F

Differential Equations

F.1 Solutions of Differential Equations

- Find general solutions of differential equations.
- Find particular solutions of differential equations.

General Solution of a Differential Equation
A differential equation is an equation involving a differentiable function and one or more of its derivatives. For instance,
\[ y' + 2y = 0 \]
Differential equation
is a differential equation. A function \( y = f(x) \) is a solution of a differential equation if the equation is satisfied when \( y \) and its derivatives are replaced by \( f(x) \) and its derivatives. For example,
\[ y = e^{-2x} \]
Solution of differential equation
is a solution of the differential equation shown above. To see this, substitute for \( y \) and \( y' = -2e^{-2x} \) in the original equation.
\[ y' + 2y = -2e^{-2x} + 2(e^{-2x}) \]
Substitute for \( y \) and \( y' \).
\[ = 0 \]
In the same way, you can show that \( y = 2e^{-2x} \), \( y = -3e^{-2x} \), and \( y = \frac{1}{2}e^{-2x} \) are also solutions of the differential equation. In fact, each function given by
\[ y = Ce^{-2x} \]
General solution
where \( C \) is a real number, is a solution of the equation. This family of solutions is called the general solution of the differential equation.

Example 1 Verifying Solutions
Determine whether each function is a solution of the differential equation \( y'' - y = 0 \).

\begin{enumerate}
  \item[\textbf{a.}] \( y = Ce^x \)  
  \item[\textbf{b.}] \( y = Ce^{-x} \)
\end{enumerate}

**SOLUTION**
\begin{enumerate}
  \item[\textbf{a.}] Because \( y' = Ce^x \) and \( y'' = Ce^x \), it follows that
    \[ y'' - y = Ce^x - Ce^x = 0. \]
    So, \( y = Ce^x \) is a solution.
  \item[\textbf{b.}] Because \( y' = -Ce^{-x} \) and \( y'' = Ce^{-x} \), it follows that
    \[ y'' - y = Ce^{-x} - Ce^{-x} = 0. \]
    So, \( y = Ce^{-x} \) is also a solution.
\end{enumerate}

**Checkpoint 1**
Determine whether \( y = Ce^{4x} \) is a solution of the differential equation \( y' = y \).  

\[ □ \]
Particular Solutions and Initial Conditions

A particular solution of a differential equation is any solution that is obtained by assigning specific values to the arbitrary constant(s) in the general solution.*

Geometrically, the general solution of a differential equation represents a family of curves known as solution curves. For instance, the general solution of the differential equation \( xy' - 2y = 0 \) is

\[
y = Cx^2.
\]

Figure F.1 shows several solution curves corresponding to different values of \( C \).

Particular solutions of a differential equation are obtained from initial conditions placed on the unknown function and its derivatives. For instance, in Figure F.1, suppose you want to find the particular solution whose graph passes through the point \((1, 3)\). This initial condition can be written as

\[
y = 3 \quad \text{when} \quad x = 1.
\]

Substituting these values into the general solution produces \( 3 = C(1)^2 \), which implies that \( C = 3 \). So, the particular solution is

\[
y = 3x^2.
\]

**Example 2  Finding a Particular Solution**

For the differential equation

\[
xy' - 3y = 0
\]

verify that \( y = Cx^3 \) is a solution. Then find the particular solution determined by the initial condition \( y = 2 \) when \( x = -3 \).

**SOLUTION**  You know that \( y = Cx^3 \) is a solution because \( y' = 3Cx^2 \) and

\[
xy' - 3y = x(3Cx^2) - 3(Cx^3)
\]

\[
= 3Cx^3 - 3Cx^3
\]

\[
= 0.
\]

Furthermore, the initial condition \( y = 2 \) when \( x = -3 \) yields

\[
y = Cx^3 \quad \text{General solution}
\]

\[
2 = C(-3)^3 \quad \text{Substitute initial condition.}
\]

\[
\frac{2}{27} = C \quad \text{Solve for } C.
\]

and you can conclude that the particular solution is

\[
y = \frac{2x^3}{27}.
\]

Try checking this solution by substituting for \( y \) and \( y' \) in the original differential equation.

**Checkpoint 2**

For the differential equation \( xy' - 2y = 0 \), verify that \( y = Cx^2 \) is a solution. Then find the particular solution determined by the initial condition \( y = 1 \) when \( x = 4 \).

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*Some differential equations have solutions other than those given by their general solutions. These are called singular solutions. In this brief discussion of differential equations, singular solutions will not be discussed.
Example 3  
Finding a Particular Solution

You are working in the marketing department of a company that is producing a new cereal product to be sold nationally. You determine that a maximum of 10 million units of the product could be sold in a year. You hypothesize that the rate of growth of the sales $x$ (in millions of units) is proportional to the difference between the maximum sales and the current sales. As a differential equation, this hypothesis can be written as

$$\frac{dx}{dt} = k(10 - x), \quad 0 \leq x \leq 10.$$  

The rate of change of $x$ is proportional to the difference between 10 and $x$.

The general solution of this differential equation is

$$x = 10 - Ce^{-kt} \quad \text{General solution}$$

where $t$ is the time in years. After 1 year, 250,000 units have been sold. Sketch the graph of the sales function over a 10-year period.

SOLUTION  
Because the product is new, you can assume that $x = 0$ when $t = 0$. So, you have two initial conditions.

$x = 0$ when $t = 0$  
First initial condition

$x = 0.25$ when $t = 1$  
Second initial condition

Substituting the first initial condition into the general solution produces

$$0 = 10 - Ce^{-k(0)}$$

which implies that

$$C = 10.$$

Substituting the second initial condition into the general solution produces

$$0.25 = 10 - 10e^{-k(1)}$$

$$-9.75 = -10e^{-k}$$

$$0.975 = e^{-k}$$

$$-\ln 0.975 = k$$

which implies that

$$k = -\ln 0.975 \approx 0.0253.$$

So, the particular solution is

$$x = 10 - 10e^{-0.0253t}, \quad \text{Particular solution}$$

The table shows the annual sales during the first 10 years, and the graph of the solution is shown in Figure F.2.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>0.25</td>
<td>0.49</td>
<td>0.73</td>
<td>0.96</td>
<td>1.19</td>
<td>1.41</td>
<td>1.62</td>
<td>1.83</td>
<td>2.04</td>
<td>2.24</td>
</tr>
</tbody>
</table>

Checkpoint 3

Repeat Example 3 using the initial conditions $x = 0$ when $t = 0$ and $x = 0.3$ when $t = 1.$

In the first three examples in this section, each solution was given in explicit form, such as \( y = f(x) \). Sometimes you will encounter solutions for which it is more convenient to write the solution in implicit form, as shown in Example 4.

