8 Complex Vector Spaces

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8.1 Complex Numbers

Use the imaginary unit \( i \) to write complex numbers.
Graphically represent complex numbers in the complex plane as points and as vectors.
Add and subtract two complex numbers, and multiply a complex number by a real scalar.
Multiply two complex numbers, and use the Quadratic Formula to find all zeros of a quadratic polynomial.
Perform operations with complex matrices, and find the determinant of a complex matrix.

COMPLEX NUMBERS

So far in the text, the scalar quantities used have been real numbers. In this chapter, you will expand the set of scalars to include complex numbers.

In algebra it is often necessary to solve quadratic equations such as \( x^2 - 3x + 2 = 0 \). The general quadratic equation is \( ax^2 + bx + c = 0 \), and its solutions are given by the Quadratic Formula

\[
 x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

where the quantity under the radical, \( b^2 - 4ac \), is called the discriminant. If \( b^2 - 4ac \geq 0 \), then the solutions are ordinary real numbers. But what can you conclude about the solutions of a quadratic equation whose discriminant is negative? For instance, the equation \( x^2 + 4 = 0 \) has a discriminant of \( b^2 - 4ac = -16 \), but there is no real number whose square is \(-16\). To overcome this deficiency, mathematicians invented the imaginary unit \( i \), defined as

\[
 i = \sqrt{-1}
\]

where \( i^2 = -1 \). In terms of this imaginary unit, \( \sqrt{-16} = 4 \sqrt{-1} = 4i \).

With this single addition of the imaginary unit \( i \) to the real number system, the system of complex numbers can be developed.

**Definition of a Complex Number**

If \( a \) and \( b \) are real numbers, then the number

\[
 a + bi
\]

is a **complex number**, where \( a \) is the **real part** and \( bi \) is the **imaginary part** of the number. The form \( a + bi \) is the **standard form** of a complex number.

Some examples of complex numbers written in standard form are \( 2 = 2 + 0i \), \( 4 + 3i \), and \( -6i = 0 - 6i \). The set of real numbers is a subset of the set of complex numbers. To see this, note that every real number \( a \) can be written as a complex number using \( b = 0 \). That is, for every real number, \( a = a + 0i \).

A complex number is uniquely determined by its real and imaginary parts. So, two complex numbers are equal if and only if their real and imaginary parts are equal. That is, if \( a + bi \) and \( c + di \) are two complex numbers written in standard form, then

\[
 a + bi = c + di
\]

if and only if \( a = c \) and \( b = d \).
8.1 Complex Numbers

THE COMPLEX PLANE

Because a complex number is uniquely determined by its real and imaginary parts, it is natural to associate the number \(a + bi\) with the ordered pair \((a, b)\). With this association, complex numbers can be represented graphically as points in a coordinate plane called the complex plane. This plane is an adaptation of the rectangular coordinate plane. Specifically, the horizontal axis is the real axis and the vertical axis is the imaginary axis. The point that corresponds to the complex number \(a + bi\) is \((a, b)\), as shown in Figure 8.1.

**EXAMPLE 1** Plotting Numbers in the Complex Plane

Plot each number in the complex plane.

\[ \textbf{a. } 4 + 3i \quad \textbf{b. } -2 - i \quad \textbf{c. } -3i \quad \textbf{d. } 5 \]

**SOLUTION**

Figure 8.2 shows the numbers plotted in the complex plane.

\[ \begin{align*}
\textbf{a.} & \quad \begin{array}{c}
\text{Imaginary axis} \\
4 & \bullet 4 + 3i \text{ or } (4, 3)
\end{array} \\
& \quad \text{Real axis} \\
& \quad -2 \quad -1 \quad 1 \quad 2 \quad 3 \quad 4
\end{align*} \]

\[ \begin{align*}
\textbf{b.} & \quad \begin{array}{c}
\text{Imaginary axis} \\
2 & \bullet -2 - i \text{ or } (-2, -1)
\end{array} \\
& \quad \text{Real axis} \\
& \quad -3 \quad -2 \quad -1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\end{align*} \]

\[ \begin{align*}
\textbf{c.} & \quad \begin{array}{c}
\text{Imaginary axis} \\
2 & \bullet -3i \text{ or } (0, -3)
\end{array} \\
& \quad \text{Real axis} \\
& \quad -3 \quad -2 \quad -1 \quad 1 \quad 2 \quad 3 \quad 4
\end{align*} \]

\[ \begin{align*}
\textbf{d.} & \quad \begin{array}{c}
\text{Imaginary axis} \\
3 & \bullet 5 \text{ or } (5, 0)
\end{array} \\
& \quad \text{Real axis} \\
& \quad -2 \quad -1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5
\end{align*} \]

Figure 8.2

Another way to represent the complex number \(a + bi\) is as a vector whose horizontal component is \(a\) and whose vertical component is \(b\). (See Figure 8.3.) (Note that the use of the letter \(i\) to represent the imaginary unit is unrelated to the use of \(i\) to represent a unit vector.)

Vector Representation of a Complex Number

Figure 8.3
Chapter 8  Complex Vector Spaces

**ADDITION, SUBTRACTION, AND SCALAR MULTIPLICATION OF COMPLEX NUMBERS**

Because a complex number consists of a real part added to a multiple of $i$, the operations of addition and multiplication are defined in a manner consistent with the rules for operating with real numbers. For instance, to add (or subtract) two complex numbers, add (or subtract) the real and imaginary parts separately.

**Definition of Addition and Subtraction of Complex Numbers**

The **sum** and **difference** of

$$a + bi \quad \text{and} \quad c + di$$

are defined as follows.

$$\begin{align*}
(a + bi) + (c + di) &= (a + c) + (b + d)i & \text{Sum} \\
(a + bi) - (c + di) &= (a - c) + (b - d)i & \text{Difference}
\end{align*}$$

**REMARK**
Note in part (a) of Example 2 that the sum of two complex numbers can be a real number.

---

**EXAMPLE 2**  Adding and Subtracting Complex Numbers

a. $(2 - 4i) + (3 + 4i) = (2 + 3) + (-4 + 4)i = 5$

b. $(1 - 3i) - (3 + i) = (1 - 3) + (-3 - 1)i = -2 - 4i$

Using the vector representation of complex numbers, you can add or subtract two complex numbers geometrically using the parallelogram rule for vector addition, as shown in Figure 8.4.

**Figure 8.4**

Many of the properties of addition of real numbers are valid for complex numbers as well. For instance, addition of complex numbers is both **associative** and **commutative**. Moreover, to find the sum of three or more complex numbers, extend the definition of addition in the natural way. For example,

\[
(2 + i) + (3 - 2i) + (-2 + 4i) = (2 + 3 - 2) + (1 - 2 + 4)i = 3 + 3i.
\]
Another property of real numbers that is valid for complex numbers is the distributive property of scalar multiplication over addition. To multiply a complex number by a real scalar, use the definition below.

**Definition of Scalar Multiplication**

If $c$ is a real number and $a + bi$ is a complex number, then the **scalar multiple** of $c$ and $a + bi$ is defined as

$$c(a + bi) = ca + cbi.$$

### Example 3

**Scalar Multiplication with Complex Numbers**

a. $3(2 + 7i) + 4(8 - i) = 6 + 21i + 32 - 4i = 38 + 17i$

b. $-4(1 + i) + 2(3 - i) - 3(1 - 4i) = -4 - 4i + 6 - 2i - 3 + 12i = -1 + 6i$

Geometrically, multiplication of a complex number by a real scalar corresponds to the multiplication of a vector by a scalar, as shown in Figure 8.5.

![Figure 8.5](image)

With addition and scalar multiplication, the set of complex numbers forms a vector space of dimension 2 (where the scalars are the real numbers). You are asked to verify this in Exercise 55.

**LINEAR ALGEBRA APPLIED**

Complex numbers have some useful applications in electronics. The state of a circuit element is described by two quantities: the voltage $V$ across it and the current $I$ flowing through it. To simplify computations, the circuit element’s state can be described by a single complex number $z = V + li$, of which the voltage and current are simply the real and imaginary parts. A similar notation can be used to express the circuit element’s capacitance and inductance.

When certain elements of a circuit are changing with time, electrical engineers often have to solve differential equations. These can often be simpler to solve using complex numbers because the equations are less complicated.
MULTIPLICATION OF COMPLEX NUMBERS

The operations of addition, subtraction, and scalar multiplication of complex numbers have exact counterparts with the corresponding vector operations. By contrast, there is no direct vector counterpart for the multiplication of two complex numbers.

Definition of Multiplication of Complex Numbers

The product of the complex numbers \(a + bi\) and \(c + di\) is defined as
\[
(a + bi)(c + di) = (ac - bd) + (ad + bc)i.
\]

Rather than try to memorize this definition of the product of two complex numbers, simply apply the distributive property, as follows.

\[
(a + bi)(c + di) = a(c + di) + bi(c + di) = ac + (ad)i + (bc)i + (bd)i^2 = ac + (ad)i + (bc)i + (bd)(-1) = ac - bd + (ad)i + (bc)i = (ac - bd) + (ad + bc)i
\]

**EXAMPLE 4**  

Multiplying Complex Numbers

a. \((-2)(1 - 3i) = -2 + 6i\)

b. \((2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2 = 8 + 2i + 3 - 4i = 11 + 2i\)

**EXAMPLE 5**  

Complex Zeros of a Polynomial

Use the Quadratic Formula to find the zeros of the polynomial \(p(x) = x^2 - 6x + 13\) and verify that \(p(x) = 0\) for each zero.

**SOLUTION**

Using the Quadratic Formula,
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.
\]

Substitute each value of \(x\) into the polynomial \(p(x)\) to verify that \(p(x) = 0\).

\[
p(3 + 2i) = (3 + 2i)^2 - 6(3 + 2i) + 13 = (3 + 2i)(3 + 2i) - 6(3 + 2i) + 13 = 9 + 6i + 6i - 4 - 18 - 12i + 13 = -16 + 12i = 0
\]

\[
p(3 - 2i) = (3 - 2i)^2 - 6(3 - 2i) + 13 = (3 - 2i)(3 - 2i) - 6(3 - 2i) + 13 = 9 - 6i - 6i - 4 - 18 + 12i + 13 = -16 - 12i = 0
\]

In Example 5, the two complex numbers \(3 + 2i\) and \(3 - 2i\) are complex conjugates of each other (together they form a conjugate pair). More will be said about complex conjugates in Section 8.2.
8.1 Complex Numbers

COMPLEX MATRICES

Now that you are able to add, subtract, and multiply complex numbers, you can apply these operations to matrices whose entries are complex numbers. Such a matrix is called complex.

Definition of a Complex Matrix

A matrix whose entries are complex numbers is called a complex matrix.

All of the ordinary operations with matrices also work with complex matrices, as demonstrated in the next two examples.

**EXAMPLE 6** Operations with Complex Matrices

Let $A$ and $B$ be the complex matrices

\[
A = \begin{bmatrix}
i & 1 + i \\
2 - 3i & 4
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
2i & 0 \\
i & 1 + 2i
\end{bmatrix}
\]

and determine each of the following.

a. $3A$  b. $(2 - i)B$  c. $A + B$  d. $BA$

**SOLUTION**

a. $3A = 3 \begin{bmatrix}
i & 1 + i \\
2 - 3i & 4
\end{bmatrix} = \begin{bmatrix}
3i & 3 + 3i \\
6 - 9i & 12
\end{bmatrix}$

b. $(2 - i)B = (2 - i) \begin{bmatrix}
2i & 0 \\
i & 1 + 2i
\end{bmatrix} = \begin{bmatrix}
2 + 4i & 0 \\
1 + 2i & 4 + 3i
\end{bmatrix}$

c. $A + B = \begin{bmatrix}
i & 1 + i \\
2 - 3i & 4
\end{bmatrix} + \begin{bmatrix}
2i & 0 \\
i & 1 + 2i
\end{bmatrix} = \begin{bmatrix}
3i & 1 + i \\
2 - 2i & 5 + 2i
\end{bmatrix}$

d. $BA = \begin{bmatrix}
i & 1 + i \\
2 - 3i & 4
\end{bmatrix} \begin{bmatrix}
2i & 0 \\
i & 1 + 2i
\end{bmatrix} = \begin{bmatrix}
-2 + 0 & 2i - 2 + 0 \\
-1 + 2 - 3i + 4i + 6 & i - 1 + 4 + 8i
\end{bmatrix} = \begin{bmatrix}
-2 & -2 + 2i \\
7 + i & 3 + 9i
\end{bmatrix}$

**EXAMPLE 7** Finding the Determinant of a Complex Matrix

Find the determinant of the matrix

\[
A = \begin{bmatrix}
2 - 4i & 2 \\
3 & 5 - 3i
\end{bmatrix}
\]

**SOLUTION**

\[
\det(A) = \begin{vmatrix}
2 - 4i & 2 \\
3 & 5 - 3i
\end{vmatrix}
\]

\[
= (2 - 4i)(5 - 3i) - (2)(3)
\]

\[
= 10 - 20i - 6i - 12 - 6
\]

\[
= -8 - 26i
\]
8.1 Exercises

**Simplifying an Expression**  In Exercises 1–6, determine the value of the expression.

1. \(\sqrt{-2}\sqrt{-3}\)  
2. \(\sqrt{8}\sqrt{-8}\)  
3. \(\sqrt{-4}\sqrt{-4}\)  
4. \(i^3\)  
5. \(i^4\)  
6. \((-i)^2\)

**Equality of Complex Numbers**  In Exercises 7–10, determine \(x\) such that the complex numbers in each pair are equal.

7. \(x + 3i, 6 + 3i\)  
8. \((2x - 8) + (x - 1)i, 2 + 4i\)  
9. \((x^2 + 6) + (2x)i, 15 + 6i\)  
10. \((-x + 4) + (x + 1)i, x + 3i\)

**Plotting Complex Numbers**  In Exercises 11–16, plot the number in the complex plane.

11. \(z = 6 - 2i\)  
12. \(z = 3i\)  
13. \(z = -5 + 5i\)  
14. \(z = 7\)  
15. \(z = 1 + 5i\)  
16. \(z = 1 - 5i\)

**Adding and Subtracting Complex Numbers**  In Exercises 17–24, find the sum or difference of the complex numbers. Use vectors to illustrate your answer.

17. \((2 + 6i) + (3 - 3i)\)  
18. \((1 + i\sqrt{2}) + (2 - i\sqrt{2})\)  
19. \((5 + i) - (5 - i)\)  
20. \(i - (3 + i)\)  
21. \(6 - (-2i)\)  
22. \((12 - 7i) - (3 + 4i)\)  
23. \((2 + i) + (2 + i)\)  
24. \((2 + i) + (2 - i)\)

**Scalar Multiplication**  In Exercises 25 and 26, use vectors to illustrate the operations geometrically. Be sure to graph the original vector.

