1. Consider the linear system of Example 7 in Section 1.2.
\[
\begin{align*}
x - 2y + 3z &= 9 \\
-x + 3y &= -4 \\
2x - 5y + 5z &= 17
\end{align*}
\]
(a) Use the MATLAB command `rref` to solve the system.
(b) Let \( A \) be the coefficient matrix, and \( B \) be the right-hand side of the system.
\[
A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ -4 \\ 17 \end{bmatrix}
\]
Use the MATLAB command `A\ B` to solve the system.

2. Enter the matrix
\[
A = \begin{bmatrix} -3 & 2 & 4 & 5 & 1 \\ -9 & 0 & 2 & -2 & 0 \end{bmatrix}
\]
Use the MATLAB command `rref` to find the reduced row-echelon form of \( A \). What is the solution to the linear system represented by the augmented matrix \( A \)?

3. Solve the linear system
\[
\begin{align*}
16x - 120y + 240z - 140w &= -4 \\
-120x + 1200y - 2700z + 1680w &= 60 \\
240x - 2700y + 6480z - 4200w &= -180 \\
-140x + 1680y - 4200z + 2800w &= 140
\end{align*}
\]
You can display more significant digits of the answer by entering `format long` before solving the system. Return to the standard format by entering `format short`.

4. Use the MATLAB command `rref` to determine which of the following matrices are row-equivalent to
\[
B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}
\]
(a) \( \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \)
(b) \( \begin{bmatrix} 1 & 3 & 2 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \)
(c) \( \begin{bmatrix} 12 & 11 & 10 & 9 \\ 8 & 7 & 6 & 5 \end{bmatrix} \)

5. Let \( A \) be the coefficient matrix, and \( B \) the right-hand side of the linear system of equations
\[
\begin{align*}
3x + 3y + 12z &= 6 \\
x + y + 4z &= 2 \\
2x + 5y + 20z &= 10 \\
-x + 2y + 8z &= 4
\end{align*}
\]
Enter the matrices \( A \) and \( B \), and form the augmented matrix \( C \) for this system by using the MATLAB command \( C = [A \ B] \). Solve the system using `rref`.

6. The MATLAB command `polyfit` allows you to fit a polynomial of degree \( n - 1 \) to a set of \( n \) data points in the plane \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\).
Find the fourth-degree polynomial that fits the five data points of Example 2 in Section 1.3 by letting
\[
x = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 4 \\ 10 \end{bmatrix}
\]
and entering the MATLAB command `polyfit(x, y, 4)`.
7. Find the second-degree polynomial that fits the points \((1, -2), (2, 4), (-4, -6)\).

8. Find the sixth-degree polynomial that fits the points \((0, 0), (-1, 4.5), (-2, 133), (-3, 1225.5), (1, -0.5), (2, 3), (3, 250.5)\).

9. The following table gives the revenues \(y\) (in billions of dollars) for General Dynamics Corporation from 2005 through 2009.

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue, (y)</td>
<td>21.2</td>
<td>24.1</td>
<td>27.2</td>
<td>29.3</td>
<td>32.0</td>
</tr>
</tbody>
</table>

Use \(p=\text{polyfit}(x,y,4)\) to fit the fourth-degree polynomial to these data. Let \(x\) represent the year, with \(x = 5\) corresponding to 2005. Then use \(f=\text{polyval}(p,10)\) to estimate the revenue in 2010. (The actual revenue in 2010 was $32.5 billion)
1. Enter the matrices
   \[ A = \begin{bmatrix} 0 & -4 & 5 \\ 3 & 1 & -2 \\ 2 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -5 & 6 & 7 \\ 0 & -1 & 2 \\ 4 & 0 & -3 \end{bmatrix}. \]
   Use MATLAB to find
   (a) \( A + B \)  \hspace{1em} (b) \( B - 3A \)  \hspace{1em} (c) \( AB \)  \hspace{1em} (d) \( BA \).

2. Enter the three matrices
   \[ A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}, \quad B = \begin{bmatrix} 1.0000 & 0.5000 & 0.3333 & 0.2500 \\ 0.5000 & 0.3333 & 0.2500 & 0.2000 \\ 0.3333 & 0.2500 & 0.2000 & 0.1667 \\ 0.2500 & 0.2000 & 0.1667 & 0.1429 \end{bmatrix}, \]
   \[ C = \begin{bmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{bmatrix}. \]
   (a) Enter \texttt{format long} and then calculate \( A - B \). Return to the standard short format with \texttt{format short}.
   (b) Calculate \( AC \) and \( BC \). Define what is meant by the inverse of a square matrix. What is the inverse of the matrix \( A \) of the matrix \( C \)?