**Example 4** Sketching Graphs of Solutions

Given that
\[
2y^2 - x^2 = C \quad \text{General solution}
\]
is the general solution of the differential equation
\[
2yy' - x = 0
\]
sketch the particular solutions represented by \( C = 0, \pm 1, \text{ and } \pm 4 \).

**SOLUTION** The particular solutions represented by \( C = 0, \pm 1, \text{ and } \pm 4 \) are shown in Figure F.3.

Graphs of Five Particular Solutions

**FIGURE F.3**

**Checkpoint 4**

Given that
\[
y = Cx^2
\]
is the general solution of
\[
xy' - 2y = 0
\]
sketch the particular solutions represented by \( C = 1, C = 2, \text{ and } C = 4 \).

**SUMMARIZE** (Section F.1)

1. Explain how to verify a solution of a differential equation (page F1). For an example of verifying a solution, see Example 1.

2. Describe the difference between a general solution of a differential equation and a particular solution (pages F1 and F2). For an example of a general solution of a differential equation and a particular solution, see Example 2.

3. Describe a real-life example of how a differential equation can be used to model the sales of a company’s product (page F3, Example 3).
Solutions of Differential Equations

In Exercises 1–4, find the first and second derivatives of the function.

1. \( y = 3x^2 + 2x + 1 \)
2. \( y = -2x^3 - 8x + 4 \)
3. \( y = -3e^{2x} \)
4. \( y = -3e^{x^2} \)

In Exercises 5 and 6, solve for \( k \).

5. \( 0.5 = 9 - 9e^{-k} \)
6. \( 14.75 = 25 - 25e^{-2k} \)

Verifying Solutions In Exercises 1–12, verify that the function is a solution of the differential equation. See Example 1.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = Ce^{4x} )</td>
<td>( y' = 4y )</td>
</tr>
<tr>
<td>( y = e^{-2x} )</td>
<td>( y' + 2y = 0 )</td>
</tr>
<tr>
<td>( y = 2x^3 )</td>
<td>( y' - \frac{3}{x}y = 0 )</td>
</tr>
<tr>
<td>( y = 4x^2 )</td>
<td>( y' - \frac{2}{x}y = 0 )</td>
</tr>
<tr>
<td>( y = Cx^3 - 3x )</td>
<td>( xy' - 3x - 2y = 0 )</td>
</tr>
<tr>
<td>( y = x^2 + 2x + \frac{C}{x} )</td>
<td>( xy' + y = x(3x + 4) )</td>
</tr>
<tr>
<td>( y = Ce^{x+4} )</td>
<td>( x(y' - 1) - (y - 4) = 0 )</td>
</tr>
<tr>
<td>( y = e^{x} )</td>
<td>( y' + (2x - 1)y = 0 )</td>
</tr>
<tr>
<td>( y = e^{x/3} )</td>
<td>( x^2y'' - 2y = 0 )</td>
</tr>
<tr>
<td>( y = C_1 \sin x + C_2 \cos x )</td>
<td>( y'' + y = 0 )</td>
</tr>
<tr>
<td>( y = Ce^{3x} + C_2e^{-x} )</td>
<td>( y'' - 3y' - 4y = 0 )</td>
</tr>
</tbody>
</table>

Determining Solutions In Exercises 13–16, determine whether the function is a solution of the differential equation \( y'' - 16y = 0 \).

13. \( y = e^{-2x} \)
14. \( y = 5 \ln x \)
15. \( y = \frac{4}{x} \)
16. \( y = 3 \sin 2x \)

Determining Solutions In Exercises 17–20, determine whether the function is a solution of the differential equation \( y''' - 3y' + 2y = 0 \).

17. \( y = \frac{2}{3}xe^{-2x} \)
18. \( y = \cos x \)
19. \( y = xe^x \)
20. \( y = x \ln x \)

Finding a Particular Solution In Exercises 21–24, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition. See Example 2.

21. General solution: \( y = Ce^{-2x} \)
   Differential equation: \( y' + 2y = 0 \)
   Initial condition: \( y = 3 \) when \( x = 0 \)

22. General solution: \( 2x^2 + 3y^2 = C \)
   Differential equation: \( 2x + 3yy' = 0 \)
   Initial condition: \( y = 2 \) when \( x = 1 \)

23. General solution: \( y = C_1 + C_2 \ln x \)
   Differential equation: \( xy'' + y' = 0 \)
   Initial condition: \( y = 5 \) and \( y' = 0.5 \) when \( x = 1 \)

24. General solution: \( y = C_1e^{4x} + C_2e^{-3x} \)
   Differential equation: \( y'' - y' - 12y = 0 \)
   Initial condition: \( y = 5 \) and \( y' = 6 \) when \( x = 0 \)

Sketching Graphs of Solutions In Exercises 25 and 26, the general solution of the differential equation is given. Sketch the particular solutions that correspond to the indicated values of \( C \). See Example 4.

<table>
<thead>
<tr>
<th>General Solution</th>
<th>Differential Equation</th>
<th>( C )-Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = C(x + 2)^2 )</td>
<td>((x + 2)y'' - 2y = 0)</td>
<td>(0, \pm 1, \pm 2)</td>
</tr>
<tr>
<td>( y = Ce^{-x} )</td>
<td>( y' + y = 0 )</td>
<td>(0, \pm 1, \pm 4)</td>
</tr>
</tbody>
</table>

Finding General Solutions In Exercises 27–34, use integration to find the general solution of the differential equation.

27. \( \frac{dy}{dx} = 3x^2 \)
28. \( \frac{dy}{dx} = 2x^3 - 3x \)
29. \( \frac{dy}{dx} = \frac{1}{1 + x} \)
30. \( \frac{dy}{dx} = \frac{x - 2}{x} \)
31. \( \frac{dy}{dx} = x\sqrt{x^2 + 6} \)
32. \( \frac{dy}{dx} = \frac{x}{1 + x^2} \)
33. \( \frac{dy}{dx} = \cos 4x \)
34. \( \frac{dy}{dx} = 4 \sin x \)
Finding a Particular Solution  In Exercises 35–38, some of the curves corresponding to different values of $C$ in the general solution of the differential equation are shown in the figure. Find the particular solution that passes through the point plotted on the graph.