25. \(-u\) and \(2u\), where \(u = 3 - i\)  
26. \(3u\) and \(-\frac{3u}{2}\), where \(u = 2 + i\)

**Multiplying Complex Numbers**  In Exercises 27–34, find the product.

27. \((5 - 5i)(1 + 3i)\)  
28. \((3 + i)\left(\frac{7}{3} + i\right)\)  
29. \((\sqrt{7} - i)(\sqrt{7} + i)\)  
30. \((4 + i\sqrt{2})(4 - i\sqrt{2})\)  
31. \((a + bi)^2\)  
32. \((a + bi)(a - bi)\)  
33. \((1 + i)^3\)  
34. \((2 - i)(2 + 2i)(4 + i)\)

**Finding Zeros**  In Exercises 35–40, determine all the zeros of the polynomial function.

35. \(p(x) = 2x^2 + 2x + 5\)  
36. \(p(x) = x^2 + x + 1\)  
37. \(p(x) = x^2 - 5x + 6\)  
38. \(p(x) = x^2 - 4x + 5\)  
39. \(p(x) = x^4 - 16\)  
40. \(p(x) = x^4 + 10x^2 + 9\)

**Finding Zeros**  In Exercises 41–44, use the given zero to find all zeros of the polynomial function.

41. \(p(x) = x^3 - 3x^2 + 4x - 2\)  
42. \(p(x) = x^3 - 2x^2 - 11x + 52\)  
43. \(p(x) = 2x^3 + 3x^2 + 50x + 75\)  
44. \(p(x) = x^3 + x^2 + 9x + 9\)

**Operations with Complex Matrices**  In Exercises 45–54, perform the indicated matrix operation using the complex matrices \(A\) and \(B\).

45. \((A + B)\)  
46. \((B - A)\)  
47. \(2A\)  
48. \(\frac{i}{2}B\)  
49. \(2iA\)  
50. \(\frac{2}{i}B\)  
51. \(\det(A + B)\)  
52. \(\det(B)\)  
53. \(SAB\)  
54. \(BA\)

55. **Proof**  Prove that the set of complex numbers, with the operations of addition and scalar multiplication (with real scalars), is a vector space of dimension 2.

56. **CAPSTONE**  Consider the functions \(p(x) = x^2 - 6x + 4\) and \(q(x) = x^2 - 6x + 10\).

(a) Without graphing either function, determine whether the graphs of \(p\) and \(q\) have \(x\)-intercepts. Explain your reasoning.

(b) For which of the given functions is \(x = 3 - i\) a zero? Without using the Quadratic Formula, find the other zero of this function and verify your answer.

57. (a) Evaluate \(i^n\) for \(n = 1, 2, 3, 4,\) and 5.

(b) Calculate \(i^{2010}\).

(c) Find a general formula for \(i^n\) for any positive integer \(n\).

58. Let \(A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\).

(a) Calculate \(A^n\) for \(n = 1, 2, 3, 4,\) and 5.

(b) Calculate \(A^{2010}\).

(c) Find a general formula for \(A^n\) for any positive integer \(n\).

**True or False?**  In Exercises 59 and 60, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

59. \(\sqrt{-2}\sqrt{-2} = \sqrt{4} = 2\)  
60. \((\sqrt{-10})^2 = \sqrt{100} = 10\)

61. **Proof**  Prove that if the product of two complex numbers is zero, then at least one of the numbers must be zero.
8.2 Conjugates and Division of Complex Numbers

Find the conjugate of a complex number.
Find the modulus of a complex number.
Divide complex numbers, and find the inverse of a complex matrix.

COMPLEX CONJUGATES

In Section 8.1, it was mentioned that the complex zeros of a polynomial with real coefficients occur in conjugate pairs. For instance, in Example 5 you saw that the zeros of \( p(x) = x^2 - 6x + 13 \) are \( 3 + 2i \) and \( 3 - 2i \).

In this section, you will examine some additional properties of complex conjugates.

You will begin with the definition of the conjugate of a complex number.

**Definition of the Conjugate of a Complex Number**

The conjugate of the complex number \( z = a + bi \) is denoted by \( \bar{z} \) and is given by \( \bar{z} = a - bi \).

**Example 1** Finding the Conjugate of a Complex Number

<table>
<thead>
<tr>
<th>Complex Number</th>
<th>Conjugate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = -2 + 3i )</td>
<td>( \bar{z} = -2 - 3i )</td>
</tr>
<tr>
<td>( z = 4 - 5i )</td>
<td>( \bar{z} = 4 + 5i )</td>
</tr>
<tr>
<td>( z = -2i )</td>
<td>( \bar{z} = 2i )</td>
</tr>
<tr>
<td>( z = 5 )</td>
<td>( \bar{z} = 5 )</td>
</tr>
</tbody>
</table>

Geometrically, two points in the complex plane are conjugates if and only if they are reflections in the real (horizontal) axis, as shown in Figure 8.6. Complex conjugates have many useful properties. Some of these are shown in Theorem 8.1.

**Theorem 8.1 Properties of Complex Conjugates**

For a complex number \( z = a + bi \), the following properties are true.

1. \( \bar{z} = a^2 + b^2 \)
2. \( \bar{z} \geq 0 \)
3. \( \bar{z} = 0 \) if and only if \( z = 0 \)
4. \( \bar{\bar{z}} = \bar{z} \)

**Proof**

To prove the first property, let \( z = a + bi \). Then \( \bar{z} = a - bi \) and

\[
\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2.
\]

The second and third properties follow directly from the first. Finally, the fourth property follows from the definition of the complex conjugate. That is,

\[
\bar{\bar{z}} = \overline{(a + bi)} = a - bi = a + bi = z.
\]

**Example 2** Finding the Product of Complex Conjugates

When \( z = 1 + 2i \), you have \( \bar{z} = (1 - 2i)(1 + 2i) = 1^2 + 2^2 = 1 + 4 = 5 \).
THE MODULUS OF A COMPLEX NUMBER

Because a complex number can be represented by a vector in the complex plane, it makes sense to talk about the length of a complex number. This length is called the modulus of the complex number.

Finding the Modulus of a Complex Number
For and determine the value of each modulus.

a. 

b. 

c. 

SOLUTION

a. 

b. 

Because 

c. Because 

Note that in Example 3, \(|zw| = |z||w|\). In Exercise 41, you are asked to prove that this multiplicative property of the modulus always holds. Theorem 8.2 states that the modulus of a complex number is related to its conjugate.

THEOREM 8.2 The Modulus of a Complex Number
For a complex number \(z\), 

PROOF

Let \(z = a + bi\), then \(\bar{z} = a - bi\) and \(z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2\).

REMARK
The modulus of a complex number is also called the absolute value of the number. In fact, when \(z\) is a real number, 

Fractals appear in almost every part of the universe. They have been used to study a wide variety of applications such as bacteria cultures, the human lungs, the economy, and galaxies. The most famous fractal is called the Mandelbrot Set, named after the Polish-born mathematician Benoit Mandelbrot (1924–2010). The Mandelbrot Set is based on the following sequence of complex numbers.

The behavior of this sequence depends on the value of the complex number \(c\). For some values of \(c\), the modulus of each term \(z_n\) in the sequence is less than some fixed number \(N\), and the sequence is bounded. This means that \(c\) is in the Mandelbrot Set, and its point is colored black. For other values of \(c\), the moduli of the terms of the sequence become infinitely large, and the sequence is unbounded. This means that \(c\) is not in the Mandelbrot Set, and its point is assigned a color based on “how quickly” the sequence diverges.
DIVISION OF COMPLEX NUMBERS

One of the most important uses of the conjugate of a complex number is in performing division in the complex number system. To define division of complex numbers, consider \( z = a + bi \) and \( w = c + di \) and assume that \( c \) and \( d \) are not both 0. For the quotient \( \frac{z}{w} = x + yi \) to make sense, it has to be true that

\[
z = w(x + yi) = (c + di)(x + yi) = (cx - dy) + (dx + cy)i.
\]

But, because \( z = a + bi \), you can form the linear system below.

\[
\begin{align*}
 cx - dy &= a \\
 dx + cy &= b 
\end{align*}
\]

Solving this system of linear equations for \( x \) and \( y \) yields

\[
x = \frac{ac + bd}{w^\overline{w}} \quad \text{and} \quad y = \frac{bc - ad}{w^\overline{w}}.
\]

Now, because \( \overline{w} = (a + bi)(c - di) = (ac + bd) + (bc - ad)i \), the following definition is obtained.

**Definition of Division of Complex Numbers**

The **quotient** of the complex numbers \( z = a + bi \) and \( w = c + di \) is defined as

\[
\frac{z}{w} = \frac{a + bi}{c + di} = \frac{ac + bd + bc - ad}{c^2 + d^2} = \frac{1}{\overline{w}}(z\overline{w})
\]

provided \( c^2 + d^2 \neq 0 \).

In practice, the quotient of two complex numbers can be found by multiplying the numerator and the denominator by the conjugate of the denominator, as follows.

\[
\begin{align*}
 \frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \left( \frac{c - di}{c - di} \right) \\
 &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\
 &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\
 &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i
\end{align*}
\]

**EXAMPLE 4** 
**Division of Complex Numbers**

\[
\begin{align*}
 a. \quad \frac{1}{1 + i} &= \frac{1}{1 + i} \left( \frac{1 - i}{1 - i} \right) = \frac{1 - i}{1^2 - i^2} = \frac{1 - i}{2} = \frac{1}{2} - \frac{1}{2}i \\
 b. \quad \frac{2 - i}{3 + 4i} &= \frac{2 - i}{3 + 4i} \left( \frac{3 - 4i}{3 - 4i} \right) = \frac{2 - 11i}{9 + 16} = \frac{2}{25} - \frac{11}{25}i
\end{align*}
\]
Now that you can divide complex numbers, you can find the (multiplicative) inverse of a complex matrix, as demonstrated in Example 5.

**EXAMPLE 5  Finding the Inverse of a Complex Matrix**

Find the inverse of the matrix

\[
A = \begin{bmatrix}
2 - i & -5 + 2i \\
3 - i & -6 + 2i
\end{bmatrix}
\]

and verify your solution by showing that \(AA^{-1} = I_2\).

**SOLUTION**

Using the formula for the inverse of a \(2 \times 2\) matrix from Section 2.3,

\[
A^{-1} = \frac{1}{|A|} \begin{bmatrix}
-6 + 2i & 5 - 2i \\
-3 + i & 2 - i
\end{bmatrix}
\]

Furthermore, because

\[
|A| = (2 - i)(-6 + 2i) - (-5 + 2i)(3 - i) = (-12 + 6i + 4i + 2) - (-15 + 6i + 5i + 2) = 3 - i
\]

it follows that

\[
A^{-1} = \frac{1}{3 - i} \begin{bmatrix}
-6 + 2i & 5 - 2i \\
-3 + i & 2 - i
\end{bmatrix}
\]

\[
= \frac{1}{3 - i} \begin{bmatrix}
1 & 1 \\
3 + i & 3 + i
\end{bmatrix} \begin{bmatrix}
-6 + 2i & 5 - 2i \\
-3 + i & 2 - i
\end{bmatrix}
\]

\[
= \frac{1}{10} \begin{bmatrix}
-20 & 17 - i \\
-10 & 7 - i
\end{bmatrix}
\]

To verify your solution, multiply \(A\) and \(A^{-1}\) as follows.

\[
AA^{-1} = \begin{bmatrix}
2 - i & -5 + 2i \\
3 - i & -6 + 2i
\end{bmatrix} \begin{bmatrix}
1 & 10 \\
10 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

The last theorem in this section summarizes some useful properties of complex conjugates.

**THEOREM 8.3  Properties of Complex Conjugates**

For the complex numbers \(z\) and \(w\), the following properties are true.

1. \(\overline{z + w} = \overline{z} + \overline{w}\)
2. \(\overline{z - w} = \overline{z} - \overline{w}\)
3. \(\overline{zw} = \overline{z} \overline{w}\)
4. \(\overline{z/w} = \overline{z}/\overline{w}\)

**PROOF**

To prove the first property, let \(z = a + bi\) and \(w = c + di\). Then

\[
z + w = (a + c) + (b + d)i = (a + c) - (b + d)i = (a - bi) + (c - di) = \overline{z} + \overline{w}.
\]

The proof of the second property is similar. The proofs of the other two properties are left to you.
8.2 Exercises

Finding the Conjugate  In Exercises 1–6, find the complex conjugate $\overline{z}$ and geometrically represent both $z$ and $\overline{z}$.
1. $z = 6 - 3i$
2. $z = 2 + 5i$
3. $z = -8i$
4. $z = 2i$
5. $z = 4$
6. $z = -3$

Finding the Modulus  In Exercises 7–12, find the indicated modulus, where $z = 2 + i$, $w = -3 + 2i$, and $v = -5i$.
7. $|z|$
8. $|z^2|$
9. $|zw|$
10. $|wz|$
11. $|v|$
12. $|\overline{wz}|$

13. Verify that $|wz| = |w||z| = |zw|$, where $z = 1 + i$ and $w = -1 + 2i$.
14. Verify that $|zv|^2 = |z||v|^2 = |zv|^2$, where $z = 1 + 2i$ and $v = -2 - 3i$.

Dividing Complex Numbers  In Exercises 15–20, perform the indicated operations.
15. $\frac{2 + i}{i}$
16. $\frac{1}{6 + 3i}$
17. $\frac{3 - \sqrt{3}i}{3 + \sqrt{3}i}$
18. $\frac{5 + i}{4 + i}$
19. $\frac{(2 + i)(3 - i)}{4 - 2i}$
20. $\frac{3 - i}{(2 - i)(5 + 2i)}$

Operations with Complex Rational Expressions  In Exercises 21–24, perform the operation and write the result in standard form.
21. $\frac{2}{1 + i} - \frac{3}{1 - i}$
22. $\frac{2i}{2 + i} + \frac{5}{2 - i}$
23. $\frac{i}{3 - i} + \frac{2i}{3 + i}$
24. $\frac{1 + i}{i} - \frac{3}{4 - i}$

Finding Zeros  In Exercises 25–28, use the given zero to find all zeros of the polynomial function.
25. $p(x) = 3x^3 - 4x^2 + 8x + 8$  Zero: $1 - \sqrt{3}i$
26. $p(x) = 4x^3 + 23x^2 + 34x - 10$  Zero: $-3 + i$
27. $p(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$  Zero: $-3 + \sqrt{2}i$
28. $p(x) = x^3 + 4x^2 + 14x + 20$  Zero: $-1 - 3i$

Powers of Complex Numbers  In Exercises 29 and 30, find each power of the complex number $z$.
(a) $z^2$  (b) $z^3$  (c) $z^{-1}$  (d) $z^{-2}$
29. $z = 2 - i$
30. $z = 1 + i$

Finding the Inverse of a Complex Matrix  In Exercises 31–36, determine whether the complex matrix has an inverse. If $A$ is invertible, find its inverse and verify that $AA^{-1} = I$.
31. $A = \begin{bmatrix} 2 & 6 & 3i \\ 2 - i & i & 0 \\ i & 3 & 3i \end{bmatrix}$
32. $A = \begin{bmatrix} 2i & -2 - i \\ 3 & 3i \end{bmatrix}$
33. $A = \begin{bmatrix} 1 - i & 2 \\ 1 & 1 + i \end{bmatrix}$
34. $A = \begin{bmatrix} 1 - i & 2 \\ 0 & 1 + i \end{bmatrix}$
35. $A = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$
36. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 - i & 0 & 0 \\ 0 & 0 & 1 + i \end{bmatrix}$

Singular Matrices  In Exercises 37 and 38, determine all values of the complex number $z$ for which $A$ is singular. (Hint: Set det$(A) = 0$ and solve for $z$.)
37. $A = \begin{bmatrix} 5 & z \\ 3i & 2 - i \end{bmatrix}$
38. $A = \begin{bmatrix} 2 & 2i & 1 - i \\ 1 - i & 1 + i & z \\ 1 & 0 & 0 \end{bmatrix}$

Proof  Prove that $z = \overline{z}$ if and only if $z$ is real.