3. Write the following system of linear equations in the form \( AX = B \) and use the MATLAB command \( \text{A} \backslash \text{B} \) to solve the system.
   \[ 3x + 3y + 4z = 2 \
   x + y + 4z = -2 \
   2x + 5y + 4z = 3 \]
   Check your answer using \texttt{rref}.

4. Enter the matrices
   \[ A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 2 & -1 & -5 \\ 7 & -5 & 0 & 6 \\ -4 & 0 & 7 & 12 \end{bmatrix} \quad \text{and} \quad B = \text{pascal(4)}. \]
   (a) Use the MATLAB command \texttt{trace} to find the traces of \( A \), \( B \), and \( A + B \). What do you observe?
   (b) What is the relationship between the trace of \( AB \) and the trace of \( BA \)?

5. Use the MATLAB command \texttt{diag} to form the \( 5 \times 5 \) diagonal matrix \( D \) with diagonal entries \( 0, -1, -2, -3, \) and \( -4 \). Find the product \( D^k = D^{DDDD} \). If \( D \) is an \( n \times n \) diagonal matrix, describe how to find the product \( D^k \) for any positive integer \( k \).

6. Enter the three matrices
   \[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -6 & 3 & 5 \\ 0 & -3 & -6 & 8 \\ 3 & 5 & 0 & 7 \\ -1 & 0 & 7 & 9 \end{bmatrix}, \quad C = \begin{bmatrix} 16 & -1 & 4 & -1 \\ -3 & 12 & -7 & 8 \\ 4 & -5 & 0 & 0 \\ -14 & 3 & 2 & 8 \end{bmatrix}. \]
   (a) Calculate \( AB - AC \) and \( A(B - C) \). What do you observe?
   (b) Calculate \( 3(AC), A(3C), \) and \( (3A)C \). What do you observe?

7. Let
   \[ A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{3} \end{bmatrix}. \]
   Compute \( A^2 \), \( A^3 \), and \( A^n \). Describe the matrix \( A^n \) for large \( n \).
8. Enter the matrices
\[ A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_{12} = \text{zeros}(2, 2), \quad A_{22} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}. \]
(a) Form the 4 \times 4 matrix \( A \) using the following MATLAB construction:
\[ A = [A_{11} \ A_{12}; \ A_{12} \ A_{22}] \]
(b) Find the smallest value of \( n \), where \( n \) is a positive integer, such that \( A^n = A \).

9. Use the MATLAB command \texttt{inv} to find the inverse of the following matrix \( A \). Then adjoin the identity matrix \( I = \text{eye}(3) \) to \( A \) to form the 3 \times 6 matrix \( B = [A \ I] \). Row-reduce \( B \) to compute the inverse of \( A \) again. What do you observe?

10. Let \( A \) and \( B \) be the following 3 \times 3 matrices.
\[ A = \begin{bmatrix} 2 & 4 & \frac{5}{4} \\ -\frac{3}{4} & 2 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -\frac{3}{4} & \frac{1}{4} \\ \frac{3}{2} & \frac{3}{2} & -2 \\ \frac{3}{4} & \frac{1}{4} & 1 \end{bmatrix} \]
(a) Calculate \( A^{-1}B^{-1} \), \( (AB)^{-1} \), and \( (BA)^{-1} \). What do you observe?
(b) Find \( (A^{-1})^T \) and \( (A^T)^{-1} \). What do you observe? Remember: The MATLAB command for the transpose of a real matrix \( A \) is \( A' \).