35. $y^2 = Cx^3$

36. $2xy' - 3y = 0$

37. $y = Ce^x$

38. $y^2 = 2Cx$

39. Biology  The limiting capacity of the habitat of a wildlife herd is 750. The growth rate of the herd is proportional to the unutilized opportunity for growth, as described by the differential equation

$$\frac{dN}{dt} = k(750 - N).$$

The general solution of this differential equation is

$$N = 750 - Ce^{-kt}.$$

When $t = 0$, the population of the herd is 100. After 2 years, the population has grown to 160.

(a) Write the population $N$ as a function of $t$.

(b) What is the population of the herd after 4 years?

40. Investment  The rate of growth of an investment is proportional to the amount in the investment at any time $t$. That is,

$$\frac{dA}{dt} = kA.$$

The general solution of this differential equation is

$$A = Ce^{kt}.$$

The initial investment is $1000$, and after 10 years the balance is $3320.12$. What is the particular solution?

41. Safety  Assume that the rate of change per hour in the number of miles $x$ of road cleared by a snowplow is inversely proportional to the depth $h$ of the snow. This rate of change is described by the differential equation

$$\frac{dx}{dh} = \frac{k}{h}.$$

Show that

$$s = 25 - \frac{13}{\ln 3} \ln \frac{h}{2}$$

is a solution of this differential equation.

42. HOW DO YOU SEE IT?  The graph shows a solution of one of the following differential equations. Determine the correct equation. Explain your reasoning.

(i) $y' = xy$

(ii) $y' = \frac{4x}{y}$

(iii) $y' = -4xy$

(iv) $y' = 4 - xy$

43. Verifying a Solution  Show that $y = a + C e^{k(1-\beta)t}$ is a solution of the differential equation

$$y = a + b(y - a) + \left( \frac{1}{k} \right) \left( \frac{dy}{dt} \right)$$

where $k$ is a constant.

44. Using a Solution  The function $y = C e^{ks}$ is a solution of the differential equation

$$\frac{dy}{dx} = 0.07y.$$

Is it possible to determine $C$ or $k$ from the information given? If so, find its value.

True or False?  In Exercises 45 and 46, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

45. A differential equation can have more than one solution.

46. If $y = f(x)$ is a solution of a differential equation, then $y = f(x) + C$ is also a solution.
F.2 Separation of Variables

Use separation of variables to solve differential equations.
Use differential equations to model and solve real-life problems.

Separation of Variables

The simplest type of differential equation is one of the form \( y' = f(x) \). You know that this type of equation can be solved by integration to obtain

\[
y = \int f(x) \, dx.
\]

In this section, you will learn how to use integration to solve another important family of differential equations—those in which the variables can be separated. This technique is called separation of variables.

**Separation of Variables**

If \( f \) and \( g \) are continuous functions, then the differential equation

\[
\frac{dy}{dx} = f(x)g(y)
\]

has a general solution of

\[
\int \frac{1}{g(y)} \, dy = \int f(x) \, dx + C.
\]

Essentially, the technique of separation of variables is just what its name implies. For a differential equation involving \( x \) and \( y \), you separate the variables by grouping the \( x \) variables on one side and the \( y \) variables on the other. After separating variables, integrate each side to obtain the general solution.

**Example 1** Solving a Differential Equation

Find the general solution of

\[
\frac{dy}{dx} = \frac{x}{y^2 + 1}.
\]

**SOLUTION** Begin by separating variables, then integrate each side.

\[
\frac{dy}{dx} = \frac{x}{y^2 + 1} \quad \text{Differential equation}
\]

\[
(y^2 + 1) \, dy = x \, dx \quad \text{Separate variables.}
\]

\[
\int (y^2 + 1) \, dy = \int x \, dx \quad \text{Integrate each side.}
\]

\[
\frac{y^3}{3} + y = \frac{x^2}{2} + C \quad \text{General solution}
\]

**Checkpoint 1**

Find the general solution of \( \frac{dy}{dx} = \frac{x^2}{y} \).
Example 2 Solving a Differential Equation

Find the general solution of
\[ \frac{dy}{dx} = \frac{x}{y}. \]

**SOLUTION** Begin by separating variables, then integrate each side.

\[
\begin{aligned}
\frac{dy}{dx} &= \frac{x}{y} & \text{Differential equation} \\
y\,dy &= x\,dx & \text{Separate variables.} \\
\int y\,dy &= \int x\,dx & \text{Integrate each side.} \\
\frac{y^2}{2} &= \frac{x^2}{2} + C_1 & \text{Find antiderivative of each side.} \\
y^2 &= x^2 + 2C_1 & \text{Multiply each side by 2.}
\end{aligned}
\]

So, the general solution is \( y^2 = x^2 + C \). Note that \( C_1 \) is used as a temporary constant of integration in anticipation of multiplying each side of the equation by 2 to produce the constant \( C \).

Checkpoint 2

Find the general solution of
\[ \frac{dy}{dx} = \frac{x + 1}{y}. \]

Example 3 Solving a Differential Equation

Find the general solution of \( e^y \frac{dy}{dx} = 2x \). Use a graphing utility to graph several solutions.

**SOLUTION** Begin by separating variables, then integrate each side.

\[
\begin{aligned}
e^y \frac{dy}{dx} &= 2x & \text{Differential equation} \\
e^y \,dy &= 2x\,dx & \text{Separate variables.} \\
\int e^y \,dy &= \int 2x\,dx & \text{Integrate each side.} \\
e^y &= x^2 + C & \text{Find antiderivative of each side.}
\end{aligned}
\]

By taking the natural logarithm of each side, you can write the general solution as
\[ y = \ln(x^2 + C). \]

The graphs of the particular solutions given by \( C = 0, 5, 10, \) and \( 15 \) are shown in Figure F.4.

Checkpoint 3

Find the general solution of
\[ 2y \frac{dy}{dx} = -2x. \]

Use a graphing utility to graph the particular solutions given by \( C = 1, 2, \) and \( 4 \).
**Example 4** Finding a Particular Solution

Solve the differential equation \( xe^{x^2} + yy' = 0 \) subject to the initial condition \( y = 1 \) when \( x = 0 \).