40. CAPSTONE  Consider the quotient $\frac{1 + i}{6 - 2i}$
(a) Without performing any calculations, describe how to find the quotient.
(b) Explain why the process described in part (a) results in a complex number of the form $a + bi$.
(c) Find the quotient.

Proof  Prove that for any two complex numbers $z$ and $w$, each of the statements below is true.
(a) $|zw| = |z||w|$
(b) If $w \neq 0$, then $|z/w| = |z|/|w|$.

42. Graphical Interpretation  Describe the set of points in the complex plane that satisfies each of the statements below.
(a) $|z| = 3$
(b) $|z - 1 + i| = 5$
(c) $|z - i| \leq 2$
(d) $2 \leq |z| \leq 5$

43. (a) Evaluate $(1/i)^n$ for $n = 1, 2, 3, 4,$ and $5$.
(b) Calculate $(1/i)^{2000}$ and $(1/i)^{2010}$.
(c) Find a general formula for $(1/i)^n$ for any positive integer $n$.

44. (a) Verify that $(1 + i)^2 = i$.
(b) Find the two square roots of $i$.
(c) Find all zeros of the polynomial $x^4 + 1$. 
8.3 Polar Form and DeMoivre’s Theorem

Determine the polar form of a complex number, convert between the polar form and standard form of a complex number, and multiply and divide complex numbers in polar form.

Use DeMoivre’s Theorem to find powers and roots of complex numbers in polar form.

POLAR FORM OF A COMPLEX NUMBER

At this point you can add, subtract, multiply, and divide complex numbers. However, there is still one basic procedure that is missing from the algebra of complex numbers. To see this, consider the problem of finding the square root of a complex number such as . When you use the four basic operations (addition, subtraction, multiplication, and division), there seems to be no reason to guess that

That is,

\[ \left( \frac{1+i}{\sqrt{2}} \right)^2 = i. \]

To work effectively with powers and roots of complex numbers, it is helpful to use a polar representation for complex numbers, as shown in Figure 8.7. Specifically, if \( z = a + bi \) is a nonzero complex number, then let \( \theta \) be the angle from the positive real axis to the radial line passing through the point \( (a, b) \) and let \( r \) be the modulus of \( a + bi \). This leads to the following.

\[
\begin{align*}
  a &= r \cos \theta \\
  b &= r \sin \theta \\
  r &= \sqrt{a^2 + b^2}
\end{align*}
\]

So, \( a + bi = (r \cos \theta) + (r \sin \theta)i \), from which the **polar form** of a complex number is obtained.

**REMARK**

The polar form of \( z = 0 \) is expressed as \( z = 0(\cos \theta + i \sin \theta) \), where \( \theta \) is any angle.

**Definition of the Polar Form of a Complex Number**

The **polar form** of the nonzero complex number \( z = a + bi \) is given by

\[
z = r(\cos \theta + i \sin \theta)
\]

where \( a = r \cos \theta, b = r \sin \theta, r = \sqrt{a^2 + b^2}, \) and \( \tan \theta = b/a \). The number \( r \) is the **modulus** of \( z \) and \( \theta \) is the **argument** of \( z \).

Because there are infinitely many choices for the argument, the polar form of a complex number is not unique. Normally, the values of \( \theta \) that lie between \(-\pi \) and \( \pi \) are used, although on occasion it is convenient to use other values. The value of \( \theta \) that satisfies the inequality

\[-\pi < \theta \leq \pi \]

is called the **principal argument** and is denoted by \( \text{Arg}(z) \). Two nonzero complex numbers in polar form are equal if and only if they have the same modulus and the same principal argument.
8.3 Polar Form and DeMoivre’s Theorem

**EXAMPLE 1** Finding the Polar Form of a Complex Number

Find the polar form of each of the complex numbers. (Use the principal argument.)

a. $z = 1 - i$  

SOLUTION

Because $a = 1$ and $b = -1$, then $r^2 = 1^2 + (-1)^2 = 2$, which implies that $r = \sqrt{2}$. From $a = r \cos \theta$ and $b = r \sin \theta$,

$$\cos \theta = \frac{a}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin \theta = \frac{b}{r} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$  

So, $\theta = -\pi/4$ and

$$z = \sqrt{2} \left[ \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right].$$

b. Because $a = 2$ and $b = 3$, then $r^2 = 2^2 + 3^2 = 13$, which implies that $r = \sqrt{13}$. So,

$$\cos \theta = \frac{a}{r} = \frac{2}{\sqrt{13}} \quad \text{and} \quad \sin \theta = \frac{b}{r} = \frac{3}{\sqrt{13}}$$

and it follows that $\theta = 0.98$. So, the polar form is

$$z = \sqrt{13} \left[ \cos(0.98) + i \sin(0.98) \right].$$

c. Because $a = 0$ and $b = 1$, it follows that $r = 1$ and $\theta = \pi/2$, so

$$z = 1 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right).$$

The polar forms derived in parts (a), (b), and (c) are depicted graphically in Figure 8.8.

![Figure 8.8](image-url)
Express the complex number in standard form.

\[ z = 8 \left[ \cos \left( -\frac{\pi}{3} \right) + i \sin \left( -\frac{\pi}{3} \right) \right] \]

**SOLUTION**

Because \( \cos(-\pi/3) = 1/2 \) and \( \sin(-\pi/3) = -\sqrt{3}/2 \), obtain the standard form

\[ z = 8 \left[ \cos \left( -\frac{\pi}{3} \right) + i \sin \left( -\frac{\pi}{3} \right) \right] = 8 \left[ \frac{1}{2} - i \frac{\sqrt{3}}{2} \right] = 4 - 4\sqrt{3}i. \]

The polar form adapts nicely to multiplication and division of complex numbers. Suppose you have two complex numbers in polar form

\[ z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2). \]

Then the product of \( z_1 \) and \( z_2 \) is expressed as

\[ z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \]
\[ = r_1 r_2 \left[ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]. \]

Using the trigonometric identities \( \cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \) and \( \sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 \), you have

\[ z_1 z_2 = r_1 r_2 \left[ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]. \]

This establishes the first part of the next theorem. The proof of the second part is left to you. (See Exercise 75.)

**THEOREM 8.4  Product and Quotient of Two Complex Numbers**

Given two complex numbers in polar form

\[ z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2) \]

the product and quotient of the numbers are as follows.

\[ z_1 z_2 = r_1 r_2 \left[ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right] \quad \text{Product} \]
\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} \left[ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right], \quad z_2 \neq 0 \quad \text{Quotient} \]

**LINEAR ALGEBRA APPLIED**

Elliptic curves are the foundation for elliptic curve cryptography (ECC), a type of public key cryptography for secure communications over the Internet. ECC has gained popularity due to its computational and bandwidth advantages over traditional public key algorithms.

One specific variety of elliptic curve is formed using Eisenstein integers. Eisenstein integers are complex numbers of the form \( z = a + bw \), where \( \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \) and \( a \) and \( b \) are integers. These numbers can be graphed as intersection points of a triangular lattice in the complex plane. Dividing the complex plane by the lattice of all Eisenstein integers results in an elliptic curve.
Theorem 8.4 says that to multiply two complex numbers in polar form, multiply moduli and add arguments. To divide two complex numbers, divide moduli and subtract arguments. (See Figure 8.9.)

Figure 8.9

**EXAMPLE 3**  Multiplying and Dividing in Polar Form

Find $z_1z_2$ and $z_1/z_2$ for the complex numbers

$$z_1 = 5 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \quad \text{and} \quad z_2 = \frac{1}{3} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right).$$

**SOLUTION**

Because $z_1$ and $z_2$ are in polar form, apply Theorem 8.4, as follows.

**Multiply**

$$z_1z_2 = \left( 5 \right) \left( \frac{1}{3} \right) \left[ \cos \left( \frac{\pi}{4} + \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{4} + \frac{\pi}{6} \right) \right] = \frac{5}{3} \left( \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right).$$

**Divide**

$$\frac{z_1}{z_2} = \frac{5/3}{1/3} \left[ \cos \left( \frac{\pi}{4} - \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{4} - \frac{\pi}{6} \right) \right] = 15 \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right).$$

Use the standard forms of $z_1$ and $z_2$ to check the multiplication in Example 3. For instance,

$$z_1z_2 = \left( \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i \right) \left( \frac{\sqrt{6}}{6} + \frac{1}{6}i \right) = \frac{5\sqrt{6}}{12} + \frac{5\sqrt{2}}{12}i + \frac{5\sqrt{6}}{12}i + \frac{5\sqrt{2}}{12}i^2$$

$$= \frac{5\sqrt{6} - \sqrt{2}}{12} + \frac{5\sqrt{6} + 5\sqrt{2}}{12}i$$

$$= \frac{5}{3} \left( \frac{\sqrt{6} - \sqrt{2}}{4} + \frac{\sqrt{6} + \sqrt{2}}{4}i \right).$$

To verify that this answer is equivalent to the result in Example 3, use the formulas for $\cos (u + v)$ and $\sin (u + v)$ to obtain

$$\cos \left( \frac{5\pi}{12} \right) = \cos \left( \frac{\pi}{6} + \frac{\pi}{4} \right) = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \text{and} \quad \sin \left( \frac{5\pi}{12} \right) = \sin \left( \frac{\pi}{6} + \frac{\pi}{4} \right) = \frac{\sqrt{6} + \sqrt{2}}{4}.$$
DEMOIVRE’S THEOREM

The final topic in this section involves procedures for finding powers and roots of complex numbers. Repeated use of multiplication in the polar form yields
\[
  z = r(\cos \theta + i \sin \theta)
\]
\[
  z^2 = r(\cos \theta + i \sin \theta)r(\cos \theta + i \sin \theta)
  = r^2(\cos 2\theta + i \sin 2\theta)
\]
and
\[
  z^3 = r(\cos \theta + i \sin \theta)r^2(\cos 2\theta + i \sin 2\theta)
  = r^3(\cos 3\theta + i \sin 3\theta).
\]
Similarly,
\[
  z^4 = r^4(\cos 4\theta + i \sin 4\theta)
\]
and
\[
  z^5 = r^5(\cos 5\theta + i \sin 5\theta).
\]
This pattern leads to the next important theorem, named after the French mathematician Abraham DeMoivre (1667–1754). You are asked to prove this theorem in Review Exercise 85.

THEOREM 8.5 DeMoivre’s Theorem

If \( z = r(\cos \theta + i \sin \theta) \) and \( n \) is any positive integer, then
\[
  z^n = r^n(\cos n\theta + i \sin n\theta).
\]

EXAMPLE 4 Raising a Complex Number to an Integer Power

Find \((-1 + \sqrt{3}i)^{12}\) and write the result in standard form.

SOLUTION

First convert to polar form. For \(-1 + \sqrt{3}i\),
\[
r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \quad \text{and} \quad \tan \theta = \frac{\sqrt{3}}{-1} = -\sqrt{3}
\]
which implies that \( \theta = 2\pi/3 \). So,
\[
-1 + \sqrt{3}i = 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right).
\]
By DeMoivre’s Theorem,
\[
(-1 + \sqrt{3}i)^{12} = \left[ 2 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \right]^{12}
  = 2^{12} \left[ \cos \frac{12(2\pi)}{3} + i \sin \frac{12(2\pi)}{3} \right]
  = 4096(\cos 8\pi + i \sin 8\pi)
  = 4096[1 + i(0)]
  = 4096.
\]
Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial of degree $n$ has $n$ zeros in the complex number system. So, a polynomial such as $x^6 - 1$ has six zeros, and in this case you can find the six zeros by factoring and using the Quadratic Formula.

$$x^6 - 1 = (x^3 - 1)(x^3 + 1) = (x - 1)(x^2 + x + 1)(x^2 + x + 1)$$

Consequently, the zeros are

$$x = \pm 1, \quad x = \frac{-1 \pm \sqrt{3}i}{2}, \quad \text{and} \quad x = \frac{1 \pm \sqrt{3}i}{2}.$$ 

Each of these numbers is called a sixth root of 1. In general, the $n$th root of any complex number is defined as follows.

**Definition of the $n$th Root of a Complex Number**

The complex number $w = a + bi$ is an $n$th root of the complex number $z$ when

$$z = w^n = (a + bi)^n.$$ 

DeMoivre’s Theorem is useful in determining roots of complex numbers. To see how this is done, let $w$ be an $n$th root of $z$, where

$$w = s(\cos \beta + i \sin \beta) \quad \text{and} \quad z = r(\cos \theta + i \sin \theta).$$

Then, by DeMoivre’s Theorem

$$w^n = s^n(\cos n\beta + i \sin n\beta)$$

and because $w^n = z$, it follows that

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Now, because the right and left sides of this equation represent equal complex numbers, equate moduli to obtain $s^n = r$, which implies that $s = \sqrt[n]{r}$, and equate principal arguments to conclude that $\theta$ and $n\beta$ must differ by a multiple of $2\pi$. Note that $r$ is a positive real number and so $s = \sqrt[n]{r}$ is a positive real number. Consequently, for some integer $k$, $n\beta = \theta + 2\pi k$, which implies that

$$\beta = \frac{\theta + 2\pi k}{n}.$$ 

Finally, substituting this value of $\beta$ into the polar form of $w$ produces the result stated in the next theorem.

**THEOREM 8.6 The $n$th Roots of a Complex Number**

For any positive integer $n$, the complex number $z = r(\cos \theta + i \sin \theta)$ has exactly $n$ distinct roots. These $n$ roots are given by

$$\sqrt[n]{r} \left[ \cos \left( \frac{\theta + 2\pi k}{n} \right) + i \sin \left( \frac{\theta + 2\pi k}{n} \right) \right],$$

where $k = 0, 1, 2, \ldots, n - 1$. 

**REMARK**

Note that when $k$ exceeds $n - 1$, the roots begin to repeat. For instance, when $k = n$, the angle is

$$\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi$$

which yields the same values for the sine and cosine as $k = 0$. 

8.3 Polar Form and DeMoivre’s Theorem
The formula for the $n$th roots of a complex number has a nice geometric interpretation, as shown in Figure 8.10. Because the $n$th roots all have the same modulus (length) $\sqrt[n]{r}$, they lie on a circle of radius $\sqrt[n]{r}$ with center at the origin. Furthermore, the $n$ roots are equally spaced around the circle, because successive $n$th roots have arguments that differ by $2\pi/n$.

