components are a set of numbers, operations with the set of numbers, and properties of the numbers (and operations).

Figure P.8 is a diagram that represents different mathematical systems. Note that the set of numbers for the system can vary. The set can consist of whole numbers, integers, rational numbers, real numbers, or algebraic expressions.

Basic Properties of Real Numbers

For the mathematical system that consists of the set of real numbers together with the operations of addition, subtraction, multiplication, and division, the resulting properties are called the properties of real numbers. In the list on page 17, a verbal description of each property is given, as well as one or two examples.
### Properties of Real Numbers

Let \( a, b, \) and \( c \) represent real numbers.

<table>
<thead>
<tr>
<th>Property</th>
<th>Verbal Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Closure Property of Addition</strong></td>
<td>The sum of two real numbers is a real number.</td>
</tr>
<tr>
<td>( a + b ) is a real number.</td>
<td>Example: ( 1 + 5 = 6, ) and ( 6 ) is a real number.</td>
</tr>
<tr>
<td><strong>Closure Property of Multiplication</strong></td>
<td>The product of two real numbers is a real number.</td>
</tr>
<tr>
<td>( ab ) is a real number.</td>
<td>Example: ( 7 \cdot 3 = 21, ) and ( 21 ) is a real number.</td>
</tr>
<tr>
<td><strong>Commutative Property of Addition</strong></td>
<td>Two real numbers can be added in either order.</td>
</tr>
<tr>
<td>( a + b = b + a )</td>
<td>Example: ( 2 + 6 = 6 + 2 )</td>
</tr>
<tr>
<td><strong>Commutative Property of Multiplication</strong></td>
<td>Two real numbers can be multiplied in either order.</td>
</tr>
<tr>
<td>( a \cdot b = b \cdot a )</td>
<td>Example: ( 3 \cdot (−5) = −5 \cdot 3 )</td>
</tr>
<tr>
<td><strong>Associative Property of Addition</strong></td>
<td>When three real numbers are added, it makes no difference which two are added first.</td>
</tr>
<tr>
<td>( (a + b) + c = a + (b + c) )</td>
<td>Example: ( (1 + 7) + 4 = 1 + (7 + 4) )</td>
</tr>
<tr>
<td><strong>Associative Property of Multiplication</strong></td>
<td>When three real numbers are multiplied, it makes no difference which two are multiplied first.</td>
</tr>
<tr>
<td>( (ab)c = a(bc) )</td>
<td>Example: ( (4 \cdot 3) \cdot 9 = 4 \cdot (3 \cdot 9) )</td>
</tr>
<tr>
<td><strong>Distributive Properties</strong></td>
<td>Multiplication distributes over addition.</td>
</tr>
<tr>
<td>( a(b + c) = ab + ac )</td>
<td>Examples: ( 2(3 + 4) = 2 \cdot 3 + 2 \cdot 4 )</td>
</tr>
<tr>
<td>( (a + b)c = ac + bc )</td>
<td>( (3 + 4)2 = 3 \cdot 2 + 4 \cdot 2 )</td>
</tr>
<tr>
<td><strong>Additive Identity Property</strong></td>
<td>The sum of zero and a real number equals the number itself.</td>
</tr>
<tr>
<td>( a + 0 = 0 + a = a )</td>
<td>Example: ( 4 + 0 = 0 + 4 = 4 )</td>
</tr>
<tr>
<td><strong>Multiplicative Identity Property</strong></td>
<td>The product of 1 and a real number equals the number itself.</td>
</tr>
<tr>
<td>( a \cdot 1 = 1 \cdot a = a )</td>
<td>Example: ( 5 \cdot 1 = 1 \cdot 5 = 5 )</td>
</tr>
<tr>
<td><strong>Additive Inverse Property</strong></td>
<td>The sum of a real number and its opposite is zero.</td>
</tr>
<tr>
<td>( a + (−a) = 0 )</td>
<td>Example: ( 5 + (−5) = 0 )</td>
</tr>
<tr>
<td><strong>Multiplicative Inverse Property</strong></td>
<td>The product of a nonzero real number and its reciprocal is 1.</td>
</tr>
<tr>
<td>( a \cdot \frac{1}{a} = 1, \ a \neq 0 )</td>
<td>Example: ( 7 \cdot \frac{1}{7} = 1 )</td>
</tr>
</tbody>
</table>