**SOLUTION**

\[
\begin{align*}
xe^{x^2} + yy' &= 0 & & \text{Differential equation} \\
\frac{dy}{dx} &= -xe^{x^2} & & \text{Subtract } xe^{x^2} \text{ from each side.} \\
y\,dy &= -xe^{x^2}\,dx & & \text{Separate variables.} \\
\int y\,dy &= \int -xe^{x^2}\,dx & & \text{Integrate each side.} \\
y^2 &= -\frac{1}{2}e^{x^2} + C_1 & & \text{Find antiderivative of each side.} \\
y^2 &= -e^{x^2} + C & & \text{Multiply each side by 2.}
\end{align*}
\]

To find the particular solution, substitute the initial condition values to obtain

\[
(1)^2 = -e^{(0)^2} + C.
\]

This implies that \( 1 = -1 + C \), or \( C = 2 \). So, the particular solution that satisfies the initial condition is

\[
y^2 = -e^{x^2} + 2. & & \text{Particular solution}
\]

**Checkpoint 4**

Solve the differential equation \( e^x + yy' = 0 \) subject to the initial condition \( y = 2 \) when \( x = 0 \).

**Example 5** Solving a Differential Equation

Example 3 in Section F.1 uses the differential equation

\[
\frac{dx}{dt} = k(10 - x), \quad 0 \leq x \leq 10
\]

to model the sales of a new product. Solve this differential equation.

**SOLUTION**

\[
\begin{align*}
\frac{dx}{dt} &= k(10 - x) & & \text{Differential equation} \\
\frac{1}{10 - x}\,dx &= k\,dt & & \text{Separate variables.} \\
\int \frac{1}{10 - x}\,dx &= \int k\,dt & & \text{Integrate each side.} \\
-\ln(10 - x) &= kt + C_1 & & \text{Find antiderivatives.} \\
\ln(10 - x) &= -kt - C_1 \\
10 - x &= e^{-kt-C_1} \\
x &= 10 - Ce^{-kt} & & \text{Exponentiate each side.} & & \text{Solve for } x.
\end{align*}
\]

**Checkpoint 5**

Solve the differential equation \( \frac{dy}{dx} = k(65 - y) \) for \( 0 \leq y \leq 65 \).
Application

**Example 6** Corporate Investing

A corporation invests part of its receipts at a rate of $P$ dollars per year in a fund for future corporate expansion. The fund earns $r$ percent interest per year compounded continuously. The rate of growth of the amount $A$ in the fund is

$$\frac{dA}{dt} = rA + P$$

where $t$ is the time (in years). Solve the differential equation for $A$ as a function of $t$, where $A = 0$ when $t = 0$.

**SOLUTION** You can solve the differential equation using separation of variables.

- $\frac{dA}{dt} = rA + P$ Differential equation
- $dA = (rA + P)\,dt$ Differential form
- $\frac{dA}{rA + P} = dt$ Separate variables.
- $\frac{1}{r}\ln|A + P| = t + C_1$ Integrate.
- $\ln(A + P) = rt + C_2$ Assume $rA + P > 0$ and multiply each side by $r$.
- $rA + P = e^{rt+C_2}$ Exponentiate each side.
- $A = \frac{Ce^{rt} - P}{r}$ Solve for $A$.
- $A = Ce^{rt} - \frac{P}{r}$ General solution

Using $A = 0$ when $t = 0$, you find the value of $C$.

$$0 = Ce^{(0)} - \frac{P}{r} \implies C = \frac{P}{r}$$

So, the differential equation for $A$ as a function of $t$ can be written as

$$A = \frac{P}{r}(e^{rt} - 1).$$

**Checkpoint 6**

Use the result of Example 6 to find $A$ when $P = $550,000, $r = 5.9\%$, and $t = 25$ years.

**SUMMARIZE** *(Section F.2)*

1. Explain how to use separation of variables to solve a differential equation *(page F7)*. For examples of solving a differential equation using separation of variables, see Examples 1, 2, 3, 4, and 5.

2. Describe a real-life example of how separation of variables can be used to solve a differential equation that models corporate investing *(page F10, Example 6).*
**SKILLS WARM UP F.2**
The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.4, 5.2, and 5.3.

In Exercises 1–6, find the indefinite integral and check your result by differentiating.

1. \( \int x^{3/2} \, dx \)
2. \( \int (r^3 - t^{1/3}) \, dt \)
3. \( \int \frac{2}{x - 5} \, dx \)
4. \( \int \frac{y}{2y^2 + 1} \, dy \)
5. \( \int e^{2x} \, dy \)
6. \( \int xe^{x^2 - x^2} \, dx \)

In Exercises 7–10, solve the equation for \( C \) or \( k \).

7. \( (3)^2 - 6(3) = 1 + C \)
8. \( (-1)^2 + (-2)^2 = C \)
9. \( 10 = 2e^{2x} \)
10. \( (6)^2 - 3(6) = e^{-k} \)

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**Exercises F.2**

**Separation of Variables** In Exercises 1–6, decide whether the variables in the differential equation can be separated.

1. \( \frac{dy}{dx} = \frac{x}{y + 3} \)
2. \( \frac{dy}{dx} = \frac{x + 1}{x} \)
3. \( \frac{dy}{dx} = \frac{1}{x} + 1 \)
4. \( \frac{dy}{dx} = \frac{x}{x + y} \)
5. \( \frac{dy}{dx} = x - y \)
6. \( \frac{dy}{dx} = \frac{1}{y} \)

**Solving a Differential Equation** In Exercises 7–26, use separation of variables to find the general solution of the differential equation. See Examples 1 and 2.