You have already found the sixth roots of 1 by factoring and the Quadratic Formula. Try solving the same problem using Theorem 8.6 to get the roots shown in Figure 8.11. When Theorem 8.6 is applied to the real number 1, the $n$th roots have a special name—the $n$th roots of unity.

**Example 5** Finding the $n$th Roots of a Complex Number

Determine the fourth roots of $i$.

**SOLUTION**

In polar form,

$$i = 1 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

so $r = 1$ and $\theta = \pi/2$. Then, by applying Theorem 8.6,

$$i^{1/4} = \sqrt[4]{1} \left[ \cos \left( \frac{\pi/2}{4} + \frac{2k\pi}{4} \right) + i \sin \left( \frac{\pi/2}{4} + \frac{2k\pi}{4} \right) \right]$$

$$= \cos \left( \frac{\pi}{8} + \frac{k\pi}{2} \right) + i \sin \left( \frac{\pi}{8} + \frac{k\pi}{2} \right).$$

Setting $k = 0, 1, 2,$ and $3$,

$$z_1 = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$$
$$z_2 = \cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8}$$
$$z_3 = \cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8}$$
$$z_4 = \cos \frac{13\pi}{8} + i \sin \frac{13\pi}{8}$$

as shown in Figure 8.12.

**Remark**

In Figure 8.12, note that when each of the four angles $\pi/8, 5\pi/8, 9\pi/8,$ and $13\pi/8$ is multiplied by 4, the result is of the form $(\pi/2) + 2k\pi$. 

![Figure 8.10](image1)

![Figure 8.11](image2)

![Figure 8.12](image3)
8.3 Exercises

Converting to Polar Form  In Exercises 1–4, express the complex number in polar form.

1. Imaginary axis
   \[ 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \]
   Real axis
   \[ -2 \quad -2 \quad -2 \quad -2 \quad -2 \]
   \[ -1 \quad -1 \quad -1 \quad -1 \quad -1 \]

2. Imaginary axis
   \[ 1 \quad 1 \quad 1 \quad 1 \quad 1 \]
   Real axis
   \[ 3 \quad 2 \quad 1 \quad 0 \quad -1 \]

3. Imaginary axis
   \[ -6 \quad -5 \quad -4 \quad -3 \quad -2 \]
   Real axis
   \[ -6 \quad -5 \quad -4 \quad -3 \quad -2 \]

4. Imaginary axis
   \[ 3 \quad 3 \quad 3 \quad 3 \quad 3 \]
   Real axis
   \[ 3 \quad 2 \quad 1 \quad 0 \quad -1 \]

Graphing and Converting to Polar Form  In Exercises 5–16, represent the complex number graphically, and give the polar form of the number. (Use the principal argument.)

5. \(-2 - 2i\)  6. \(2 + 2i\)
7. \(-2(1 + \sqrt{3}i)\)  8. \(\frac{1}{2}(\sqrt{3} - i)\)
9. \(6i\)  10. \(-2i\)
11. 7  12. 4
13. \(3 + \sqrt{3}i\)  14. \(2\sqrt{2} - i\)
15. \(-1 - 2i\)  16. \(5 + 2i\)

Graphing and Converting to Standard Form  In Exercises 17–26, represent the complex number graphically, and give the standard form of the number.

17. \(2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})\)
18. \(5(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})\)
19. \(\frac{3}{2}(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3})\)
20. \(\frac{3}{4}(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4})\)
21. \(\frac{15}{4}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})\)
22. \(8(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})\)
23. \(4(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})\)
24. \(6(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})\)
25. \(7(\cos 0 + i \sin 0)\)
26. \(9(\cos \pi + i \sin \pi)\)

Multiplying and Dividing in Polar Form  In Exercises 27–34, perform the indicated operation and leave the result in polar form.

27. \(3(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})\)
28. \(\frac{3}{4}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})\)
29. \([0.5(\cos \pi + i \sin \pi)]\)
30. \([\frac{3}{5}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})]^3\)
31. \(2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})\)
32. \(\frac{1}{2}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})\)
33. \(\frac{1}{2}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})\)
34. \(\frac{1}{2}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})\)

Finding Powers of Complex Numbers  In Exercises 35–44, use DeMoivre’s Theorem to find the indicated powers of the complex number. Express the result in standard form.

35. \((1 + i)^4\)  36. \((2 + 2i)^6\)
37. \((-1 + i)^{10}\)  38. \((\sqrt{3} + i)^7\)
39. \((1 - \sqrt{3}i)^3\)
40. \(5(\cos \frac{\pi}{9} + i \sin \frac{\pi}{9})^3\)
41. \(3(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})^4\)
42. \(\frac{5}{4}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})^{10}\)
43. \(2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})^8\)
44. \(5(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2})^4\)

Finding Square Roots of a Complex Number  In Exercises 45–52, find the square roots of the complex number.

45. \(2i\)  46. \(5i\)
47. \(-3i\)  48. \(-6i\)
49. \(2 - 2i\)  50. \(2 + 2i\)
51. \(1 + \sqrt{3}i\)  52. \(-1 - \sqrt{3}i\)
Finding and Graphing \( n \)th Roots. In Exercises 65–72,

(a) use Theorem 8.6 to find the indicated roots, (b) represent each of the roots graphically, and (c) express each of the roots in standard form.

53. Square roots: 16 \( \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \)

54. Square roots: 9 \( \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \)

55. Fourth roots: 16 \( \left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) \)

56. Fifth roots: 32 \( \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \)

57. Square roots: \(-25i\)

58. Fourth roots: \(625i\)

59. Cube roots: \(-\frac{125}{2} \left(1 + \sqrt{3}i\right)\)

60. Cube roots: \(-4\sqrt{2}(1 - i)\)

61. Cube roots: 8

62. Fourth roots: \(81i\)

63. Fourth roots: 1

64. Cube roots: 1000

Finding and Graphing Solutions. In Exercises 65–72, find all the solutions of the equation and represent your solutions graphically.

65. \( x^4 - 256i = 0 \)  
66. \( x^4 + 16i = 0 \)

67. \( x^3 + 1 = 0 \)  
68. \( x^3 - 27 = 0 \)

69. \( x^3 + 243 = 0 \)  
70. \( x^4 - 81 = 0 \)

71. \( x^3 + 64i = 0 \)  
72. \( x^4 + i = 0 \)

73. Electrical Engineering. In an electric circuit, the formula \( V = I \cdot Z \) relates voltage drop \( V \), current \( I \), and impedance \( Z \), where complex numbers can represent each of these quantities. Find the impedance when the voltage drop is 5 + 5i and the current is 2 + 4i.

74. CAPSTONE. Use the graph of the roots of a complex number.

(a) Write each of the roots in trigonometric form.

(b) Identify the complex number whose roots are given. Use a graphing utility to verify your results.

\[
\begin{align*}
\text{(i) Imaginary axis} & \quad \text{(ii) Imaginary axis} \\
\text{[Diagram with roots]} & \quad \text{[Diagram with roots]}
\end{align*}
\]

75. Proof. When provided with two complex numbers \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \), with \( z_2 \neq 0 \), prove that

\[
\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).
\]

76. Proof. Show that the complex conjugate of \( \bar{z} = r(\cos \theta + i \sin \theta) \) is \( \bar{z} = r(\cos(-\theta) + i \sin(-\theta)) \).

77. Use the polar forms of \( z \) and \( \bar{z} \) in Exercise 76 to find each of the following.

(a) \( z\bar{z} \)

(b) \( z/\bar{z}, \ z \neq 0 \)

78. Proof. Show that the negative of \( z = r(\cos \theta + i \sin \theta) \) is \( -z = r(\cos(\theta + \pi) + i \sin(\theta + \pi)) \).

79. Writing

(a) Let \( z = r(\cos \theta + i \sin \theta) \).

Sketch \( z, i z, \) and \( z/i \) in the complex plane.

(b) What is the geometric effect of multiplying a complex number \( z \) by \( i \)? What is the geometric effect of dividing \( z \) by \( i \)?

80. Calculus. Recall that the Maclaurin series for \( e^x, \sin x, \) and \( \cos x \) are

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

(a) Substitute \( x = i\theta \) in the series for \( e^x \) and show that \( e^{i\theta} = \cos \theta + i \sin \theta \).

(b) Show that any complex number \( z = a + bi \) can be expressed in polar form as \( z = re^{i\theta} \).

(c) Prove that if \( z = re^{i\theta} \), then \( \bar{z} = re^{-i\theta} \).

(d) Prove the formula \( e^{i\pi} = -1 \).

True or False? In Exercises 81 and 82, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

81. Although the square of the complex number \( bi \) is given by \( (bi)^2 = -b^2 \), the absolute value of the complex number \( z = a + bi \) is defined as \( |a + bi| = \sqrt{a^2 + b^2} \).

82. Geometrically, the \( n \)th roots of any complex number \( z \) are all equally spaced around the unit circle centered at the origin.
8.4 Complex Vector Spaces and Inner Products

Recognize and perform vector operations in complex vector spaces $C^n$, represent a vector in $C^n$ by a basis, and find the Euclidean inner product, Euclidean norm, and Euclidean distance in $C^n$.

**COMPLEX VECTOR SPACES**

All the vector spaces studied so far in the text have been real vector spaces because the scalars have been real numbers. A complex vector space is one in which the scalars are complex numbers. So, if $v_1, v_2, \ldots, v_m$ are vectors in a complex vector space, then a linear combination is of the form

$$c_1v_1 + c_2v_2 + \ldots + c_mv_m$$

where the scalars $c_1, c_2, \ldots, c_m$ are complex numbers. The complex version of $R^n$ is the complex vector space $C^n$ consisting of ordered $n$-tuples of complex numbers. So, a vector in $C^n$ has the form

$$v = (a_1 + bi, a_2 + bj, \ldots, a_n + bi).$$

It is also convenient to represent vectors in $C^n$ by column matrices of the form

$$v = \begin{bmatrix} a_1 + bi \\ a_2 + bj \\ \vdots \\ a_n + bi \end{bmatrix}.$$

As with $R^n$, the operations of addition and scalar multiplication in $C^n$ are performed component by component.

**EXAMPLE 1** Vector Operations in $C^n$

Let

$$v = (1 + 2i, 3 - i) \text{ and } u = (-2 + i, 4)$$

be vectors in the complex vector space $C^2$. Determine each vector.

a. $v + u$

b. $(2 + i)v$

c. $3v - (5 - i)u$

**SOLUTION**

a. In column matrix form, the sum $v + u$ is

$$v + u = \begin{bmatrix} 1 + 2i \\ 3 - i \end{bmatrix} + \begin{bmatrix} -2 + i \\ 4 \end{bmatrix} = \begin{bmatrix} -1 + 3i \\ 7 - i \end{bmatrix}.$$

b. Because $(2 + i)(1 + 2i) = 5i$ and $(2 + i)(3 - i) = 7 + i$,

$$v = (2 + i)(1 + 2i, 3 - i)$$

$$= (5i, 7 + i).$$

c. $3v - (5 - i)u = 3(1 + 2i, 3 - i) - (5 - i)(-2 + i, 4)$

$$= (3 + 6i, 9 - 3i) - (-9 + 7i, 20 - 4i)$$

$$= (12 - i, -11 + i).$$
Many of the properties of $\mathbb{R}^n$ are shared by $\mathbb{C}^n$. For instance, the scalar multiplicative identity is the scalar $1$ and the additive identity in $\mathbb{C}^n$ is $\mathbf{0} = (0, 0, \ldots, 0)$. The standard basis for $\mathbb{C}^n$ is simply

\[ \mathbf{e}_1 = (1, 0, 0, \ldots, 0) \]
\[ \mathbf{e}_2 = (0, 1, 0, \ldots, 0) \]
\[ \vdots \]
\[ \mathbf{e}_n = (0, 0, 0, \ldots, 1) \]

which is the standard basis for $\mathbb{R}^n$. Because this basis contains $n$ vectors, it follows that the dimension of $\mathbb{C}^n$ is $n$. Other bases exist; in fact, any linearly independent set of $n$ vectors in $\mathbb{C}^n$ can be used, as demonstrated in Example 2.

### Example 2 Verifying a Basis

Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(i, 0, 0), (i, i, 0), (0, 0, i)\}$ is a basis for $\mathbb{C}^3$.

**Solution**

Because the dimension of $\mathbb{C}^3$ is 3, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis if it is linearly independent. To check for linear independence, set a linear combination of the vectors in $S$ equal to $\mathbf{0}$, as follows.

\[ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (0, 0, 0) \]
\[ (c_1i, 0, 0) + (c_2i, c_2i, 0) + (0, 0, c_3i) = (0, 0, 0) \]
\[ ((c_1 + c_2)i, c_2i, c_3i) = (0, 0, 0) \]

This implies that

\[ (c_1 + c_2)i = 0 \]
\[ c_2i = 0 \]
\[ c_3i = 0. \]

So, $c_1 = c_2 = c_3 = 0$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

### Example 3 Representing a Vector in $\mathbb{C}^n$ by a Basis

Use the basis $S$ in Example 2 to represent the vector $\mathbf{v} = (2, i, 2 - i)$.

**Solution**

By writing

\[ \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \]
\[ = ((c_1 + c_2)i, c_2i, c_3i) \]
\[ = (2, i, 2 - i) \]

you can obtain

\[ (c_1 + c_2)i = 2 \]
\[ c_2i = i \]
\[ c_3i = 2 - i \]

which implies that

\[ c_2 = 1, \quad c_1 = \frac{2 - i}{i} = -1 - 2i, \quad \text{and} \quad c_3 = \frac{2 - i}{i} = -1 - 2i. \]

So, $\mathbf{v} = (-1 - 2i)\mathbf{v}_1 + \mathbf{v}_2 + (-1 - 2i)\mathbf{v}_3$. 

**Remark**

Try verifying that this linear combination yields $(2, i, 2 - i)$.
8.4 Complex Vector Spaces and Inner Products

Other than \( \mathbb{C}^n \), there are several additional examples of complex vector spaces. For instance, the set of \( m \times n \) complex matrices with matrix addition and scalar multiplication forms a complex vector space. Example 4 describes a complex vector space in which the vectors are functions.

**EXAMPLE 4**

**The Space of Complex-Valued Functions**

Consider the set \( S \) of complex-valued functions of the form

\[ f(x) = f_1(x) + i f_2(x) \]

where \( f_1 \) and \( f_2 \) are real-valued functions of a real variable. The set of complex numbers forms the scalars for \( S \), and vector addition is defined by

\[ f(x) + g(x) = [f_1(x) + f_2(x)] + [i g_1(x) + i g_2(x)] \]

\[ = [f_1(x) + g_1(x)] + i [f_2(x) + g_2(x)] . \]

It can be shown that \( S \), scalar multiplication, and vector addition form a complex vector space. For instance, to show that \( S \) is closed under scalar multiplication, let \( c = a + bi \) be a complex number. Then

\[ c f(x) = (a + bi)[f_1(x) + i f_2(x)] \]

\[ = [a f_1(x) - b f_2(x)] + i [b f_1(x) + a f_2(x)] \]

is in \( S \).