11. In this exercise, you will use MATLAB to find the least squares regression line for the set of data \((1, 1), (2, 2), (3, 4), (4, 4), \) and \((5, 6)\) from Example 10, Section 2.5.
(a) Form the following matrices.
\[ X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix} \]
(b) Let \( A = (X^TX)^{-1}X^TY \).
(c) Compare your answer with the MATLAB least squares command \texttt{polyfit}. (\textit{Hint: Let} \( X1 = X(:, 2) \) and enter \texttt{polyfit(X1,Y,1)}.)
(d) Plot the data using the following MATLAB commands.
\[ t = (0:0.1:6); \quad \text{plot}(X1,Y,'+') \]
(e) Plot the least squares line for the data by using the following MATLAB commands.
\[ t = (0:0.1:6); \quad p=\text{polyfit}(X1,Y,1); \quad f=\text{polyval}(p,t); \quad \text{plot}(t,f,'*') \]
(f) You can combine the previous plots by using the following MATLAB command.
\[ \text{plot}(X1,Y,+'t,f, '*') \]

12. Repeat Exercise 11 for the data \((0, 6), (4, 3), (5, 0), (8, -4), (10, -5)\). Plot the data and the least squares line on the interval \([0, 10]\). That is, use \( t = (0:0.1:10) \).
1. Use MATLAB to calculate the determinants of the following matrices.

(a) \[
\begin{bmatrix}
2 & -3 \\
-6 & 9
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
-5 & 6 & 7 \\
0 & -1 & 2 \\
4 & 0 & -3
\end{bmatrix}
\]
(c) \texttt{pascal(4)}
(d) \texttt{hilb(8)}

2. Let
\[
A = \begin{bmatrix}
1 & 4 \\
2 & -1
\end{bmatrix}
\]
(a) Compute \(\text{det}(2 \cdot \text{eye}(2) - A)\).
(b) Find an integer value of \(t\) such that \(\det(tI - A) = 0\).

3. Choose arbitrary \(4 \times 4\) matrices \(A\) and \(B\). Compute \(\det(A)\), \(\det(B)\) and \(\det(AB)\). What do you observe? Do the same for \(\det(A) + \det(B)\) and \(\det(A + B)\).

4. Choose an arbitrary real number \(t\). Form the matrix
\[
\begin{bmatrix}
\cos(t) & \sin(t) \\
-\sin(t) & \cos(t)
\end{bmatrix}
\]
and calculate its determinant. Does the value of the determinant depend on \(t\)?

5. Consider the matrices
\[
A = \begin{bmatrix}
2 & 0 & 1 \\
1 & -1 & 2 \\
3 & 1 & 1
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
2 & -1 & 4 \\
0 & -1 & 3 \\
3 & -2 & 1
\end{bmatrix}
\]
(a) Verify that \(\det(A) \det(B) = \det(AB)\).
(b) Verify that \(\det(A^T) = \det(A)\).
(c) Verify that \(\det(A^{-1}) = 1/\det(A)\).

6. This exercise uses Cramer’s Rule to solve the linear system \(Ax = b\) from Example 3, Section 3.4. Let
\[
A = \begin{bmatrix}
4 & -2 \\
3 & -5
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
10 \\
11
\end{bmatrix}
\]
be the coefficient matrix and the right-hand side, respectively, of the system. To form the matrix \(A_1\), replace the first column of \(A\) with \(b\). To do this, enter the following.

\[
A1 = A
\]
\[
A1(:,[1]) = b
\]
Obtain the solution \(x_1\) by entering the following.

\[
\text{det}(A1)/\text{det}(A)
\]
Calculate \(x_2\) in a similar manner.

\[
A2 = A
\]
\[
A2(:,[2]) = b
\]
\[
\text{det}(A2)/\text{det}(A)
\]

7. Use the Cramer’s Rule algorithm from Exercise 6 to solve the following linear system. Compare your answer with that obtained using \texttt{rref}.

\[
\begin{align*}
3x + 3y + 4z &= 2 \\
x + y + 4z &= -2 \\
2x + 5y + 4z &= 3
\end{align*}
\]
1. Let \( \mathbf{u}_1 = (1, 1, 2, 2) \), \( \mathbf{u}_2 = (2, 3, 5, 6) \), and \( \mathbf{u}_3 = (2, -1, 3, 6) \). Use MATLAB to write (if possible) the vector \( \mathbf{v} \) as a linear combination of the vectors \( \mathbf{u}_1 \), \( \mathbf{u}_2 \), and \( \mathbf{u}_3 \).
   (a) \( \mathbf{v} = (0, 5, 3, 0) \)
   (b) \( \mathbf{v} = (-1, 6, 1, -4) \)