7. \( \frac{dy}{dx} = 2x \)
8. \( \frac{dy}{dx} = \frac{1}{x} \)
9. \( \frac{dr}{ds} = 0.05r \)
10. \( \frac{dr}{ds} = 0.05s \)
11. \( \frac{dy}{dx} = \frac{x - 1}{y^3} \)
12. \( \frac{dy}{dx} = \frac{x^2 - 3}{y^2} \)
13. \( 3y^2 \frac{dy}{dx} = 1 \)
14. \( \frac{dy}{dx} = x^2y \)
15. \( x^2 + 4y \frac{dy}{dx} = 0 \)
16. \( (1 + y) \frac{dy}{dx} - 4x = 0 \)
17. \( y' - xy = 0 \)
18. \( y' - y = 5 \)
19. \( e^x \frac{dy}{dx} = 3r^2 + 1 \)
20. \( e^{(y' + 1)} = 1 \)
21. \( \frac{dy}{dx} = \frac{1}{\sqrt{1 - y}} \)
22. \( y' = \frac{x}{y} - \frac{x}{1 + y} \)
23. \( (2 + x)y' = 2y \)
24. \( y' - y(x + 1) = 0 \)
25. \( \frac{dy}{dx} = \sin x \)
26. \( \frac{dy}{dx} = 6 \cos(\pi x) \)

**Finding a Particular Solution** In Exercises 31–38, use the initial condition to find the particular solution of the differential equation. See Example 4.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>31. ( yy' - e^t = 0 )</td>
<td>( y = 4 ) when ( x = 0 )</td>
</tr>
<tr>
<td>32. ( \sqrt{x} + \sqrt{y}y' = 0 )</td>
<td>( y = 4 ) when ( x = 1 )</td>
</tr>
<tr>
<td>33. ( x(y + 4) + y' = 0 )</td>
<td>( y = -5 ) when ( x = 0 )</td>
</tr>
<tr>
<td>34. ( \frac{dy}{dx} = x^2(1 + y) )</td>
<td>( y = 3 ) when ( x = 0 )</td>
</tr>
<tr>
<td>35. ( \sqrt{x^2 - 16y}y' = 5x )</td>
<td>( y = -2 ) when ( x = 5 )</td>
</tr>
<tr>
<td>36. ( y' = e^{x-y} )</td>
<td>( y = 0 ) when ( x = 0 )</td>
</tr>
<tr>
<td>37. ( \frac{dy}{dx} = y \cos x )</td>
<td>( y = 1 ) when ( x = 0 )</td>
</tr>
<tr>
<td>38. ( \frac{dy}{dx} = 2x \sin x^2 )</td>
<td>( y = 1 ) when ( x = 0 )</td>
</tr>
</tbody>
</table>

**Finding an Equation** In Exercises 39 and 40, find an equation of the form that passes through the point and has the specified slope. Then graph the equation.

39. Point: \((-1, 1)\) \( y' = \frac{-9x}{16y} \)
40. Point: \((8, 2)\) \( y' = \frac{2y}{3x} \)
Velocity In Exercises 41 and 42, solve the differential equation to find velocity \( v \) as a function of time \( t \) if \( v = 0 \) when \( t = 0 \). The differential equation models the motion of two people on a toboggan after consideration of the forces of gravity, friction, and air resistance.

41. \( 12.5 \frac{dv}{dt} = 43.2 - 1.25v \)

42. \( 12.5 \frac{dv}{dt} = 43.2 - 1.75v \)

43. Biology: Cell Growth The growth rate of a spherical cell with volume \( V \) is proportional to its surface area \( S \). For a sphere, the surface area and volume are related by \( S = kV^{2/3} \). So, a model for the cell’s growth is

\[
\frac{dV}{dt} = kV^{2/3}.
\]

Solve this differential equation.

44. HOW DO YOU SEE IT? The differential equation

\[
\frac{dN}{dt} = k(30 - N)
\]

represents the rate of change of the number of units produced per day by a new employee, where \( N \) is the number of units and \( t \) is the time (in days). The general solution of this differential equation is

\[
N = 30 - 30e^{-kt}.
\]

The graphs below show the particular solutions for \( k = 0.3, 0.6, \) and \( 1 \). Match the value of \( k \) with each graph. Explain your reasoning.

45. Radioactive Decay The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 25 years?

46. Radioactive Decay The rate of decomposition of radioactive einsteinium is proportional to the amount present at any time. The half-life of radioactive einsteinium is 276 days. After 100 days, 0.5 gram remains. What was the initial amount?

47. Weight Gain A calf that weighed 60 pounds at birth gains weight at the rate

\[
\frac{dw}{dt} = k(1200 - w)
\]

where \( w \) is the weight (in pounds) and \( t \) is the time (in years).

(a) Solve this differential equation.

(b) Use a graphing utility to graph the particular solutions for

\( k = 0.8, 0.9, \) and \( 1 \).

(c) The animal is sold when its weight reaches 800 pounds. Find the time of sale for each of the models in part (b).

(d) What is the maximum weight of the animal for each of the models in part (b)?

Business Capsule After finding that the camera he wanted was sold out at a local store, Jack Abraham was inspired to start Milo.com. Named after his dog, the site shows buyers which nearby stores currently have a product in stock. This benefits not only shoppers but also retailers, as Milo drives foot traffic into their stores. In just one year, Milo.com grew to cover more than 140 retailers in 50,000 locations across the United States. In 2010, the company was bought by eBay, where Abraham now leads the local division.

48. Research Project Use your school’s library, the Internet, or some other reference source to gather information about a company that offers innovative products or services. Collect data about the revenue that the company has generated and find a mathematical model of the data. Write a short paper that summarizes your findings.
Take this quiz as you would take a quiz in class. When you are done, check your work against the answers given in the back of the book.

In Exercises 1–4, verify that the function is a solution of the differential equation.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( y = Ce^{-x^2} )</td>
<td>( 2y' + y = 0 )</td>
</tr>
<tr>
<td>2. ( y = C_1 \cos x + C_2 \sin x )</td>
<td>( y'' + y = 0 )</td>
</tr>
<tr>
<td>3. ( y = \frac{1}{x} )</td>
<td>( xy'' + 2y' = 0 )</td>
</tr>
<tr>
<td>4. ( y = \frac{x^3}{5} - x + C \sqrt{x} )</td>
<td>( 2xy' - y = x^3 - x )</td>
</tr>
</tbody>
</table>

In Exercises 5 and 6, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

5. General solution: \( y = C_1 \sin 3x + C_2 \cos 3x \)
   
   Differential equation: \( y'' + 9y = 0 \)
   
   Initial condition: \( y = 2 \) and \( y' = 1 \) when \( x = \pi/6 \)

6. General solution: \( y = C_1 x + C_2 x^3 \)
   
   Differential equation: \( x^2 y'' - 3xy' + 3y = 0 \)
   
   Initial condition: \( y = 0 \) and \( y' = 4 \) when \( x = 2 \)

In Exercises 7–10, use separation of variables to find the general solution of the differential equation.