The definition of the Euclidean inner product in \( \mathbb{C}^n \) is similar to the standard dot product in \( \mathbb{R}^n \), except that here the second factor in each term is a complex conjugate.

**Definition of the Euclidean Inner Product in \( \mathbb{C}^n \)**

Let \( u \) and \( v \) be vectors in \( \mathbb{C}^n \). The **Euclidean inner product** of \( u \) and \( v \) is given by

\[ u \cdot v = u_1 \overline{v_1} + u_2 \overline{v_2} + \cdots + u_n \overline{v_n} . \]

**EXAMPLE 5**

**Finding the Euclidean Inner Product in \( \mathbb{C}^3 \)**

Determine the Euclidean inner product of the vectors

\[ u = (2 + i, 0, 4 - 5i) \quad \text{and} \quad v = (1 + i, 2 + i, 0) . \]

**SOLUTION**

\[ u \cdot v = u_1 \overline{v_1} + u_2 \overline{v_2} + u_3 \overline{v_3} \]

\[ = (2 + i)(1 - i) + 0(2 - i) + (4 - 5i)(0) \]

\[ = 3 - i \]

Several properties of the Euclidean inner product in \( \mathbb{C}^n \) are stated in the following theorem.

**THEOREM 8.7**

**Properties of the Euclidean Inner Product**

Let \( u, v, \) and \( w \) be vectors in \( \mathbb{C}^n \) and let \( k \) be a complex number. Then the following properties are true.

1. \( u \cdot v = \overline{v \cdot u} \)
2. \( (u + v) \cdot w = u \cdot w + v \cdot w \)
3. \( (k u) \cdot v = k (u \cdot v) \)
4. \( u \cdot (k v) = k (u \cdot v) \)
5. \( u \cdot u \geq 0 \)
6. \( u \cdot u = 0 \) if and only if \( u = 0 \).
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PROOF

The proof of the first property is shown below, and the proofs of the remaining properties have been left to you (see Exercises 59–63). Let

\[ \mathbf{u} = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, \ldots, v_n). \]

Then

\[ \mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + \cdots + v_nu_n \]

The Euclidean inner product in \( \mathbb{C}^n \) is used to define the Euclidean norm (or length) of a vector in \( \mathbb{C}^n \) and the Euclidean distance between two vectors in \( \mathbb{C}^n \).

Definitions of the Euclidean Norm and Distance in \( \mathbb{C}^n \)

The Euclidean norm (or length) of \( \mathbf{u} \) in \( \mathbb{C}^n \) is denoted by \( \| \mathbf{u} \| \) and is

\[ \| \mathbf{u} \| = (\mathbf{u} \cdot \mathbf{u})^{1/2}. \]

The Euclidean distance between \( \mathbf{u} \) and \( \mathbf{v} \) is

\[ d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \|. \]

The Euclidean norm and distance may be expressed in terms of components, as follows (see Exercise 51).

\[ \| \mathbf{u} \| = \left( |u_1|^2 + |u_2|^2 + \cdots + |u_n|^2 \right)^{1/2} \]
\[ d(\mathbf{u}, \mathbf{v}) = \left( |u_1 - v_1|^2 + |u_2 - v_2|^2 + \cdots + |u_n - v_n|^2 \right)^{1/2} \]

EXAMPLE 6  Finding the Euclidean Norm and Distance in \( \mathbb{C}^n \)

Let \( \mathbf{u} = (2 + i, 0, 4 - 5i) \) and \( \mathbf{v} = (1 + i, 2 + i, 0) \).

a. Find the norms of \( \mathbf{u} \) and \( \mathbf{v} \).  

b. Find the distance between \( \mathbf{u} \) and \( \mathbf{v} \).

SOLUTION

a. \[ \| \mathbf{u} \| = \left( |u_1|^2 + |u_2|^2 + |u_3|^2 \right)^{1/2} \]
\[ = \left[ (2^2 + 1^2) + (0^2 + 0^2) + (4^2 + (-5)^2) \right]^{1/2} \]
\[ = (5 + 0 + 41)^{1/2} \]
\[ = \sqrt{46} \]

b. \[ \| \mathbf{v} \| = \left( |v_1|^2 + |v_2|^2 + |v_3|^2 \right)^{1/2} \]
\[ = \left[ (1^2 + 1^2) + (2^2 + 1^2) + (0^2 + 0^2) \right]^{1/2} \]
\[ = (2 + 5 + 0)^{1/2} \]
\[ = \sqrt{7} \]

b. \[ d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| \]
\[ = \| (1, -2 - i, 4 - 5i) \| \]
\[ = \left[ (1^2 + 0^2) + ((-2)^2 + (-1)^2) + (4^2 + (-5)^2) \right]^{1/2} \]
\[ = (1 + 5 + 41)^{1/2} \]
\[ = \sqrt{47} \]
8.4 Complex Vector Spaces and Inner Products

COMPLEX INNER PRODUCT SPACES

The Euclidean inner product is the most commonly used inner product in $C^n$. On occasion, however, it is useful to consider other inner products. To generalize the notion of an inner product, use the properties listed in Theorem 8.7.

Definition of a Complex Inner Product

Let $u$ and $v$ be vectors in a complex vector space. A function that associates $u$ and $v$ with the complex number $\langle u, v \rangle$ is called a complex inner product when it satisfies the following properties.

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle ku, v \rangle = k\langle u, v \rangle$
4. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

A complex vector space with a complex inner product is called a complex inner product space or unitary space.

EXAMPLE 7 A Complex Inner Product Space

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in the complex space $C^2$. Show that the function defined by

$$\langle u, v \rangle = u_1 \overline{v_1} + 2u_2 \overline{v_2}$$

is a complex inner product.

SOLUTION

Verify the four properties of a complex inner product, as follows.

1. $\langle v, u \rangle = \overline{\langle u, v \rangle} = \overline{u_1 \overline{v_1} + 2u_2 \overline{v_2}} = u_1 \overline{v_1} + 2u_2 \overline{v_2} = \langle u, v \rangle$
2. $\langle u + v, w \rangle = (u_1 + v_1) \overline{w_1} + 2(u_2 + v_2) \overline{w_2}$
   $$= (u_1 \overline{w_1} + 2u_2 \overline{w_2}) + (v_1 \overline{w_1} + 2v_2 \overline{w_2})$$
   $$= \langle u, w \rangle + \langle v, w \rangle$$
3. $\langle ku, v \rangle = (ku_1) \overline{w_1} + 2(ku_2) \overline{w_2} = k(u_1 \overline{w_1} + 2u_2 \overline{w_2}) = k\langle u, v \rangle$
4. $\langle u, u \rangle = u_1 \overline{u_1} + 2u_2 \overline{u_2} = |u_1|^2 + 2|u_2|^2 \geq 0$
   Moreover, $\langle u, u \rangle = 0$ if and only if $u_1 = u_2 = 0$.

Because all the properties hold, $\langle u, v \rangle$ is a complex inner product.

Complex vector spaces and inner products have an important application called the Fourier transform, which decomposes a function into a sum of orthogonal basis functions. The given function is projected onto the standard basis functions for varying frequencies to get the Fourier amplitudes for each frequency. Like Fourier coefficients and the Fourier approximation, this transform is named after the French mathematician Jean-Baptiste Joseph Fourier (1768–1830).

The Fourier transform is integral to the study of signal processing. To understand the basic premise of this transform, imagine striking two piano keys simultaneously. Your ear receives only one signal, the mixed sound of the two notes, and yet your brain is able to separate the notes. The Fourier transform gives a mathematical way to take a signal and separate out its frequency components.
### 8.4 Exercises

**Vector Operations** In Exercises 1–8, perform the indicated operation using
\[ u = (i, 3 - i), \quad v = (2 + i, 3 + i), \text{ and } w = (4i, 6). \]

1. \( 3u \)
2. \( 4iw \)
3. \( (1 + 2i)w \)
4. \( iv + 3w \)
5. \( u - (2 - i)v \)
6. \( (6 + 3i)v - (2 + 2i)w \)
7. \( u + iv + 2iw \)
8. \( 2iv - (3 - i)w + u \)

**Linear Dependence or Independence** In Exercises 9–12, determine whether the set of vectors is linearly independent or linearly dependent.

9. \( \{(1, i), (i, -1)\} \)
10. \( \{(1 + i, 1 - i, 1), (i, 0, 1), (-2, -1 + i, 0)\} \)
11. \( \{(1, i, 1 + i), (0, i, -i), (0, 0, 1)\} \)
12. \( \{(1 + i, 1 - i, 0), (1 - i, 0, 0), (0, 1, 1)\} \)

**Verifying a Basis** In Exercises 13–16, determine whether \( S \) is a basis for \( \mathbb{C}^n \).

13. \( S = \{(1, -i), (i, 1)\} \)
14. \( S = \{(1, i), (i, 1)\} \)
15. \( S = \{(i, 0, 0), (0, i, i), (0, 0, 1)\} \)
16. \( S = \{(1 - i, 0, 1), (2, i, 1 + i), (1 - 1, 1, 1)\} \)

**Representing a Vector by a Basis** In Exercises 17–20, express \( v \) as a linear combination of each of the following basis vectors.

(a) \( \{(i, 0, 0), (i, 0, i), (i, i, i)\} \)
(b) \( \{(1, 0, 0), (1, 1, 0), (0, 0, 1)\} \)
17. \( v = (1, 2, 0) \)
18. \( v = (1 - i, 1 + i, -3) \)
19. \( v = (-i, 2 + i, -1) \)
20. \( v = (i, i, i) \)

**Finding Euclidean Inner Products** In Exercises 21 and 22, determine the Euclidean inner product \( u \cdot v \).

21. \( u = (-i, 2i, 1 - i) \)
22. \( u = (4 + i, i, 0) \)
\[ v = (3i, 0, 1 + 2i) \]
\[ v = (3i, 0, 1 + 2i) \]
\[ v = (3i, 0, 1 + 2i) \]
\[ v = (3i, 0, 1 + 2i) \]

**Properties of Euclidean Inner Products** In Exercises 23–26, let \( u = (1 - i, 3i), v = (2i, 2 + i), w = (1 + i, 0) \), and \( k = -i \). Evaluate the expressions in parts (a) and (b) to verify that they are equal.

23. \( a) u \cdot v \quad b) \overrightarrow{u} \cdot \overrightarrow{v} \)
26. \( a) u \cdot (kv) \quad b) k(u \cdot v) \)

**Finding the Euclidean Norm** In Exercises 27–34, determine the Euclidean norm of \( v \).

27. \( v = (i, 2) \)
28. \( v = (1, 0) \)
29. \( v = (2 + i, 2 - i) \)
30. \( v = (2 + 3i, 2 - 3i) \)
31. \( v = (1, 2 + i, -i) \)
32. \( v = (0, 0, 0) \)
33. \( v = (1 - 2i, i, 3i, 1 + i) \)
34. \( v = (2, -1 + i, 2 - i, 4i) \)

**Finding the Euclidean Distance** In Exercises 35–40, determine the Euclidean distance between \( u \) and \( v \).

35. \( u = (1, 0), v = (i, i) \)
36. \( u = (2 + i, 4, -i), v = (2 + i, 4, -i) \)
37. \( u = (i, 2i, 2i), v = (0, 1, 0) \)
38. \( u = (\sqrt{2}, 2i, -1), v = (i, i, i) \)
39. \( u = (1, 0), v = (0, 1) \)
40. \( u = (1, 2, 1, -2i), v = (i, 2i, i, 2) \)

**Complex Inner Products** In Exercises 41–44, determine whether the function is a complex inner product, where \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \).

41. \( \langle u, v \rangle = u_1^* + u_2v_2 \)
42. \( \langle u, v \rangle = (u_1 + v_1) + 2(u_2 + v_2) \)
43. \( \langle u, v \rangle = 4u_1v_1^* + 6u_2v_2^* \)
44. \( \langle u, v \rangle = u_1v_1 - u_2v_2 \)

**Finding Complex Inner Products** In Exercises 45–48, use the inner product \( \langle u, v \rangle = u_1v_1^* + 2u_2v_2^* \) to find \( \langle u, v \rangle \).

45. \( u = (2i, -i) \) and \( v = (i, 4i) \)
46. \( u = (3 + i, i) \) and \( v = (2 - i, 2i) \)
47. \( u = (2 - i, 2 + i) \) and \( v = (3 - i, 3 + 2i) \)
48. \( u = (4 + 2i, 3) \) and \( v = (2 - 3i, -2) \)

**Finding Complex Inner Products** In Exercises 49 and 50, use the inner product
\[ \langle u, v \rangle = u_{11}v_{11}^* + u_{12}v_{12}^* + u_{21}v_{21}^* + u_{22}v_{22}^* \]
where
\[ u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \text{ and } v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \]
to find \( \langle u, v \rangle \).

49. \( u = \begin{bmatrix} 0 & i \\ 1 & -2i \end{bmatrix} \) and \( v = \begin{bmatrix} 1 & 1 - 2i \\ 0 & i \end{bmatrix} \)
50. \( u = \begin{bmatrix} 1 & 2i \\ 1 + i & 0 \end{bmatrix} \) and \( v = \begin{bmatrix} i & -2i \\ 3i & -1 \end{bmatrix} \)
51. Let \( u = (a_1 + b_1i, a_2 + b_2i, \ldots, a_n + b_ni) \).
   (a) Use the definitions of Euclidean norm and Euclidean inner product to show that
   \[
   |u| = (|a_1|^2 + |b_1|^2 + \cdots + |a_n|^2 + |b_n|^2)^{1/2}.
   \]
   (b) Use the results of part (a) to show that
   \[
   d(u, v) = (|a_1 - v_1|^2 + |a_2 - v_2|^2 + \cdots + |a_n - v_n|^2)^{1/2}.
   \]

52. CAPSTONE The complex Euclidean inner product of \( u \) and \( v \) is sometimes called the complex dot product. Compare the properties of the complex dot product in \( \mathbb{C}^n \) and those of the dot product in \( \mathbb{R}^n \).
   (a) Which properties are the same? Which properties are different?
   (b) Explain the reasons for the differences.

53. Let \( v_1 = (i, 0, 0) \) and \( v_2 = (i, i, 0) \). If \( v_1 = (z_1, z_2, z_3) \) and the set \( \{v_1, v_2, v_3\} \) is not a basis for \( \mathbb{C}^3 \), what does this imply about \( z_1, z_2, \) and \( z_3 \)?

54. Let \( v_1 = (i, i, 2) \) and \( v_2 = (1, 0, 1) \). Determine a vector \( v_3 \) such that \( \{v_1, v_2, v_3\} \) is a basis for \( \mathbb{C}^3 \).

Properties of Complex Inner Products In Exercises 55–58, verify the statement using the properties of a complex inner product.