2. Use MATLAB to determine whether the given set of vectors spans \( \mathbb{R}^4 \):
   (a) \( \{ (1, -2, 3, 4), (2, 4, 5, 0), (-2, 0, 0, 4), (3, 2, 1, -4) \} \)
   (b) \( \{ (0, 1, -1, 1), (2, -2, 3, 1), (7, 0, 1, 0), (5, 2, -2, -1) \} \)

3. Use MATLAB to determine whether the set is linearly independent or dependent.
   (a) \( \{ (0, 1, -3, 4), (-1, 0, 0, 2), (0, 5, 3, 0), (-1, 7, -3, -6) \} \)
   (b) \( \{ (0, 0, 1, 2, 3), (0, 0, 2, 3, 1), (1, 2, 3, 4, 5), (2, 1, 0, 0, 0), (-1, -3, -5, 0, 0) \} \)

4. Use MATLAB to determine whether the set of vectors forms a basis of \( \mathbb{R}^4 \).
   (a) \( \{ (1, -2, 3, 4), (2, 4, 5, 0), (-2, 0, 0, 4), (3, 2, 1, -4) \} \)
   (b) \( \{ (0, 1, -1, 1), (2, -2, 3, 1), (7, 0, 1, 0), (5, 2, -2, -1) \} \)
   (c) \( \{ (0, 1, -3, 4), (-1, 0, 0, 2), (0, 5, 3, 0), (-1, 7, -3, -6) \} \)
   (d) \( \{ (0, 0, 1, 2), (0, 2, 3, 1), (1, 3, 4, 5), (2, 1, 0, 0), (-3, -5, 0, 0) \} \)

5. Suppose you want to find a basis for \( \mathbb{R}^4 \) that contains the vectors \( \mathbf{v}_1 = (1, 1, 0, 0) \) and \( \mathbf{v}_2 = (1, 0, 1, 0) \). One way to do this is to consider the set of vectors consisting of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \), together with the standard basis vectors, \( \mathbf{e}_1 = (1, 0, 0, 0) \), \( \mathbf{e}_2 = (0, 1, 0, 0) \), \( \mathbf{e}_3 = (0, 0, 1, 0) \), and \( \mathbf{e}_4 = (0, 0, 0, 1) \). Let \( A \) be the matrix whose columns consist of the vectors \( \mathbf{v}_1 \), \( \mathbf{v}_2 \), \( \mathbf{e}_1 \), \( \mathbf{e}_2 \), \( \mathbf{e}_3 \), and \( \mathbf{e}_4 \), and apply \texttt{rref} to \( A \) to obtain

\[
A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Because the leading ones of the reduced matrix on the right are in columns 1, 2, 3, and 6, a basis for \( \mathbb{R}^4 \) consists of the corresponding column vectors of \( A \): \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_4 \} \).

A convenient way to construct the matrix \( A \) is to define the matrix \( B \) whose columns are the given vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Then \( A \) is the matrix obtained by adjoining the \( 4 \times 4 \) identity matrix to \( B : A = [B \ \text{eye}(4)] \).

Use this algorithm to find a basis for \( \mathbb{R}^4 \) that contains the given vectors.
   (a) \( \mathbf{v}_1 = (2, 1, 0, 0, 0) \), \( \mathbf{v}_2 = (-1, 0, 1, 0, 0) \)
   (b) \( \mathbf{v}_1 = (1, 0, 2, 0, 0) \), \( \mathbf{v}_2 = (1, 1, 2, 0, 0) \), \( \mathbf{v}_3 = (1, 1, 1, 0, 1) \)

6. Use MATLAB to find a subset of the given set of vectors that forms a basis for the span of the vectors.
   (a) \( \{ (1, 2, -1, 0), (-3, -6, 3, 0), (1, 0, 0, 1), (-2, -2, 1, -1) \} \)
   (b) \( \{ (0, 0, 1, 1, 0), (1, 1, 0, 0, 1), (1, 1, 1, 1, 1), (1, 1, 2, 2, 1), (0, 0, 3, 3, 1), (0, 0, 0, 0, 1) \} \)

7. Let

\[
A = \begin{bmatrix}
-1 & 2 & 0 & 0 & 3 \\
0 & 2 & 3 & -1 & 2 \\
-1 & 4 & 3 & -1 & 5 \\
2 & -4 & 0 & 0 & -6 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

   (a) Find a basis for the row space of \( A \).
   (b) Find a basis for the column space of \( A \).
   (c) Use the MATLAB command \texttt{rank} to find the rank of \( A \).
8. Find a basis for the nullspace of the given matrix \( A \). Then verify that the sum of the rank and nullity of \( A \) equals the number of columns.