7. \( \frac{dy}{dx} = -4x + 4 \)

8. \( y' = (x + 2)(y - 1) \)

9. \( y \frac{dy}{dx} = \frac{1}{2x + 1} \)

10. \( \frac{dy}{dx} = \frac{x}{3y^2 + 1} \)

In Exercises 11 and 12, (a) find the general solution of the differential equation and (b) use a graphing utility to graph the particular solutions given by \( C = 0 \) and \( C = \pm 1 \).

11. \( \frac{dy}{dx} = \frac{x^2 + 1}{2y} \)

12. \( \frac{dy}{dx} = \frac{y}{x - 3} \)

In Exercises 13 and 14, use the initial condition to find the particular solution of the differential equation.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>13. ( y' + 2y - 1 = 0 )</td>
<td>( y = 1 ) when ( x = 0 )</td>
</tr>
<tr>
<td>14. ( \frac{dy}{dx} = y \sin \pi x )</td>
<td>( y = -3 ) when ( x = \frac{1}{2} )</td>
</tr>
</tbody>
</table>

15. Find an equation of the graph that passes through the point \((0, 2)\) and has a slope of \( y' = 3x^2y \). Then graph the equation.

16. Ignoring resistance, a sailboat starting from rest accelerates at a rate proportional to the difference between the velocities of the wind and the boat. With a 20-knot wind, this acceleration is described by the differential equation \( \frac{dv}{dt} = k(20 - v) \), where \( v \) is the velocity of the boat (in knots) and \( t \) is the time (in hours). After half an hour, the boat is moving at 10 knots. Write the velocity as a function of time.
F.3 First-Order Linear Differential Equations

- Solve first-order linear differential equations.
- Use first-order linear differential equations to model and solve real-life problems.

First-Order Linear Differential Equations

In this section, you will see how to solve a very important class of differential equations—first-order linear differential equations. The term “first-order” refers to the fact that the highest-order derivative of y in the equation is the first derivative.

**Definition of a First-Order Linear Differential Equation**

A **first-order linear differential equation** is an equation of the form

\[ y' + P(x)y = Q(x) \]

where P and Q are functions of x. An equation that is written in this form is said to be in **standard form**.

To solve a linear differential equation, write it in standard form to identify the functions P(x) and Q(x). Then integrate P(x) and form the expression

\[ u(x) = e^{\int P(x) \, dx} \]

which is called an **integrating factor**. The general solution of the equation is

\[ y = \frac{1}{u(x)} \int Q(x)u(x) \, dx. \]

**Example 1** Solving a Linear Differential Equation

Find the general solution of \( y' + y = e^x \).

**SOLUTION** For this equation, \( P(x) = 1 \) and \( Q(x) = e^x \). So, the integrating factor is

\[ u(x) = e^{\int P(x) \, dx} = e^{\int dx} = e^x. \]

This implies that the general solution is

\[ y = \frac{1}{u(x)} \int Q(x)u(x) \, dx \]

\[ = \frac{1}{e^x} \int e^x(e^x) \, dx \]

\[ = e^{-x} \left( \frac{1}{2}e^{2x} + C \right) \]

\[ = \frac{1}{2}e^x + Ce^{-x}. \]

**Checkpoint 1**

Find the general solution of

\[ y' - y = 10. \]
In Example 1, the differential equation was given in standard form. For equations that are not written in standard form, you should first convert to standard form so that you can identify the functions \( P(x) \) and \( Q(x) \).

**Example 2**  
**Solving a Linear Differential Equation**

Find the general solution of \( xy' - 2y = x^2 \). Assume \( x > 0 \).

**SOLUTION**  
Begin by writing the equation in standard form.

\[
y' + \left(-\frac{2}{x}\right)y = x \quad \text{Standard form, } y' + P(x)y = Q(x)
\]

In this form, you can see that \( P(x) = -2/x \) and \( Q(x) = x \). So,

\[
\int P(x) \, dx = -\int \frac{2}{x} \, dx = -2 \ln x = -\ln x^2
\]

which implies that the integrating factor is

\[
u(x) = e^{\int P(x) \, dx} = e^{-\ln x^2} = \frac{1}{x^2}
\]

Integrating factor

This implies that the general solution is

\[
y = \frac{1}{u(x)} \int Q(x)u(x) \, dx \quad \text{Form of general solution}
\]

\[
= \frac{1}{1/x^2} \int \frac{1}{x^2} \, dx \quad \text{Substitute.}
\]

\[
= x^2 \int \frac{1}{x} \, dx \quad \text{Simplify.}
\]

\[
= x^2 \ln x + C. \quad \text{General solution}
\]

**Checkpoint 2**

Find the general solution of \( xy' - y = x \). Assume \( x > 0 \).

**Guidelines for Solving a Linear Differential Equation**

1. Write the equation in standard form.
   \[
y' + P(x)y = Q(x)
   \]

2. Find the integrating factor.
   \[
u(x) = e^{\int P(x) \, dx}
   \]

3. Evaluate the integral below to find the general solution.
   \[
y = \frac{1}{u(x)} \int Q(x)u(x) \, dx
   \]
In Intravenous Feeding

Glucose is added intravenously to the bloodstream at the rate of \( q \) units per minute, and the body removes glucose from the bloodstream at a rate proportional to the amount present. Assume that \( A \) is the amount of glucose in the bloodstream at time \( t \) and that the rate of change of the amount of glucose is

\[
\frac{dA}{dt} = q - kA
\]

where \( k \) is a constant. Find the general solution of the differential equation.

**SOLUTION** In standard form, this linear differential equation is

\[
\frac{dA}{dt} + kA = q
\]

which implies that \( P(t) = k \) and \( Q(t) = q \). So, the integrating factor is

\[
u(t) = e^{\int P(t)dt} = e^{k \cdot dt} = e^{kt}
\]

and the general solution is

\[
A = \frac{1}{u(t)} \int Q(t)u(t) \, dt = e^{-kt} \int q e^{kt} \, dt = e^{-kt} \left( \frac{q}{k} e^{kt} + C \right) = \frac{q}{k} + Ce^{-kt}.
\]

**Checkpoint 3**

Use the general solution

\[
A = \frac{q}{k} + Ce^{-kt}
\]

from Example 3 to find the particular solution determined by the initial condition \( A = 0 \) when \( t = 0 \). (Assume \( k = 0.05 \) and \( q = 0.05 \).)
SKILLS WARM UP F.3

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Sections 4.2, 4.4, and 5.1–5.3.