The operations of subtraction and division are not listed above because they fail to possess many of the properties described in the list. For instance, subtraction and division are not commutative. To see this, consider \( 4 − 3 \neq 3 − 4 \) and \( 15 ÷ 5 \neq 5 ÷ 15. \) Similarly, the examples \( 8 − (6 − 2) \neq (8 − 6) − 2 \) and \( 20 ÷ (4 ÷ 2) \neq (20 ÷ 4) ÷ 2 \) illustrate the fact that subtraction and division are not associative.
Example 1  Identifying Properties of Real Numbers

Name the property of real numbers that justifies each statement. (Note: \( a \) and \( b \) are real numbers.)

(a) \( 9 \cdot 5 = 5 \cdot 9 \)
(b) \( 4(a + 3) = 4 \cdot a + 4 \cdot 3 \)
(c) \( 6 \cdot \frac{1}{6} = 1 \)
(d) \( -3 + (2 + b) = (-3 + 2) + b \)
(e) \( (b + 8) + 0 = b + 8 \)

**Solution**

(a) This statement is justified by the Commutative Property of Multiplication.
(b) This statement is justified by the Distributive Property.
(c) This statement is justified by the Multiplicative Inverse Property.
(d) This statement is justified by the Associative Property of Addition.
(e) This statement is justified by the Additive Identity Property.

Now try Exercise 1.

Example 2  Identifying Properties of Real Numbers

The area of the rectangle in Figure P.9 can be represented in two ways: as the area of a single rectangle, or as the sum of the areas of the two rectangles.

(a) Find this area in both ways.

(b) What property of real numbers does this demonstrate?

**Solution**

(a) The area of the single rectangle with width 3 and length \( x + 2 \) is
\[ A = 3(x + 2). \]
The areas of the two rectangles are
\[ A_1 = 3(x) \text{ and } A_2 = 3(2). \]
The sum of these two areas represents the area of the single rectangle. That is,
\[ A = A_1 + A_2 \]
\[ 3(x + 2) = 3(x) + 3(2) \]
\[ = 3x + 6. \]

(b) Because the area of the single rectangle and the sum of the areas of the two rectangles are equal, you can write \( 3(x + 2) = 3x + 6 \). This demonstrates the Distributive Property.

Now try Exercise 85.

To help you understand each property of real numbers, try stating the properties in your own words.
Example 3 ■ Using the Properties of Real Numbers
Complete each statement using the specified property of real numbers.

(a) Multiplicative Identity Property
\[(4a)1 = \underline{4a}\]

(b) Associative Property of Addition
\[(a + 9) + 1 = \underline{a + (9 + 1)}\]

(c) Additive Inverse Property
\[0 = 5c + \underline{(-5c)}\]

(d) Distributive Property
\[4 \cdot b + 4 \cdot 5 = \underline{4(b + 5)}\]

Solution
(a) By the Multiplicative Identity Property, you can write
\[(4a)1 = 4a\].

(b) By the Associative Property of Addition, you can write
\[(a + 9) + 1 = a + (9 + 1)\].

(c) By the Additive Inverse Property, you can write
\[0 = 5c + (-5c)\].

(d) By the Distributive Property, you can write
\[4 \cdot b + 4 \cdot 5 = 4(b + 5)\].

Now try Exercise 29.

Additional Properties of Real Numbers
Once you have determined the basic properties of a mathematical system (called the axioms of the system), you can go on to develop other properties of the system. These additional properties are often called theorems, and the formal arguments that justify the theorems are called proofs. The list on page 20 summarizes several additional properties of real numbers.

Example 4 ■ Proof of a Property of Equality
Prove that if \(a + c = b + c\), then \(a = b\). (Use the Addition Property of Equality.)