(a) \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \)

(b) \( A = \text{hilb}(5) \)

(c) \( A = \text{pascal}(5) \)

(d) \( A = \text{magic}(6) \)

9. Let \( \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\} \) be a (nonstandard) basis for \( \mathbb{R}^3 \). You can find the coordinate matrix of \( x = (1, 2, -1) \) relative to this basis by writing \( x \) as a linear combination of the basis vectors. That is, the coordinate matrix is the solution vector to the linear system \( Bc = x \), where the basis vectors form the columns of \( B \). Use MATLAB to solve this system and compare your answer to Section 4.7, Example 3.

10. Let \( B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \) and \( BPRIME = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\} \) be the two bases of \( R^3 \) given in Section 4.7, Example 4. You can use MATLAB to find the transition matrix from \( B \) to \( BPRIME \) by first forming the two matrices

\[
B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad BPRIME = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}
\]

Adjoin \( B \) and \( BPRIME \) by using the MATLAB command \( \text{C} = [\text{BPRIME} \ B] \). Let \( A \) be the reduced row-echelon form of \( C \), \( A = \text{rref}(C) \). Finally, obtain \( PINV = P^{-1} \) by deleting the first three columns of this reduced matrix using the MATLAB command \( \text{PINV} = A(:,4:6) \).

Find the transition matrix from \( B \) to \( BPRIME \).

(a) \( B = \{(-3, 2), (4, -2)\} \), \( BPRIME = \{(-1, 2), (2, -2)\} \)

(b) \( B = \{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\} \), \( BPRIME = \{(1, 0, 1, 0), (1, 0, -1, 0), (0, 1, 0, 1), (0, 1, 0, -1)\} \)
CHAPTER 5 MATLAB EXERCISES

1. Use the MATLAB command \texttt{norm(v)} to find
   (a) the length of the vector \( v = (0, -2, 1, 4, -2) \).
   (b) a unit vector in the direction of \( v = (-3, 2, 4, -5, 0, 1) \).
   (c) the distance between the vectors \( u = (0, 2, 2, -3) \) and \( v = (-4, 7, 10, 1) \).

2. The dot product of the vectors (written as columns)
   \[
   \begin{bmatrix}
   u_1 \\
   u_2 \\
   \vdots \\
   u_n
   \end{bmatrix}
   \quad \text{and} \quad
   \begin{bmatrix}
   v_1 \\
   v_2 \\
   \vdots \\
   v_n
   \end{bmatrix}
   \]

   is given by the matrix product
   \[
   \begin{bmatrix}
   u_1 \\
   u_2 \\
   \vdots \\
   u_n
   \end{bmatrix}
   \begin{bmatrix}
   v_1 \\
   v_2 \\
   \vdots \\
   v_n
   \end{bmatrix}
   \]

   So, you can compute the dot product of \( u \) and \( v \) by multiplying the transpose of \( u \) times the vector \( v \).

   Let \( u = (2, -5, 0, 4, 8) \), \( v = (0, -3, 2, -1, 1) \) and \( w = (1, -1, 0, 0, 7) \), and use MATLAB to find the following.
   (a) \( u \cdot v \)
   (b) \( (u \cdot v)w \)
   (c) \( u \cdot (2v - 3w) \)
   (d) \( v \cdot v \)

3. The angle \( \theta \) between two nonzero vectors is given by
   \[
   \cos \theta = \frac{u \cdot v}{\|u\|\|v\|}
   \]

   Use MATLAB to find the angle between \( u = (-3, 4, 0) \) and \( v = (1, 1, 4) \). (Hint: Use the built-in inverse cosine function, \texttt{acos}.)