In Exercises 1–4, simplify the expression.
1. \( e^{-x}(e^{2x} + e^x) \)
2. \( \frac{1}{x}(e^{-x} + e^{2x}) \)
3. \( e^{-\ln x^3} \)
4. \( e^2 \ln x + x \)

In Exercises 5–10, find the indefinite integral.
5. \( \int 4e^{2x} \, dx \)
6. \( \int xe^{3x} \, dx \)
7. \( \int \frac{1}{2x + 5} \, dx \)
8. \( \int \frac{x + 1}{x^2 + 2x + 3} \, dx \)
9. \( \int (4x - 3)^2 \, dx \)
10. \( \int (1 - x^2)^2 \, dx \)

Exercises F.3

Writing in Standard Form
In Exercises 1–6, write the first-order linear differential equation in standard form.
1. \( x^2 - 2xy' + 3y = 0 \)
2. \( y' - 5(2x - y) = 0 \)
3. \( xy' + x = e^{4x} \)
4. \( xy' + x = x^y \)
5. \( y + 1 = (x - 1)y' \)
6. \( x = x^2(y' + y) \)

Solving a Linear Differential Equation
In Exercises 7–18, find the general solution of the first-order linear differential equation. See Examples 1 and 2.
7. \( \frac{dy}{dx} + 3y = 6 \)
8. \( \frac{dy}{dx} - 5y = 15 \)
9. \( \frac{dy}{dx} - y = e^{4x} \)
10. \( \frac{dy}{dx} + 3y = e^{-3x} \)
11. \( \frac{dy}{dx} = \frac{x^2 + 3}{x} \)
12. \( \frac{dy}{dx} = \frac{e^{-2x}}{1 + e^{-2x}} \)
13. \( y' + 2xy = 10x \)
14. \( y' + 5y = e^{5x} \)
15. \( (x - 1)y' + y = x^2 - 1 \)
16. \( xy' + y = x^2 + 1 \)
17. \( x^2y' + 2y = e^{x^2} \)
18. \( xy' + y = x^2 \ln x \)

Using Two Methods
In Exercises 19–22, solve for \( y \) in two ways.
19. \( y' + y = 4 \)
20. \( y' - 3y = -2 \)
21. \( y' - 2xy = 2x \)
22. \( y' + 4xy = x \)

Matching
In Exercises 23–26, match the differential equation with its solution without solving the differential equation. Explain your reasoning.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>23. ( y' - 2x = 0 )</td>
<td>(a) ( y = Ce^{2x} )</td>
</tr>
<tr>
<td>24. ( y' - 2y = 0 )</td>
<td>(b) ( y = -\frac{1}{2} + Ce^{4t} )</td>
</tr>
<tr>
<td>25. ( y' - 2xy = 0 )</td>
<td>(c) ( y = x^2 + C )</td>
</tr>
<tr>
<td>26. ( y' - 2xy = 0 )</td>
<td>(d) ( y = Ce^{2t} )</td>
</tr>
</tbody>
</table>

Finding a Particular Solution
In Exercises 27–34, find the particular solution that satisfies the initial condition.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>27. ( y' + y = 6e^x )</td>
<td>( y = 3 ) when ( x = 0 )</td>
</tr>
<tr>
<td>28. ( y' + 2y = e^{-2x} )</td>
<td>( y = 4 ) when ( x = 1 )</td>
</tr>
<tr>
<td>29. ( xy' + y = 0 )</td>
<td>( y = 2 ) when ( x = 2 )</td>
</tr>
<tr>
<td>30. ( y' + y = x )</td>
<td>( y = 4 ) when ( x = 0 )</td>
</tr>
<tr>
<td>31. ( y' + 3x^2y = 3x^2 )</td>
<td>( y = 6 ) when ( x = 0 )</td>
</tr>
<tr>
<td>32. ( y' + (2x - 1)y = 0 )</td>
<td>( y = 2 ) when ( x = 1 )</td>
</tr>
<tr>
<td>33. ( xy' + 2y = 3x^2 - 5x )</td>
<td>( y = 3 ) when ( x = -1 )</td>
</tr>
<tr>
<td>34. ( 2xy' - y = x^3 - x )</td>
<td>( y = 2 ) when ( x = 4 )</td>
</tr>
</tbody>
</table>

Sales
The rate of change (in thousands of units) in sales \( S \) of a biomedical syringe is modeled by
\[
\frac{dS}{dt} = 0.2(100 - S) + 0.2t
\]
where \( t \) is the time (in years). Solve this differential equation and use the result to complete the table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
36. **HOW DO YOU SEE IT?** The rate of change in the spread of a rumor in a school is modeled by

\[
\frac{dP}{dt} = k(1 - P)
\]

where \( P \) is the percent (in decimal form) of the students who have heard the rumor and \( t \) is the time (in hours), with \( t = 0 \) corresponding to 8:00 A.M. The graph shows the particular solution for this differential equation.

(a) What percent of the students have heard the rumor by 8:00 A.M.?
(b) At what time have 50% of the students heard the rumor?
(c) What percent of the students have heard the rumor by 3:00 P.M.?

37. **Vertical Motion** A falling object encounters air resistance that is proportional to its velocity \( v \). The acceleration due to gravity is \(-9.8 \text{ meters per second per second}\). The rate of change in velocity is

\[
\frac{dv}{dt} = kv - 9.8.
\]

Solve this differential equation to find \( v \) as a function of time \( t \).

38. **Velocity** A booster rocket carrying an observation satellite is launched into space. The rocket and satellite have mass \( m \) and are subject to air resistance proportional to the velocity \( v \) at any time \( t \). A differential equation that models the velocity of the rocket and satellite is

\[
m \frac{dv}{dt} = -mg - kv
\]

where \( g \) is the acceleration due to gravity. Solve the differential equation for \( v \) as a function of \( t \).

39. **Learning Curve** The management at a medical supply factory has found that the maximum number of units an employee can produce in a day is 40. The rate of increase in the number of units \( N \) produced with respect to time \( t \) (in days) by a new employee is proportional to \( 40 - N \). This rate of change of performance with respect to time can be modeled by

\[
\frac{dN}{dt} = k(40 - N).
\]

(a) Solve this differential equation.
(b) Find the particular solution for a new employee who produced 10 units on the first day at the factory and 19 units on the twentieth day.