55. \( \langle u, kv+w \rangle = \overline{k} \langle u, v \rangle + \langle u, w \rangle \)
56. \( \langle u, 0 \rangle = 0 \)
57. \( \langle u, v \rangle + \langle v, u \rangle = \langle v, 2u \rangle \)
58. \( \langle u, \overline{k}v \rangle = \overline{k} \langle u, v \rangle \)

Proof In Exercises 59–63, prove the property, where \( u, v, \) and \( w \) are vectors in \( \mathbb{C}^n \) and \( k \) is a complex number.

59. \( \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \)
60. \( \langle ku, v \rangle = k \langle u, v \rangle \)
61. \( \langle u, \overline{k}v \rangle = \overline{k} \langle u, v \rangle \)
62. \( u \cdot u \geq 0 \)
63. \( u \cdot u = 0 \) if and only if \( u = 0 \).

64. Writing Let \( \langle u, v \rangle \) be a complex inner product and let \( k \) be a complex number. How are \( \langle u, v \rangle \) and \( \langle ku, v \rangle \) related?

Finding a Linear Transformation In Exercises 65 and 66, determine the linear transformation \( T : \mathbb{C}^n \to \mathbb{C}^n \) that has the given characteristics.

65. \( T(1, 0) = (2 + i, 1) \)
   \( T(0, 1) = (0, -i) \)
66. \( T(i, 0) = (2 + i, 1) \)
   \( T(0, i) = (0, -i) \)

8.4 Exercises 419

Finding an Image and a Preimage In Exercises 67–70, the linear transformation \( T : \mathbb{C}^n \to \mathbb{C}^n \) is shown by \( T(v) = Av \). Find the image of \( v \) and the preimage of \( w \).

67. \( A = \begin{bmatrix} 1 & 0 \\ i & i \end{bmatrix}, \ v = \begin{bmatrix} 1 + i \\ i - 1 \end{bmatrix}, \ w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)
68. \( A = \begin{bmatrix} 0 & i & 1 \\ i & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \ v = \begin{bmatrix} 0 \\ i \\ 1 + i \end{bmatrix}, \ w = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix} \)
69. \( A = \begin{bmatrix} 1 & 0 \\ i & i \end{bmatrix}, \ v = \begin{bmatrix} 2 - i \\ 3 + 2i \end{bmatrix}, \ w = \begin{bmatrix} 2 \\ 3i \end{bmatrix} \)
70. \( A = \begin{bmatrix} 0 & 1 & 1 \\ i & i & -1 \\ 0 & 0 & 0 \end{bmatrix}, \ v = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \ w = \begin{bmatrix} 1 - i \\ 1 + i \end{bmatrix} \)

71. Find the kernel of the linear transformation in Exercise 68.
72. Find the kernel of the linear transformation in Exercise 69.

Finding an Image In Exercises 73 and 74, find the image of \( v = (i, i) \) for the indicated composition, where \( T_1 \) and \( T_2 \) are the matrices below.

73. \( T_2 \circ T_1 \)
74. \( T_1 \circ T_2 \)

75. Determine which of the sets below are subspaces of the vector space of \( 2 \times 2 \) complex matrices.
   (a) The set of \( 2 \times 2 \) symmetric matrices.
   (b) The set of \( 2 \times 2 \) matrices \( A \) satisfying \( (\overline{A})^T = A \).
   (c) The set of \( 2 \times 2 \) matrices in which all entries are real.
   (d) The set of \( 2 \times 2 \) diagonal matrices.

76. Determine which of the sets below are subspaces of the vector space of complex-valued functions (see Example 4).
   (a) The set of all functions \( f \) satisfying \( f(i) = 0 \).
   (b) The set of all functions \( f \) satisfying \( f(0) = 1 \).
   (c) The set of all functions \( f \) satisfying \( f(i) = f(-i) \).

True or False? In Exercises 77 and 78, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

77. Using the Euclidean inner product of \( u \) and \( v \) in \( \mathbb{C}^n \),\( u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n \).
78. The Euclidean norm of \( u \) in \( \mathbb{C}^n \) denoted by \( ||u|| \) is \( (u \cdot u)^{1/2} \).
Find the conjugate transpose $A^*$ of a complex matrix $A$.

Determine if a matrix $A$ is unitary.

Find the eigenvalues and eigenvectors of a Hermitian matrix, and diagonalize a Hermitian matrix.

### CONJUGATE TRANSPOSE OF A MATRIX

Problems involving diagonalization of complex matrices and the associated eigenvalue problems require the concepts of unitary and Hermitian matrices. These matrices roughly correspond to orthogonal and symmetric real matrices. In order to define unitary and Hermitian matrices, the concept of the conjugate transpose of a complex matrix must first be introduced.

#### Definition of the Conjugate Transpose of a Complex Matrix

The **conjugate transpose** of a complex matrix $A$, denoted by $A^*$, is given by

$$A^* = \overline{A}^T$$

where the entries of $\overline{A}$ are the complex conjugates of the corresponding entries of $A$.

Note that if $A$ is a matrix with real entries, then $A^* = A^T$. To find the conjugate transpose of a matrix, first calculate the complex conjugate of each entry and then take the transpose of the matrix, as shown in the following example.

#### EXAMPLE 1 Finding the Conjugate Transpose of a Complex Matrix

Determine $A^*$ for the matrix

$$A = \begin{bmatrix} 3 + 7i & 0 \\ 2i & 4 - i \end{bmatrix}$$

**SOLUTION**

$$\overline{A} = \begin{bmatrix} 3 + 7i & 0 \\ 2i & 4 - i \end{bmatrix} = \begin{bmatrix} 3 - 7i & 0 \\ -2i & 4 + i \end{bmatrix}$$

$$A^* = \overline{A}^T = \begin{bmatrix} 3 - 7i & -2i \\ 0 & 4 + i \end{bmatrix}$$

Several properties of the conjugate transpose of a matrix are listed in the following theorem. The proofs of these properties are straightforward and are left for you to supply in Exercises 47–50.

#### THEOREM 8.8 Properties of the Conjugate Transpose

If $A$ and $B$ are complex matrices and $k$ is a complex number, then the following properties are true.

1. $(A^*)^* = A$
2. $(A + B)^* = A^* + B^*$
3. $(kA)^* = \overline{k} A^*$
4. $(AB)^* = B^*A^*$
UNITARY MATRICES

Recall that a real matrix $A$ is orthogonal if and only if $A^{-1} = A^T$. In the complex system, matrices having the property that $A^{-1} = A^*$ are more useful, and such matrices are called unitary.

**Definition of Unitary Matrix**

A complex matrix $A$ is unitary when

$$A^{-1} = A^*.$$ 

**EXAMPLE 2**  

**A Unitary Matrix**

Show that the matrix $A$ is unitary.

$$A = \frac{1}{2} \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{bmatrix}$$

**SOLUTION**

Begin by finding the product $AA^*$.

$$AA^* = \frac{1}{2} \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{bmatrix} \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because

$$AA^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

it follows that $A^* = A^{-1}$. So, $A$ is a unitary matrix.

Recall from Section 7.3 that a real matrix is orthogonal if and only if its row (or column) vectors form an orthonormal set. For complex matrices, this property characterizes matrices that are unitary. Note that a set of vectors

$\{v_1, v_2, \ldots, v_m\}$

in $\mathbb{C}^n$ (a complex Euclidean space) is called orthonormal when the statements below are true.

1. $\|v_i\| = 1$, $i = 1, 2, \ldots, m$
2. $v_i \cdot v_j = 0$, $i \neq j$

The proof of the next theorem is similar to the proof of Theorem 7.8 presented in Section 7.3.

**THEOREM 8.9**  

**Unitary Matrices**

An $n \times n$ complex matrix $A$ is unitary if and only if its row (or column) vectors form an orthonormal set in $\mathbb{C}^n$. 
EXAMPLE 3  The Row Vectors of a Unitary Matrix

Show that the complex matrix $A$ is unitary by showing that its set of row vectors forms an orthonormal set in $C^3$.

$$A = \begin{bmatrix}
\frac{1}{2} & \frac{1 + i}{2} & -\frac{1}{2} \\
-\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{5i}{2\sqrt{15}} & \frac{3 + i}{2\sqrt{15}} & \frac{4 + 3i}{2\sqrt{15}}
\end{bmatrix}$$

**SOLUTION**

Let $r_1$, $r_2$, and $r_3$ be defined as follows.

$$r_1 = \left(\frac{1}{2}, \frac{1 + i}{2}, -\frac{1}{2}\right), \quad r_2 = \left(-\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$$

$$r_3 = \left(\frac{5i}{2\sqrt{15}}, \frac{3 + i}{2\sqrt{15}}, \frac{4 + 3i}{2\sqrt{15}}\right)$$

Begin by showing that $r_1$, $r_2$, and $r_3$ are unit vectors.

$$\|r_1\| = \left(\frac{1}{2} + \frac{1 + i}{2} + \frac{-1}{2}\right)^{\frac{1}{2}} = \left[1 + \frac{2}{4} + \frac{1}{4}\right]^\frac{1}{2} = 1$$

$$\|r_2\| = \left(-\frac{i}{\sqrt{3}} + \frac{i}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)^{\frac{1}{2}} = \left[\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right]^{\frac{1}{2}} = 1$$

$$\|r_3\| = \left(\frac{5i}{2\sqrt{15}} + \frac{3 + i}{2\sqrt{15}} + \frac{4 + 3i}{2\sqrt{15}}\right)^{\frac{1}{2}} = \left[\frac{25}{60} + \frac{10}{60} + \frac{25}{60}\right]^{\frac{1}{2}} = 1$$

Then show that all pairs of distinct vectors are orthogonal.

$$r_1 \cdot r_2 = \frac{1}{2} \left(-\frac{i}{\sqrt{3}} + \frac{1 + i}{2\sqrt{3}} + \frac{-1}{2\sqrt{3}}\right)$$

$$= -\frac{i}{2\sqrt{3}} - \frac{i}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}}$$

$$= 0$$

$$r_1 \cdot r_3 = \frac{1}{2} \left(\frac{5i}{2\sqrt{15}} + \frac{3 + i}{2\sqrt{15}} + \frac{4 + 3i}{2\sqrt{15}}\right)$$

$$= -\frac{5i}{4\sqrt{15}} + \frac{4 + 2i}{4\sqrt{15}} + \frac{-4 + 3i}{4\sqrt{15}}$$

$$= 0$$

$$r_2 \cdot r_3 = \left(-\frac{i}{\sqrt{3}} + \frac{3 + i}{2\sqrt{15}} + \frac{1}{\sqrt{3}}\right)$$

$$= -\frac{5}{6\sqrt{5}} + \frac{1 + 3i}{6\sqrt{5}} + \frac{4 - 3i}{6\sqrt{5}}$$

$$= 0$$

**REMARK**

Try showing that the column vectors of $A$ also form an orthonormal set in $C^3$.

So, $\{r_1, r_2, r_3\}$ is an orthonormal set.
HERMITIAN MATRICES

A real matrix is symmetric when it is equal to its own transpose. In the complex system, the more useful type of matrix is one that is equal to its own conjugate transpose. Such a matrix is called **Hermitian** after the French mathematician Charles Hermite (1822–1901).

As with symmetric matrices, you can recognize Hermitian matrices by inspection. To see this, consider the matrix

\[
A = \begin{bmatrix}
  a_{11} + a_{22} & a_{12} + a_{22} & \cdots & a_{1n} + a_{2n} \\
  a_{21} + a_{22} & a_{22} + a_{22} & \cdots & a_{2n} + a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} + a_{n2} & a_{n2} + a_{n2} & \cdots & a_{nn} + a_{nn}
\end{bmatrix}
\]

The conjugate transpose of \( A \) has the form

\[
A^* = A^T^*
= \begin{bmatrix}
  a_{11} + a_{22} & c_{12} + c_{22} & \cdots & c_{1n} + c_{2n} \\
  c_{21} + c_{22} & a_{22} + a_{22} & \cdots & c_{2n} + c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n1} + c_{n2} & c_{n2} + c_{n2} & \cdots & a_{nn} + a_{nn}
\end{bmatrix}
= \begin{bmatrix}
  a_{11} - a_{22} & c_{12} - c_{22} & \cdots & c_{1n} - c_{2n} \\
  c_{21} - c_{22} & a_{22} - a_{22} & \cdots & c_{2n} - c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n1} - c_{n2} & c_{n2} - c_{n2} & \cdots & a_{nn} - a_{nn}
\end{bmatrix}
\]

If \( A \) is Hermitian, then \( A = A^* \). So, \( A \) must be of the form

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

Similar results can be obtained for Hermitian matrices of order \( n \times n \). In other words, a square matrix \( A \) is Hermitian if and only if the following two conditions are met.

1. The entries on the main diagonal of \( A \) are real.
2. The entry \( a_{ij} \) in the \( i \)th row and the \( j \)th column is the complex conjugate of the entry \( a_{ji} \) in the \( j \)th row and the \( i \)th column.

### EXAMPLE 4 Hermitian Matrices

Which matrices are Hermitian?

a. \[
\begin{bmatrix}
  1 & 3 - i \\
  3 + i & i
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
  0 & 3 - 2i \\
  3 - 2i & 4
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
  3 & 2 - i & -3i \\
  2 + i & 0 & 1 - i \\
  3i & 1 + i & 0
\end{bmatrix}
\]

d. \[
\begin{bmatrix}
  -1 & 2 & 3 \\
  2 & 0 & -1 \\
  3 & -1 & 4
\end{bmatrix}
\]

**SOLUTION**

a. This matrix is not Hermitian because it has an imaginary entry on its main diagonal.

b. This matrix is symmetric but not Hermitian because the entry in the first row and second column is not the complex conjugate of the entry in the second row and first column.

c. This matrix is Hermitian.

d. This matrix is Hermitian because all real symmetric matrices are Hermitian.
One of the most important characteristics of Hermitian matrices is that their eigenvalues are real. This is formally stated in the next theorem.

**THEOREM 8.10  The Eigenvalues of a Hermitian Matrix**

If \( A \) is a Hermitian matrix, then its eigenvalues are real numbers.

**PROOF**

Let \( \lambda \) be an eigenvalue of \( A \) and let

\[
\mathbf{v} = \begin{bmatrix}
  a_1 + b_1i \\
  a_2 + b_2i \\
  \vdots \\
  a_n + b ni
\end{bmatrix}
\]

be its corresponding eigenvector. If both sides of the equation \( A\mathbf{v} = \lambda \mathbf{v} \) are multiplied by the row vector \( \mathbf{v}^* \), then

\[
\mathbf{v}^*A\mathbf{v} = \mathbf{v}^*(\lambda \mathbf{v}) = \lambda (\mathbf{v}^*\mathbf{v}) = \lambda (a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots + a_n^2 + b_n^2).
\]

Furthermore, because

\[
(\mathbf{v}^*A\mathbf{v})^* = \mathbf{v}^*A^*(\mathbf{v}^*) = \mathbf{v}^*A\mathbf{v}
\]

it follows that \( \mathbf{v}^*A\mathbf{v} \) is a Hermitian \( 1 \times 1 \) matrix. This implies that \( \mathbf{v}^*A\mathbf{v} \) is a real number, so \( \lambda \) is real.