4. You can find the orthogonal projection of the column vector \( x \) onto the column vector \( y \) by computing
   \[
   \frac{x^Ty}{y^Ty} y.
   \]

   Use MATLAB to find the following projections of \( x \) onto \( y \).
   (a) \( x = (3, 1, 2) \), \( y = (7, 1, -2) \)
   (b) \( x = (1, 1, 1) \), \( y = (-1, 1, 1) \)
   (c) \( x = (0, 1, 3, -3) \), \( y = (4, 0, 0, 1) \)

5. Use the MATLAB command \texttt{cross(u, v)} to find the cross products of the following vectors.
   (a) \( u = (1, -2, 1) \), \( v = (3, 1, -2) \)
   (b) \( u = (0, 1, -2) \), \( v = (-5, 14, 6) \)

6. Let \( u = (-3, 2, 4) \), \( v = (5, 0, -7) \), and \( w = (-1, -5, 6) \). Use MATLAB to illustrate the following properties of the cross product.
   (a) \( u \times v = -(v \times u) \)
   (b) \( u \times u = \mathbf{0} \)
   (c) \( u \times (v + w) = (u \times v) + (u \times w) \)
   (d) \( u \cdot (v \times w) = (u \times v) \cdot w \)
7. The MATLAB command $A\cdot b$ finds the least squares solution to the linear system of equations $Ax = b$. For instance, if

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

then the command $A\cdot b$ gives the answer

$$\begin{bmatrix} 0.6000 \\ 0.5000 \end{bmatrix}$$

Use MATLAB to solve the least squares problem $Ax = b$ for the given matrices.

(a) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

8. Use MATLAB to find bases for the four fundamental subspaces of the following matrices.

(a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 1 & 4 \\ 0 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 3 & -2 & -1 & 0 \\ 4 & 1 & -1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$
1. Find the kernel and range of the linear transformation given by \( T(x) = Ax \) for these matrices \( A \).

   (a) \( A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & 0 & -2 \end{bmatrix} \)
   (b) \( A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & -1 & -2 & 2 \\ 1 & 2 & 4 & -5 \end{bmatrix} \)
   (c) \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \\ -13 & -14 & -15 & -16 \end{bmatrix} \)
   (d) \( A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \end{bmatrix} \)

2. Let \( B \) be the upper triangular matrix generated by the MATLAB command \( B = \text{triu(ones(6))} \).

   Let \( A = BB^T - B \) and determine the rank and nullity of the linear transformation

   \( L: \mathbb{R}^6 \to \mathbb{R}^6, L(x) = Ax \).

3. Which of these linear transformations defined by \( T(x) = Ax \) are one-to-one? Which are onto?

   (a) \( A = \text{magic}(6) \)
   (b) \( A = \text{hilb}(6) \)
   (c) \( A = \text{tril(ones(6))} \)

4. Let \( T: \mathbb{R}^n \to \mathbb{R}^m \) be a linear transformation. Let \( B = \{ v_1, v_2, \ldots, v_n \} \) and \( B' = \{ w_1, w_2, \ldots, w_m \} \) be bases for \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively. You can use MATLAB to find the matrix of \( T \) relative to the bases \( B \) and \( B' \) as follows.

   (a) Form the matrices \( B \) and \( B' \) whose columns are the given basis vectors.
   (b) Let \( A \) be the \( m \times n \) standard matrix of \( T \).
   (c) Adjoin \( B' \) to \( AB \) to form the \( m \times (m + n) \) matrix \( \mathbf{C} = [B' \ A^T \ B] \).
   (d) Use \( \text{ref}(\mathbf{C}) \) to calculate the reduced row-echelon form of \( \mathbf{C} \). The \( m \times n \) matrix composed of the right-hand \( n \) columns of the reduced row-echelon form of \( \mathbf{C} \) form the matrix of \( T \) relative to the bases \( B \) and \( B' \).

   Use this algorithm to find the matrix of the following linear transformations relative to the given bases.

   (a) \( T: \mathbb{R}^2 \to \mathbb{R}^3, T(x, y) = (x + y, x, y) \),
   \( B = \{ (1, -1), (0, 1) \}, \ B' = \{ (1, 1, 0), (0, 1, 1), (1, 0, 1) \}\)
   (b) \( T: \mathbb{R}^3 \to \mathbb{R}^2, T(x, y, z) = (2x - z, y - 2x) \),
   \( B = \{ (2, 0, 1), (0, 2, 1), (1, 2, 1) \}, \ B' = \{ (1, 1), (2, 0) \}\)
   (c) \( T: \mathbb{R}^4 \to \mathbb{R}^4, T(x, y, z) = (2x, x + y, y + z, x + z) \),
   \( B = \{ (2, 0, 1), (0, 2, 1), (1, 2, 1) \}, \ B' = \{ (1, 0, 1, 0), (0, 1, 0, 1), (1, 0, 1, 0) \}\)

5. Use the results of Exercise 4 to find the image of the given vector \( v \) two ways: first by calculating \( T(v) = Av \), and second by using the matrix of \( T \) relative to the bases \( B \) and \( B' \).