Solution
\[a + c = b + c\]
\[(a + c) + (-c) = (b + c) + (-c)\]
\[a + [c + (-c)] = b + [c + (-c)]\]
\[a + 0 = b + 0\]
\[a = b\]

Write original equation.
Addition Property of Equality
Associative Property of Addition
Additive Inverse Property
Additive Identity Property

Now try Exercise 69.
### Additional Properties of Real Numbers

Let \( a, b, \) and \( c \) be real numbers.

#### Properties of Equality

**Addition Property of Equality**

If \( a = b \), then \( a + c = b + c \).

**Multiplication Property of Equality**

If \( a = b \), then \( ac = bc, \) \( c \neq 0 \).

**Cancellation Property of Addition**

If \( a + c = b + c \), then \( a = b \).

**Cancellation Property of Multiplication**

If \( ac = bc \) and \( c \neq 0 \), then \( a = b \).

**Reflexive Property of Equality**

\( a = a \)

**Symmetric Property of Equality**

If \( a = b \), then \( b = a \).

**Transitive Property of Equality**

If \( a = b \) and \( b = c \), then \( a = c \).

#### Properties of Zero

**Multiplication Property of Zero**

\( 0 \cdot a = 0 \)

**Division Property of Zero**

\( \frac{0}{a} = 0, \) \( a \neq 0 \)

**Division by Zero Is Undefined**

\( \frac{a}{0} \) is undefined.

#### Properties of Negation

**Multiplication by \(-1\)**

\( (-1)a = -a \)

\( (-1)(-a) = a \)

**Placement of Negative Signs**

\( -(ab) = (-a)(b) = (a)(-b) \)

**Product of Two Opposites**

\( (-a)(-b) = ab \)
Example 5  ■  Proof of a Property of Negation

Prove that

\((-1)a = -a.\)

(You may use any of the properties of equality and properties of zero.)

**Solution**

At first glance, it is a little difficult to see what you are being asked to prove. However, a good way to start is to consider carefully the definitions of each of the three numbers in the equation.

\[ a = \text{given real number} \]
\[ -1 = \text{the additive inverse of } 1 \]
\[ -a = \text{the additive inverse of } a \]

By showing that \((-1)a\) has the same properties as the additive inverse of \(a\), you will be showing that \((-1)a\) must be the additive inverse of \(a\).

\[
(-1)a + a = (-1)a + (1)(a) \quad \text{Multiplicative Identity Property}
\]
\[
= (-1 + 1)a \quad \text{Distributive Property}
\]
\[
= (0)a \quad \text{Additive Inverse Property}
\]
\[
= 0 \quad \text{Multiplication Property of Zero}
\]

Because you have shown that \((-1)a + a = 0\), you can now use the fact that \(-a + a = 0\) to conclude that \((-1)a + a = -a + a\). From this, you can complete the proof as follows.

\[
(-1)a + a = -a + a \quad \text{Shown in first part of proof}
\]
\[
(-1)a = -a \quad \text{Cancellation Property of Addition}
\]

Now try Exercise 70.

The list of additional properties of real numbers forms a very important part of algebra. Knowing the names of the properties is not especially important, but knowing how to use each property is extremely important. The next two examples show how several of the properties are used to solve common problems in algebra.

Example 6  ■  Applying Properties of Real Numbers

In the solution of the equation \(b + 2 = 6\), identify the property of real numbers that justifies each step.

**Solution**

\[
b + 2 = 6 \quad \text{Original equation}
\]

**Solution Step**  
**Property**

\[
(b + 2) + (-2) = 6 + (-2) \quad \text{Addition Property of Equality}
\]

\[
b + [2 + (-2)] = 4 \quad \text{Associative Property of Addition}
\]

\[
b + 0 = 4 \quad \text{Additive Inverse Property}
\]

\[
b = 4 \quad \text{Additive Identity Property}
\]

Now try Exercise 71.
Example 7  ■ Applying the Properties of Real Numbers

In the solution of the equation $3a = 9$, identify the property of real numbers that justifies each step.