To find the eigenvalues of complex matrices, follow the same procedure as for real matrices.

**EXAMPLE 5  Finding the Eigenvalues of a Hermitian Matrix**

Find the eigenvalues of the matrix \( A \).

\[
A = \begin{bmatrix}
  3 & 2 - i \\
  2 + i & -1 - i \\
  3i & 1 + i \\
\end{bmatrix}
\]

**SOLUTION**

The characteristic polynomial of \( A \) is

\[
|\lambda I - A| = \begin{vmatrix}
  \lambda - 3 & -2 + i & 3i \\
  -2 - i & \lambda - 1 + i \\
  -3i & -1 - i & \lambda \\
\end{vmatrix}
\]

\[
= (\lambda - 3)(\lambda^2 - 2) - (-2 + i)(-2 - i)\lambda - (3i + 3) \\
+ 3i[(1 + 3i) + 3\lambda] \\
= (\lambda^3 - 3\lambda^2 - 2\lambda + 6) - (5\lambda + 9 + 3i) + (3i - 9 - 9\lambda) \\
= \lambda^3 - 3\lambda^2 - 16\lambda - 12 \\
= (\lambda + 1)(\lambda - 6)(\lambda + 2).
\]

So, the characteristic equation is \( (\lambda + 1)(\lambda - 6)(\lambda + 2) = 0 \), and the eigenvalues of \( A \) are \(-1, 6, \) and \(-2\).
To find the eigenvectors of a complex matrix, use a procedure similar to that used for a real matrix. For instance, in Example 5, to find the eigenvector corresponding to the eigenvalue substitute the value for into the equation
to obtain

Solve this equation using Gauss-Jordan elimination, or a graphing utility or software program, to obtain the eigenvector corresponding to which is shown below.

Eigenvectors for and can be found in a similar manner. They are and respectively.

Quantum mechanics had its start in the early 20th century as scientists began to study subatomic particles and light. Collecting data on energy levels of atoms, and the rates of transition between levels, they found that atoms could be induced to more excited states by the absorption of light.

German physicist Werner Heisenburg (1901–1976) laid a mathematical foundation for quantum mechanics using matrices. Studying the dispersion of light, he used vectors to represent energy levels of states and Hermitian matrices to represent “observables” such as momentum, position, and acceleration. He noticed that a measurement yields precisely one real value and leaves the system in precisely one of a set of mutually exclusive (orthogonal) states. So, the eigenvalues are the possible values that can result from a measurement of an observable, and the eigenvectors are the corresponding states of the system following the measurement.

Let matrix be a diagonal Hermitian matrix that represents an observable. Then consider a physical system whose state is represented by the column vector . To measure the value of the observable in the system of state you can find the product

Because is Hermitian and its values along the diagonal are real, is a real number. It represents the average of the values given by measuring the observable on a system in the state a large number of times.
Just as real symmetric matrices are orthogonally diagonalizable, Hermitian matrices are **unitarily diagonalizable**. A square matrix \( A \) is unitarily diagonalizable when there exists a unitary matrix \( P \) such that
\[
P^{-1}AP
\]
is a diagonal matrix. Because \( P \) is unitary, \( P^{-1} = P^* \), so an equivalent statement is that \( A \) is unitarily diagonalizable when there exists a unitary matrix \( P \) such that \( P^*AP \) is a diagonal matrix. The next theorem states that Hermitian matrices are unitarily diagonalizable.

**THEOREM 8.11 Hermitian Matrices and Diagonalization**

If \( A \) is an \( n \times n \) Hermitian matrix, then
1. eigenvectors corresponding to distinct eigenvalues are orthogonal.
2. \( A \) is unitarily diagonalizable.

**PROOF**

To prove part 1, let \( v_1 \) and \( v_2 \) be two eigenvectors corresponding to the distinct (and real) eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Because \( Av_1 = \lambda_1 v_1 \) and \( Av_2 = \lambda_2 v_2 \), you have the equations shown below for the matrix product \( (Av_1)^*v_2 \).

\[
(Av_1)^*v_2 = v_1^*A^*v_2 = v_1^*\lambda_2 v_2 = \lambda_2 v_1^*v_2
\]
\[
(Av_2)^*v_2 = (\lambda_1 v_1)^*v_2 = v_1^*\lambda_1 v_2 = \lambda_1 v_1^*v_2
\]

So,
\[
\lambda_2 v_1^*v_2 - \lambda_1 v_1^*v_2 = 0
\]
\[
(\lambda_2 - \lambda_1)v_1^*v_2 = 0
\]
\[
v_1^*v_2 = 0 \quad \text{because} \quad \lambda_1 \neq \lambda_2
\]
and this shows that \( v_1 \) and \( v_2 \) are orthogonal. Part 2 of Theorem 8.11 is often called the **Spectral Theorem**, and its proof is left to you.

**EXAMPLE 6 The Eigenvectors of a Hermitian Matrix**

The eigenvectors of the Hermitian matrix shown in Example 5 are mutually orthogonal because the eigenvalues are distinct. Verify this by calculating the Euclidean inner products \( v_1 \cdot v_2, v_1 \cdot v_3, \) and \( v_2 \cdot v_3 \). For example,

\[
v_1 \cdot v_2 = (-1)(1 - 2i) + (1 + 2i)(6 - 9i) + (1)(13)
\]
\[
= (-1)(1 + 2i) + (1 + 2i)(6 + 9i) + 13
\]
\[
= -1 - 2i + 6 + 9i + 12i - 18 + 13
\]
\[
= 0.
\]

The other two inner products \( v_1 \cdot v_3 \) and \( v_2 \cdot v_3 \) can be shown to equal zero in a similar manner.

The three eigenvectors in Example 6 are mutually orthogonal because they correspond to distinct eigenvalues of the Hermitian matrix \( A \). Two or more eigenvectors corresponding to the same eigenvalue may not be orthogonal. Once any set of linearly independent eigenvectors is obtained for an eigenvalue, however, the Gram-Schmidt orthonormalization process can be used to find an orthogonal set.
8.5 Unitary and Hermitian Matrices

**Example 7**

**Diagonalization of a Hermitian Matrix**

Find a unitary matrix $P$ such that $P^*AP$ is a diagonal matrix where

$$A = \begin{bmatrix}
  3 & 2 - i & -3i \\
  2 + i & 0 & 1 - i \\
  3i & 1 + i & 0
\end{bmatrix}.$$

**Solution**

The eigenvectors of $A$ are shown after Example 5. Form the matrix $P$ by normalizing these three eigenvectors and using the results to create the columns of $P$.

$$\|v_1\| = \|(-1, 1 + 2i, 1)\| = \sqrt{1 + 5 + 1} = \sqrt{7}$$
$$\|v_2\| = \|(1 - 21i, 6 - 9i, 13)\| = \sqrt{442 + 117 + 169} = \sqrt{728}$$
$$\|v_3\| = \|(1 + 3i, -2 - i, 5)\| = \sqrt{10 + 5 + 25} = \sqrt{40}$$

So,

$$P = \begin{bmatrix}
  \frac{1}{\sqrt{7}} & \frac{1 - 21i}{\sqrt{728}} & \frac{1 + 3i}{\sqrt{40}} \\
  \frac{1 + 2i}{\sqrt{7}} & \frac{6 - 9i}{\sqrt{728}} & \frac{-2 - i}{\sqrt{40}} \\
  \frac{1}{\sqrt{7}} & \frac{13}{\sqrt{728}} & \frac{5}{\sqrt{40}}
\end{bmatrix}.$$}

Try computing the product $P^*AP$ for the matrices $A$ and $P$ in Example 7 to see that

$$P^*AP = \begin{bmatrix}
  -1 & 0 & 0 \\
  0 & 6 & 0 \\
  0 & 0 & -2
\end{bmatrix}$$

where $-1$, $6$, and $-2$ are the eigenvalues of $A$.

You have seen that Hermitian matrices are unitarily diagonalizable. It turns out that there is a larger class of matrices, called normal matrices, that are also unitarily diagonalizable. A square complex matrix $A$ is normal when it commutes with its conjugate transpose: $AA^* = A^*A$. The main theorem of normal matrices states that a complex matrix $A$ is normal if and only if it is unitarily diagonalizable. You are asked to explore normal matrices further in Exercise 56.

The properties of complex matrices described in this section are comparable to the properties of real matrices discussed in Chapter 7. The summary below indicates the correspondence between unitary and Hermitian complex matrices when compared with orthogonal and symmetric real matrices.

**Comparison of Symmetric and Hermitian Matrices**

<table>
<thead>
<tr>
<th>A is a symmetric matrix (real)</th>
<th>A is a Hermitian matrix (complex)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Eigenvalues of $A$ are real.</td>
<td>1. Eigenvalues of $A$ are real.</td>
</tr>
<tr>
<td>2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.</td>
<td>2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.</td>
</tr>
<tr>
<td>3. There exists an orthogonal matrix $P$ such that $P^TAP$ is diagonal.</td>
<td>3. There exists a unitary matrix $P$ such that $P^*AP$ is diagonal.</td>
</tr>
</tbody>
</table>
8.5 Exercises

Finding the Conjugate Transpose In Exercises 1–4, determine the conjugate transpose of the matrix.

1. \[
\begin{bmatrix}
  i & -i \\
  2 & 3i
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
  1 + 2i & 2 - i \\
  1 & 1
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
  0 & 5 + i \\
  \sqrt{2}i & 4 \\
  -\sqrt{2}i & 0
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
  2 & i \\
  5 & 3i \\
  0 & 6 - i
\end{bmatrix}
\]

Finding the Conjugate Transpose In Exercises 5 and 6, use a software program or graphing utility to find the conjugate transpose of the matrix.

5. \[
\begin{bmatrix}
  1 + i & 0 & 1 & -i \\
  2 + i & 1 & 0 & 2i \\
  1 - i & i & 2 & 4i \\
  i & 2 + i & -1 & 0
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
  2 + i & 1 & -1 & 2i \\
  0 & 2 - i & 2i & 1 - i \\
  i & 2 + i & -i & 1 \\
  1 + 2i & 4 & 0 & -2i
\end{bmatrix}
\]

Non-Unitary Matrices In Exercises 7–10, explain why the matrix is not unitary.

7. \[
A = \begin{bmatrix}
  i & 0 \\
  0 & 0
\end{bmatrix}
\]

8. \[
A = \begin{bmatrix}
  1 & i \\
  i & -1
\end{bmatrix}
\]

9. \[
A = \begin{bmatrix}
  \frac{1 + i}{\sqrt{2}} & 0 & -i \\
  0 & 1 & 0 \\
  \frac{1}{2} & \frac{1}{2} & \frac{1 + i}{2}
\end{bmatrix}
\]

10. \[
A = \begin{bmatrix}
  -\frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1 + i}{\sqrt{3}} \\
  \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1 + i}{\sqrt{2}} \\
  \frac{1}{2} & \frac{1}{2} & \frac{1 + i}{2}
\end{bmatrix}
\]

Identifying Unitary Matrices In Exercises 11–16, determine whether A is unitary by calculating $AA^*$.

11. \[
A = \begin{bmatrix}
  1 + i & 1 + i \\
  1 - i & 1 - i
\end{bmatrix}
\]

12. \[
A = \begin{bmatrix}
  1 + i & 1 - i \\
  1 - i & 1 + i
\end{bmatrix}
\]

13. \[
A = \begin{bmatrix}
  -i & 0 \\
  0 & i
\end{bmatrix}
\]

14. \[
A = \begin{bmatrix}
  \frac{i}{\sqrt{2}} & i \\
  i & \frac{-i}{\sqrt{2}} \\
  \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\
  \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}}
\end{bmatrix}
\]

15. \[
A = \begin{bmatrix}
  \frac{-4}{5} & \frac{3}{5} \\
  \frac{3}{5} & \frac{4}{5}
\end{bmatrix}
\]

16. \[
A = \begin{bmatrix}
  -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\
  \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\
  0 & \frac{i}{\sqrt{3}} & \frac{-i}{\sqrt{6}}
\end{bmatrix}
\]

Row Vectors of a Unitary Matrix In Exercises 17–20, (a) verify that A is unitary by showing that its rows are orthonormal, and (b) determine the inverse of A.

17. \[
A = \begin{bmatrix}
  -\frac{4}{5} & \frac{3}{5} \\
  \frac{3}{5} & \frac{4}{5}
\end{bmatrix}
\]

18. \[
A = \begin{bmatrix}
  1 + i & -1 + i \\
  \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix}
\]

19. \[
A = \begin{bmatrix}
  \frac{1 - \sqrt{3}i}{2} & \frac{1 + \sqrt{3}i}{2} \\
  \frac{1 + \sqrt{3}i}{2} & \frac{1 - \sqrt{3}i}{2}
\end{bmatrix}
\]

20. \[
A = \begin{bmatrix}
  \frac{-1 + i}{\sqrt{6}} & 0 & 1 - i \\
  \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}}
\end{bmatrix}
\]

Identifying Hermitian Matrices In Exercises 21–26, determine whether the matrix is Hermitian.

21. \[
A = \begin{bmatrix}
  0 & i \\
  -i & 0
\end{bmatrix}
\]

22. \[
A = \begin{bmatrix}
  i & 0 \\
  0 & -i
\end{bmatrix}
\]

23. \[
A = \begin{bmatrix}
  0 & 2 + i \\
  2 - i & i
\end{bmatrix}
\]

24. \[
A = \begin{bmatrix}
  0 & i & 1 \\
  2 + i & i & 0 \\
  1 & 0 & 1
\end{bmatrix}
\]

25. \[
A = \begin{bmatrix}
  1 & 2 + i & 3 - i \\
  2 - i & 2 & 3 + i \\
  \sqrt{2} - i & 2 & 3 + i \\
  5 & 3 - i & 6
\end{bmatrix}
\]

Finding Eigenvalues of a Hermitian Matrix In Exercises 27–32, determine the eigenvalues of the matrix A.

27. \[
A = \begin{bmatrix}
  0 & i \\
  -i & 0
\end{bmatrix}
\]

28. \[
A = \begin{bmatrix}
  3 & i \\
  -i & 3
\end{bmatrix}
\]

29. \[
A = \begin{bmatrix}
  3 & 1 - i \\
  1 + i & 2
\end{bmatrix}
\]

30. \[
A = \begin{bmatrix}
  0 & 2 + i \\
  2 - i & 4
\end{bmatrix}
\]

31. \[
A = \begin{bmatrix}
  1 & 4 & 1 - i \\
  0 & i & 3i \\
  0 & 0 & 2 + i
\end{bmatrix}
\]
Exercise 31

36. In Exercises 33–36, determine the eigenvectors of the matrix in the indicated exercise.

37. Let be a complex number with modulus 1. Show that is unitary by computing 

\[ A = \begin{bmatrix} 2 & \frac{i}{\sqrt{2}} & i \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix} \]

Finding Eigenvectors of a Hermitian Matrix In Exercises 33–36, determine the eigenvectors of the matrix.