   (a) \( v = (5, 4) \)
   (b) \( v = (0, -5, 7) \)
   (c) \( v = (1, -5, 2) \)

6. Let \( B = \{ v_1, v_2, \ldots, v_n \} \) and \( B' = \{ w_1, w_2, \ldots, w_m \} \) be two ordered bases for \( \mathbb{R}^n \).

   Recall from Section 4.7 that you find the transition matrix \( P^{-1} \) from \( B \) to \( B' \) as follows.

   (a) Form the matrices \( B \) and \( B' \) whose columns are the given basis vectors.
   (b) Adjoin \( B \) to \( B' \), forming the \( n \times 2n \) matrix \( \mathbf{C} = [B' \ A^T \ B] \).
   (c) Let \( D \) be the reduced row-echelon form of \( \mathbf{C} \), \( D = \text{ref}(\mathbf{C}) \).
   (d) \( P^{-1} \) is the \( n \times n \) matrix consisting of the right-hand \( n \) columns of \( D \).

   Use MATLAB to find the matrix \( \mathbf{AP} \) of the linear transformation \( T: \mathbb{R}^n \to \mathbb{R}^n \) relative to the basis \( B' \).

   (a) \( T: \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (2x - y, y - x) \),
   \( B' = \{ (1, -2), (0, 3) \}\)
   (b) \( T: \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (x + y, x + y + 3z) \),
   \( B' = \{ (1, -1, 0), (0, 0, 1), (0, 1, -1) \}\)
7. Let \( B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \) and \( \text{Bprime} = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\} \) be bases for \( \mathbb{R}^3 \), and let

\[
A = \begin{bmatrix}
1 & 3 & 0 \\
3 & 1 & 0 \\
0 & 0 & -2 \\
\end{bmatrix}
\]

be the matrix of \( T: \mathbb{R}^3 \to \mathbb{R}^3 \) relative to \( B \), the standard basis.

(a) Find the transition matrix \( P \) from \( \text{Bprime} \) to \( B \).
(b) Find the transition matrix \( P^{-1} \) from \( B \) to \( \text{Bprime} \).
(c) Find \( \text{Aprime} \), the matrix of \( T \) relative to \( \text{Bprime} \).
(d) Let

\[
[v]_{\text{Bprime}} = \begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}
\]

and find \([v]_{B}\) and \([T(v)]_{B}\).
(e) Find \([T(v)]_{\text{Bprime}}\) two ways: first as \( P^{-1}[T(v)]_{B} \) and then as \( \text{Aprime}[v]_{\text{Bprime}} \).
1. The MATLAB command `poly(A)` produces the coefficients of the characteristic polynomial of the square matrix `A`, beginning with the highest degree term. Find the characteristic polynomial of the following matrices.

(a) \[ A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \]

(b) \[ A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix} \]

(c) \[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \]

2. If you set \( p = \text{poly}(A) \), then the command `roots(p)` calculates the roots of the characteristic polynomial of the matrix `A`. Use this sequence of commands to find the eigenvalues of the matrices in Exercise 1.

3. The MATLAB command \([V D] = \text{eig}(A)\) produces a diagonal matrix `D` containing the eigenvalues of `A` on the diagonal and a matrix `V` whose columns are the corresponding eigenvectors. Use this command to find the eigenvalues and corresponding eigenvectors of the three matrices in Exercise 1.

4. Let

\[ A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \]

Use MATLAB to find the eigenvalues and corresponding eigenvectors of \( A, A^T \), and \( A^{-1} \). What do you observe?

5. Let

\[ A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \]

Use MATLAB to diagonalize `A` as follows. First, compute the eigenvalues and eigenvectors of `A` using the command \([P D] = \text{eig}(A)\). The diagonal matrix `D` contains the eigenvalues of `A`, and the corresponding eigenvectors form the columns of `P`. Verify that `P` diagonalizes `A` by showing that `P^{-1}AP = D`.