Solution

$3a = 9$  

$\frac{1}{3}(3a) = \frac{1}{3}(9)$  

$\frac{1}{3} \cdot 3(a) = 3$  

$(1)(a) = 3$  

$a = 3$

Solution Step  

Property

NOW TRY EXERCISE 73.

P.2  ■ Exercises

VOCABULARY CHECK: Fill in the blanks.

1. The three components of a ______ are a set of numbers, operations with the set of numbers, and properties of the numbers and operations.

2. The basic properties of a mathematical system are often called ______.

3. The formal argument that justifies a theorem is called a ______.

In Exercises 1–28, name the property of real numbers that justifies the statement.

1. $3 + (-5) = -5 + 3$  
2. $-5(7) = 7(-5)$  
3. $25 - 25 = 0$  
4. $5 + 0 = 5$  
5. $6(-10) = -10(6)$  
6. $2(6 \cdot 3) = (2 \cdot 6)3$  
7. $7 \cdot 1 = 7$  
8. $4 \cdot \frac{1}{4} = 1$  
9. $25 + 35 = 35 + 25$  
10. $(-4 \cdot 10) \cdot 8 = -4(10 \cdot 8)$  
11. $3 + (12 - 9) = (3 + 12) - 9$  
12. $(16 + 8) - 5 = 16 + (8 - 5)$  
13. $(8 - 5)(10) = 8 \cdot 10 - 5 \cdot 10$  
14. $7(9 + 15) = 7 \cdot 9 + 7 \cdot 15$  
15. $(10 + 8) + 3 = 10 + (8 + 3)$  
16. $(5 + 10)(8) = 8(5 + 10)$  
17. $5(2a) = (5 \cdot 2)a$  
18. $10(2x) = (10 \cdot 2)x$  
19. $1 \cdot (5t) = 5t$  
20. $8y \cdot 1 = 8y$  
21. $3x + 0 = 3x$  
22. $0 + 8w = 8w$  
23. $\frac{1}{y} \cdot y = 1$  
24. $10x \cdot \frac{1}{10x} = 1$  
25. $3(6 + b) = 3 \cdot 6 + 3 \cdot b$  
26. $(x + 1) - (x + 1) = 0$  
27. $3(2 + x) = 3 \cdot 2 + 3x$  
28. $(6 + x) - m = 6 + (x - m)$

In Exercises 29–38, use the property of real numbers to fill in the missing part of the statement.

29. Associative Property of Multiplication  
30. Commutative Property of Addition  
31. Commutative Property of Multiplication  
32. Associative Property of Addition  
33. Distributive Property  
34. Distributive Property  
35. Commutative Property of Addition
36. Additive Inverse Property
\[ 13x - 13x = 0 \]

37. Multiplicative Identity Property
\[ (x + 8) \cdot 1 = x + 8 \]

38. Additive Identity Property
\[ 8x + 0 = 8x \]

In Exercises 39–46, give (a) the additive inverse and (b) the multiplicative inverse of the quantity.

39. 10 \hspace{1cm} 40. 18
41. -16 \hspace{1cm} 42. -52
43. 6\epsilon, \epsilon \neq 0 \hspace{1cm} 44. 2y, y \neq 0
45. x + 1, x \neq -1 \hspace{1cm} 46. y - 4, y \neq 4

In Exercises 47–54, rewrite the expression using the Associative Property of Addition or the Associative Property of Multiplication.

47. \((x + 5) - 3\) \hspace{1cm} 48. \((z - 6) + 10\)
49. \(32 + (-4 + y)\) \hspace{1cm} 50. \(15 + (3 + x)\)
51. \(3(4 \cdot 5)\) \hspace{1cm} 52. \((10 \cdot 8) \cdot 5\)
53. \(6(2y)\) \hspace{1cm} 54. \(8(3x)\)

In Exercises 55–62, rewrite the expression using the Distributive Property.

55. \(20(2 + 5)\) \hspace{1cm} 56. \((-3(4 - 8))\)
57. \(5(3x - 4)\) \hspace{1cm} 58. \(6(2x + 5)\)
59. \((x + 6)(-2)\) \hspace{1cm} 60. \((z - 10)(12)\)
61. \(-6(2y - 5)\) \hspace{1cm} 62. \(-4(10 - b)\)

In Exercises 63–68, the right side of the equation is not equal to the left side. Change the right side so that it is equal to the left side.