41. Consider the following matrix.

\[ A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \]

\[ A = \begin{bmatrix} 0 & 2 + i \\ 2 - i & 4 \end{bmatrix} \]

\[ A = \begin{bmatrix} 2 & \frac{-i}{\sqrt{2}} & i \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix} \]

\[ A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 + i \\ 0 & 1 - i & 0 \end{bmatrix} \]

Diagonalization of a Hermitian Matrix In Exercises 37–41, find a unitary matrix \( P \) that diagonalizes the matrix \( A \).

42. **CAPSTONE** Consider the following matrix.

\[ A = \begin{bmatrix} -2 & 3 - i & 4 - i \\ 3 + i & 1 & 1 + i \\ 4 + i & 1 - i & 3 \end{bmatrix} \]

(a) Is \( A \) unitary? Explain.

(b) Is \( A \) Hermitian? Explain.

(c) Are the row vectors of \( A \) orthonormal? Explain.

(d) The eigenvalues of \( A \) are distinct. Is it possible to determine the inner products of the pairs of eigenvectors by inspection? If so, state the value(s). If not, explain why not.

(e) Is \( A \) unitarily diagonalizable? Explain.

43. Show that \( A = I_3 \) is unitary by computing \( AA^* \).

44. Let \( z \) be a complex number with modulus 1. Show that the matrix \( A \) is unitary.

\[ A = \begin{bmatrix} 1 & \frac{z}{\sqrt{2}} & \frac{\bar{z}}{\sqrt{2}} \end{bmatrix} \]

8.5 Exercises

Unitary Matrices In Exercises 45 and 46, use the result of Exercise 44 to determine \( a, b, \) and \( c \) such that \( A \) is unitary.

45. \( A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & a \\ -b & c \end{bmatrix} \)

46. \( A = \frac{1}{\sqrt{2}} \begin{bmatrix} a & 6 + 3i \\ b & 45 \end{bmatrix} \)

Proof In Exercises 47–50, prove the formula, where \( A \) and \( B \) are \( n \times n \) complex matrices.

47. \( (A^*)^* = A \)

48. \( (A + B)^* = A^* + B^* \)

49. \( (kA)^* = \overline{k} A^* \)

50. \( (AB)^* = B^* A^* \)

51. Proof Let \( A \) be a matrix such that \( A^* + A = O \). Prove that \( iA \) is Hermitian.

52. Show that \( \det(iA) = \det(A) \), where \( A \) is a \( 2 \times 2 \) matrix.

Determinants In Exercises 53 and 54, assume that the result of Exercise 52 is true for matrices of any size.

53. Show that \( \det(A^*) = \det(A) \).

54. Prove that if \( A \) is unitary, then \( |\det(A)| = 1 \).

55. (a) Prove that every Hermitian matrix \( A \) can be written as the sum \( A = B + iC \), where \( B \) is a real symmetric matrix and \( C \) is real and skew-symmetric.

(b) Use part (a) to write the matrix

\[ A = \begin{bmatrix} 2 & 1 + i \\ 1 - i & 3 \end{bmatrix} \]

as the sum \( A = B + iC \), where \( B \) is a real symmetric matrix and \( C \) is real and skew-symmetric.

(c) Prove that every \( n \times n \) complex matrix \( A \) can be written as \( A = B + iC \), where \( B \) and \( C \) are Hermitian.

(d) Use part (c) to write the complex matrix

\[ A = \begin{bmatrix} i & 2 \\ 2 & 1 - 2i \end{bmatrix} \]

as the sum \( A = B + iC \), where \( B \) and \( C \) are Hermitian.

56. (a) Prove that every Hermitian matrix is normal.

(b) Prove that every unitary matrix is normal.

(c) Find a \( 2 \times 2 \) matrix that is Hermitian, but not unitary.

(d) Find a \( 2 \times 2 \) matrix that is unitary, but not Hermitian.

(e) Find a \( 2 \times 2 \) matrix that is normal, but neither Hermitian nor unitary.

(f) Find the eigenvalues and corresponding eigenvectors of your matrix in part (e).

(g) Show that the complex matrix

\[ \begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix} \]

is not diagonalizable. Is this matrix normal?
8 Review Exercises

Operations with Complex Numbers In Exercises 1–6, perform the operation.
1. Find \(u + z: u = 2 - 4i, z = 4i\)
2. Find \(u - z: u = 4, z = 8i\)
3. Find \(uz: u = 4 - 2i, z = 4 + 2i\)
4. Find \(u/z: u = 2i, z = 1 - 2i\)
5. Find \(\frac{u}{z}: u = 6 - 2i, z = 3 - 3i\)
6. Find \(\frac{u}{z}: u = 7 + i, z = i\)

Finding Zeros In Exercises 7–10, use the given zero to find all zeros of the polynomial function.
7. \(p(x) = x^3 + 2x^2 + 2x + 4\) Zero: \(\sqrt{2}i\)
8. \(p(x) = x^3 - 2x - 4\) Zero: \(-1 + i\)
9. \(p(x) = x^3 + x^2 + 3x + 5x - 10\) Zero: \(\sqrt{5}i\)
10. \(p(x) = x^3 - x^2 + x - 3 - 6\) Zero: \(\sqrt{3}i\)

Operations with Complex Matrices In Exercises 11–18, perform the operation using
\[A = \begin{bmatrix} 4 - i & 2 \\ 3 & 3 + i \end{bmatrix}\] and \[B = \begin{bmatrix} 1 + i & 2i \\ 2i & 2 + i \end{bmatrix}\]
11. \(A + B\)
12. \(A - B\)
13. \(2iB\)
14. \(-iA\)
15. \(\det(A - B)\)
16. \(\det(A + B)\)
17. \(3BA\)
18. \(2AB\)

Operations with Conjugates and Moduli In Exercises 19–24, perform the operation using \(w = 2 - 2i, v = 3 + i,\) and \(z = -1 + 2i.\)
19. \(\overline{z}\)
20. \(\overline{v}\)
21. \(|w|\)
22. \(|v|\)
23. \(|wv|\)
24. \(|\overline{wv}|\)

Dividing Complex Numbers In Exercises 25–28, perform the indicated operation.
25. \(\frac{2 + i}{2 - i}\)
26. \(\frac{1 + i}{-1 + 2i}\)
27. \(\frac{(1 - 2i)(1 + 2i)}{3 - 3i}\)
28. \(\frac{5 + 2i}{(-2 + 2i)(2 - 3i)}\)

Finding the Inverse of a Complex Matrix In Exercises 29 and 30, find \(A^{-1}\) (if it exists).
29. \(A = \begin{bmatrix} 3 - i & -1 - 2i \\ -2 + 5i & 2 + 3i \end{bmatrix}\)
30. \(A = \begin{bmatrix} 5 & 1 - i \\ 0 & i \end{bmatrix}\)

Converting to Polar Form In Exercises 31–36, determine the polar form of the complex number.
31. \(4 + 4i\)
32. \(2 - 2i\)
33. \(\sqrt{3} + i\)
34. \(1 + \sqrt{3}i\)
35. \(7 - 4i\)
36. \(3 + 2i\)

Converting to Standard Form In Exercises 37–42, find the standard form of the complex number.
37. \(\left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right]\)
38. \(\left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right]\)
39. \(4\left[\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)\right]\)
40. \(6\left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right]\)
41. \(7\left[\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)\right]\)
42. \(4\left[\cos\pi + i \sin\pi\right]\)

Multiplying and Dividing in Polar Form In Exercises 43–46, perform the indicated operation. Leave the result in polar form.
43. \(\left[\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right]\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]\)
44. \(\left[\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right]\right]\left[2\left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right]\right]\)
45. \(\left[9\cos(\pi/2) + i \sin(\pi/2)\right]\left[6\cos(2\pi/3) + i \sin(2\pi/3)\right]\)
46. \(\left[4\cos(\pi/4) + i \sin(\pi/4)\right]\left[7\cos(\pi/3) + i \sin(\pi/3)\right]\)

Finding Powers of Complex Numbers In Exercises 47–50, find the indicated power of the number and express the result in polar form.
47. \((-1 - i)^4\)
48. \((2i)^3\)
49. \(\left[\sqrt{2}\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]\right]^7\)
50. \(\left[5\left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right]\right]^4\)

Finding Roots of Complex Numbers In Exercises 51–54, express the roots in standard form.
51. Square roots: \(25\left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right]\)
52. Cube roots: \(27\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]\)
53. Cube roots: \(i\)
54. Fourth roots: \(16\left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right]\)
Vector Operations in $C^n$ In Exercises 55–58, find the indicated vector using $u = (4i, 2 + i)$, $v = (3, -i)$, and $w = (3 - i, 4 + i)$.

55. $7u - v$ 
56. $3iw + (4 - i)v$
57. $iu + iv - iw$ 
58. $(3 + 2i)u - (2 - i)w$

Finding the Euclidean Norm In Exercises 59 and 60, determine the Euclidean norm of the vector.

59. $v = (3 - 5i, 2i)$ 
60. $v = (3i, -1 - 5i, 3 + 2i)$

Finding the Euclidean Distance In Exercises 61 and 62, find the Euclidean distance between the vectors.

61. $v = (2 - i, i)$, $u = (i, 2 - i)$
62. $v = (2 + i, 2 + 2i, 3i)$, $u = (4 - 2i, 3 + 2i, 4)$

Finding Complex Inner Products In Exercises 63–66, use the inner product $\langle u, v \rangle = u^*_1v_1^* + 2u_2v_2^*$ to find $\langle u, v \rangle$.

63. $u = (-i, 3i)$ and $v = (2i, -2i)$
64. $u = (i, 2 + i)$ and $v = (5 - i, i)$
65. $u = (1 + i, 1 - 2i)$ and $v = (2 + i, 1 - 2i)$
66. $u = (2 - 2i, 1)$ and $v = (3 - 4i, 2)$

Finding the Conjugate Transpose In Exercises 67–70, determine the conjugate transpose of the matrix.

67. $A = \begin{bmatrix} -1 + 4i & 3 + i \\ 3 - i & 2 + i \end{bmatrix}$
68. $A = \begin{bmatrix} 2 + i & 2 - i \\ 1 + 2i & 2 - 2i \end{bmatrix}$
69. $A = \begin{bmatrix} 2 + 2i & 3 - 2i \\ 3i & 2 + i - 1 - 2i \end{bmatrix}$
70. $A = \begin{bmatrix} 2 & 1 + i \\ -i & 2 + 2i \\ 1 & 1 + i - 2i \end{bmatrix}$

Identifying Unitary Matrices In Exercises 71–74, determine whether the matrix is unitary.

71. $\begin{bmatrix} i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & i \end{bmatrix}$ 
72. $\begin{bmatrix} \frac{2 + i}{4} & \frac{1 + i}{4} \\ \frac{i}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$

73. $\begin{bmatrix} 1 & 0 \\ i & -i \end{bmatrix}$ 
74. $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

Identifying Hermitian Matrices In Exercises 75 and 76, determine whether the matrix is Hermitian.

75. $\begin{bmatrix} 1 & -1 + i & 2 - i \\ 1 - i & 3 & i \\ 2 + i & -i & 4 \end{bmatrix}$

Finding the Conjugate Transpose

76. $\begin{bmatrix} 9 & 2 - i \\ 2 + i & 0 - 1 - i \\ 2 & -1 + i & 3 \end{bmatrix}$

Finding Eigenvalues and Eigenvectors In Exercises 77 and 78, find the eigenvalues and corresponding eigenvectors of the matrix.

77. $\begin{bmatrix} -2 & 2 - i \\ 2 + i & 0 \end{bmatrix}$ 
78. $\begin{bmatrix} 2 & 0 - i \\ 2 + i & 0 \end{bmatrix}$

Proof

79. Prove that if $A$ is an invertible matrix, then $A^*$ is also invertible.
80. Determine all complex numbers $z$ such that $z = -\bar{z}$.

Proof

81. Prove that if $z$ is a zero of a polynomial equation with real coefficients, then the conjugate of $z$ is also a zero.

82. (a) Find the determinant of the Hermitian matrix

$\begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}$

(b) Prove that the determinant of any Hermitian matrix is real.

83. Prove that $A$ and $B$ be Hermitian matrices. Prove that $AB = BA$ if and only if $AB$ is Hermitian.

84. Prove that $H = I - 2uu^*$ is a Hermitian matrix and unitary matrix.

85. Use mathematical induction to prove DeMoivre’s Theorem.

86. Show that if $z_1 + z_2$ and $z_1z_2$ are both nonzero real numbers, then $z_1$ and $z_2$ are both real numbers.

87. Prove that if $z$ and $w$ are complex numbers, then $|z + w| \leq |z| + |w|$. 

88. Prove that for all vectors $u$ and $v$ in a complex inner product space, 

$|\langle u, v \rangle| = \frac{1}{2} \left( \|u\|^2 + \|v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2 \right)$

True or False? In Exercises 89 and 90, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

89. A square complex matrix $A$ is called normal if it commutes with its conjugate transpose so that $AA^* = A^*A$.

90. A square complex matrix $A$ is called Hermitian if $A = A^*$. 

### 8 Project

#### 1 Population Growth and Dynamical Systems (II)

In the projects for Chapter 7, you were asked to model the populations of two species using a system of differential equations of the form

\[
\begin{align*}
y_1'(t) &= ay_1(t) + by_2(t) \\
y_2'(t) &= cy_1(t) + dy_2(t).
\end{align*}
\]

The constants \(a, b, c, \) and \(d\) depend on the particular species being studied. In Chapter 7, you looked at an example of a predator-prey relationship, in which \(a = 0.5, b = 0.6, c = -0.4, \) and \(d = 3.0.\) Now consider a slightly different model.

\[
\begin{align*}
y_1'(t) &= 0.6y_1(t) + 0.8y_2(t), & y_1(0) &= 36 \\
y_2'(t) &= -0.8y_1(t) + 0.6y_2(t), & y_2(0) &= 121
\end{align*}
\]

1. Use the diagonalization technique to find the general solutions \(y_1(t)\) and \(y_2(t)\) at any time \(t > 0.\) Although the eigenvalues and eigenvectors of the matrix

\[
A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}
\]

are complex, the same principles apply, and you can obtain complex exponential solutions.

2. Convert the complex solutions to real solutions by observing that if \(\lambda = a + bi\) is a (complex) eigenvalue of \(A\) with (complex) eigenvector \(v,\) then the real and imaginary parts of \(e^{\lambda t}v\) form a linearly independent pair of (real) solutions. Use the formula \(e^{\lambda t} = \cos \theta + i \sin \theta.\)

3. Use the initial conditions to find the explicit form of the (real) solutions of the original equations.

4. If you have access to a graphing utility or software program, graph the solutions obtained in part 3 over the domain \(0 \leq t \leq 3.\) At what moment are the two populations equal?

5. Interpret the solution in terms of the long-term population trend for the two species. Does one species ultimately disappear? Why or why not? Contrast this solution to that obtained for the model in Chapter 7.

6. If you have access to a graphing utility or software program that can numerically solve differential equations, use it to graph the solutions of the original system of equations. Does this numerical approximation appear to be accurate?