6. Follow the procedure outlined in Exercise 5 to show that the matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \]

is not diagonalizable.

7. Follow the procedure outlined in Exercise 5 to diagonalize (if possible) the following matrices.

(a) \[ A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \]

(b) \[ A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix} \]

(c) \[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \]
8. For a symmetric matrix $A$, the MATLAB command $[P \ D] = \text{eig}(A)$ will produce a diagonal matrix $D$ containing the eigenvalues of $A$, and an *orthogonal* matrix $P$ containing the corresponding eigenvectors. For instance, if

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

is the matrix from Section 7.3, Example 8, then the command $[P \ D] = \text{eig}(A)$ yields

$$P = \begin{bmatrix} -0.8944 & -0.4472 \\ 0.4472 & -0.8944 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix},$$

which is equivalent to the solution given in the text. Use this procedure to orthogonally diagonalize the following symmetric matrices.

(a) $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & -3 \end{bmatrix}$
CHAPTER 8  ❑  MATLAB EXERCISES

MATLAB handles complex numbers and matrices in much the same way as real ones. The imaginary unit $i = \sqrt{-1}$ is a built-in constant. Verify the result of Example 5, Section 8.2, by entering the matrix $A$,

$$A = \begin{bmatrix} 2 - i & -5 + 2i \\ -5 + 2i & 3 - i \end{bmatrix}$$

and then entering $\text{inv}(A)$.

Note: In MATLAB, $A^T$ designates the transpose $A^T$ of a complex matrix $A$. If you do not include the period, then MATLAB returns the complex conjugate transpose $A^*$. Use MATLAB to verify the following using the matrix $A$ from Example 5, Section 8.2.

$$A^* = \begin{bmatrix} 2 - i & 3 - i \\ -5 + 2i & -6 + 2i \end{bmatrix}$$

1. Use MATLAB to perform the following matrix operations, given

$$A = \begin{bmatrix} 1 & 2 - i \\ 2 + i & i \end{bmatrix}, \quad B = \begin{bmatrix} 3i & 4 \\ -4 & -i \end{bmatrix}, \quad C = \begin{bmatrix} i & -i & 0 \\ 2 & 0 & 2 + 3i \end{bmatrix}$$

(a) $AB$    (b) $3iC$    (c) $A^{-1}$
(d) $C^*C$  (e) $\text{det}(A + B)$  (f) $iAB^2 + (1 - i)CCT$

2. Use MATLAB to solve the system of linear equations $Ax = b$.

(a) $A = \begin{bmatrix} i & 2 - i \\ 3 - 2i & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 + i \\ 6 - 4i \end{bmatrix}$
(b) $A = \begin{bmatrix} 1 & 0 & i \\ -2 & 1 + i & -i \\ 1 - i & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} i \\ 0 \\ 2 - i \end{bmatrix}$

3. Use MATLAB to determine which of the following matrices are Hermitian and which are normal.

(a) $\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$  (b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 3 - 4i & 2 \\ 1 + i & 4 - i \end{bmatrix}$  (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{bmatrix}$

4. The MATLAB command $[P \quad D] = \text{eig}(A)$ will produce a diagonal matrix $D$ containing the eigenvalues of the complex matrix $A$, and a matrix $P$ containing the corresponding eigenvectors. For instance, if

$$A = \begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}$$

is the matrix from Example 7, Section 8.5, then the command $[P \quad D] = \text{eig}(A)$ yields

$$P = \begin{bmatrix} 0.1581 + 0.4743i & -0.3780 - 0.0000i \\ -0.3162 - 0.1581i & 0.3780 + 0.7559i \\ 0.7906 & 0.3780 \end{bmatrix}$$

and

$$D = \begin{bmatrix} -2.0000 & 0 & 0 \\ 0 & -1.0000 & 0 \\ 0 & 0 & 6.0000 \end{bmatrix},$$

which is equivalent to the solution given in the text. Use this procedure to diagonalize the following matrices.

(a) $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$  (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & i \\ 0 & i & 2 \end{bmatrix}$  (c) $A = \begin{bmatrix} 1 & 1 + i & 1 - i \\ 1 - i & 0 & i \\ 1 + i & -i & 0 \end{bmatrix}$