63. \(3(x + 5) \neq 3x + 5\)
64. \(4(x + 2) \neq 4x + 2\)
65. \(-2(x + 8) \neq -2x + 16\)
66. \(-9(x + 4) \neq -9x + 36\)
67. \(3(\frac{1}{3}) \neq 1\)
68. \(6(\frac{1}{6}) \neq 0\)

In Exercises 69 and 70, use the properties of real numbers to prove the statement.

69. If \(ac = bc\) and \(c \neq 0\), then \(a = b\).
70. \((-1)(-a) = a\)

In Exercises 71–74, identify the property of real numbers that justifies each step.

71. \(x + 5 = 3\) \hspace{1cm} \text{Original equation}
\((x + 5) + (-5) = 3 + (-5)\)
\[ x + 0 = -2 \]
\[ x = -2 \]

72. \(x - 8 = 20\) \hspace{1cm} \text{Original equation}
\((x - 8) + 8 = 20 + 8\)
\[ x + 0 = 28 \]
\[ x = 28 \]

73. \(2x - 5 = 6\) \hspace{1cm} \text{Original equation}
\((2x - 5) + 5 = 6 + 5\)
\[ 2x + 0 = 11 \]
\[ 2x = 11 \]
\[ \frac{1}{2}(2x) = \frac{1}{2}(11) \]
\[ x = 5.5 \]

74. \(3x + 4 = 10\) \hspace{1cm} \text{Original equation}
\((3x + 4) + (-4) = 10 + (-4)\)
\[ 3x + 0 = 6 \]
\[ 3x = 6 \]
\[ \frac{1}{3}(3x) = \frac{1}{3}(6) \]
\[ x = 2 \]

In Exercises 75–80, use the Distributive Property to perform the arithmetic mentally. For example, you work in an industry where the wage is $14 per hour with “time and a half” for overtime. So, your hourly wage for overtime is

75. \(16(1.75) = 16(2 - \frac{1}{2})\)
76. \(15(1\frac{1}{3}) = 15(2 - \frac{1}{3})\)
77. \(7(62) = 7(60 + 2)\)
78. \(5(49) = 5(50 - 1)\)
Number of Warehouses  In Exercises 81–84, the number of Costco warehouses for the years 1997 through 2004 are approximated by the expression

\[23.4t + 89.\]

In this expression, \(t\) represents the year, with \(t = 7\) corresponding to 1997 (see figure). (Source: Costco Wholesale)

81. Use the graph to approximate the number of warehouses in 2000.

82. Use the expression to approximate the annual increase in the number of warehouses.

83. Use the expression to predict the number of warehouses in 2007.

84. In 2003, the actual number of warehouses was 397. Compare this with the approximation given by the expression.

85. Geometry  The figure shows two adjoining rectangles. Find the total area of the rectangles in two ways.

86. Geometry  The figure shows two adjoining rectangles. Find the total area of the two rectangles in two ways.

Synthesis

True or False?  In Exercises 87–90, determine whether the statement is true or false. Justify your answer.

87. \(-6x + 6x = 0\)

88. \(-9 + 5 = -5 + 9\)

89. \(6(7 + 2) = 6(7) + 2\)

90. \(-4(8 + 1) = -4(8) - 4(1)\)

91. Think About It  Does every real number have a multiplicative inverse? Explain.

92. What is the additive inverse of a real number? Give an example of the Additive Inverse Property.

93. What is the multiplicative inverse of a real number? Give an example of the Multiplicative Inverse Property.

94. State the Multiplication Property of Zero.

95. Writing  Explain how the Addition Property of Equality can be used to allow you to subtract the same number from each side of an equation.

96. Investigation  You define a new mathematical operation using the symbol \(\odot\). This operation is defined as \(a \odot b = 2 \cdot a + b\).

(a) Is this operation commutative? Explain.

(b) Is this operation associative? Explain.