40. **Investment** Let \( A \) be the amount in a fund earning interest for \( t \) years at the annual rate of \( r \) (in decimal form), compounded continuously. If a continuous cash flow of \( P \) dollars per year is withdrawn from the fund, then the rate of decrease of \( A \) is given by the differential equation

\[
\frac{dA}{dt} = rA - P
\]

where \( A = A_0 \) when \( t = 0 \).

(a) Solve this equation for \( A \) as a function of \( t \).
(b) Use the result from part (a) to find \( A \) when \( A_0 = $2,000,000, \ r = 0.07, \ P = $250,000, \) and \( t = 5 \) years.

41. **Project: Weight Loss** For a project analyzing a person’s weight loss, visit this text’s website at [www.cengagebrain.com](http://www.cengagebrain.com). *(Data Source: The College Mathematics Journal)*
F.4 Applications of Differential Equations

Use differential equations to model and solve real-life problems.

Applications of Differential Equations

**Example 1** Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of substance A. When \( t = 0 \), 60 grams of A are present, and after 1 hour \((t = 1)\), only 10 grams of A remain unconverted. How much of A is present after 2 hours?

**SOLUTION** Let \( y \) be the amount of unconverted substance A at any time \( t \). From the given assumption about the conversion rate, you can write the differential equation as shown.

\[
\frac{dy}{dt} = ky^2
\]

Rate of change of \( y \) is proportional to the square of \( y \).

Using separation of variables or a symbolic integration utility, you can find the general solution to be

\[
y = \frac{-1}{kt + C}.
\]

General solution

To solve for the constants \( C \) and \( k \), use the initial conditions. That is, because \( y = 60 \) when \( t = 0 \), you can determine that \( C = \frac{1}{60} \). Similarly, because \( y = 10 \) when \( t = 1 \), it follows that

\[
10 = \frac{-1}{k - (1/60)}
\]

which implies that \( k = -\frac{1}{12} \). So, the particular solution is

\[
y = \frac{-1}{(-1/12)t - (1/60)}
\]

Substitute for \( k \) and \( C \).

\[
y = \frac{60}{5t + 1}.
\]

Particular solution

Using the model, you can determine that the amount of unconverted substance A after 2 hours is

\[
y = \frac{60}{5(2) + 1} \\
\approx 5.45 \text{ grams.}
\]

In Figure F.5, note that the chemical conversion is occurring rapidly during the first hour. Then, as more and more of substance A is converted, the conversion rate slows down.

**Checkpoint 1**

Use the chemical reaction model in Example 1 to find the amount \( y \) of substance A (in grams) as a function of \( t \) (in hours) given that \( y = 40 \) grams when \( t = 0 \) and \( y = 5 \) grams when \( t = 2 \).
Example 2  Modeling Advertising Awareness

The new cereal product from Example 3 in Section F.1 is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years?

SOLUTION  Let $y$ be the number (in millions) of people at time $t$ who have heard of the product. This means that $(1 - y)$ is the number (in millions) of people who have not heard of it, and $\frac{dy}{dt}$ is the rate at which the population hears about the product. From the given assumption, you can write the differential equation as shown.

$$\frac{dy}{dt} = k(1 - y)$$

Using separation of variables or a symbolic integration utility, you can find the general solution to be

$$y = 1 - Ce^{-kt}.$$  General solution

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 0$ when $t = 0$, you can determine that $C = 1$. Similarly, because $y = 0.5$ when $t = 1$, it follows that $0.5 = 1 - e^{-k}$, which implies that

$$k = -\ln 0.5 \approx 0.693.$$  

So, the particular solution is

$$y = 1 - e^{-0.693t}.$$  Particular solution

This model is shown graphically in Figure F.6. Using the model, you can determine that the number of people who have heard of the product after 2 years is

$$y = 1 - e^{-0.693(2)} 
\approx 0.75 \text{ or 750,000 people.}$$

\[\text{FIGURE F.6}\]

Checkpoint 2  
Repeat Example 2 given that by the end of 1 year, only one-fourth of the population have heard of the product.
Appendix F.4 Applications of Differential Equations

Earlier in the text, you studied two models for population growth: exponential growth, which assumes that the rate of change of $y$ is proportional to $y$, and logistic growth, which assumes that the rate of change of $y$ is proportional to $y$ and $1 - y/L$, where $L$ is the population limit.

The next example describes a third type of growth model called a Gompertz growth model. This model assumes that the rate of change of $y$ is proportional to $y$ and the natural log of $L/y$, where $L$ is the population limit.

**Example 3 Modeling Population Growth**

A population of 20 wolves has been introduced into a national park. The forest service estimates that the maximum population the park can sustain is 200 wolves. After 3 years, the population is estimated to be 40 wolves. If the population follows a Gompertz growth model, how many wolves will there be 10 years after their introduction?

**SOLUTION**

Let $y$ be the number of wolves at any time $t$. From the given assumption about the rate of growth of the population, you can write the differential equation as shown.

$$\frac{dy}{dt} = ky \ln \frac{200}{y}$$

Using separation of variables or a symbolic integration utility, you can find the general solution to be

$$y = 200e^{-C e^{-kt}}.$$  

General solution

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 20$ when $t = 0$, you can determine that

$$C = \ln 10 
\approx 2.3026.$$  

Similarly, because $y = 40$ when $t = 3$, it follows that

$$40 = 200e^{-2.3026e^{-0.1194(3)}}$$

which implies that $k \approx 0.1194$. So, the particular solution is

$$y = 200e^{-2.3026e^{-0.1194t}}.$$  

Particular solution

Using the model, you can estimate the wolf population after 10 years to be

$$y = 200e^{-2.3026e^{-0.1194(10)}} \approx 100$$ wolves.

In Figure F.7, note that after 10 years the population has reached about half of the estimated maximum population. Try checking the growth model to see that it yields $y = 20$ when $t = 0$ and $y = 40$ when $t = 3$.

**Checkpoint 3**

A population of 10 wolves has been introduced into a national park. The forest service estimates that the maximum population the park can sustain is 150 wolves. After 3 years, the population is estimated to be 25 wolves. If the population follows a Gompertz growth model, how many wolves will there be 10 years after their introduction?
In genetics, a commonly used hybrid selection model is based on the differential equation
\[
\frac{dy}{dt} = k y (1 - y) (a - b y).
\]
In this model, \(y\) represents the portion of the population that has a certain characteristic and \(t\) represents the time (measured in generations). The numbers \(a\), \(b\), and \(k\) are constants that depend on the genetic characteristic that is being studied.

**Example 4  
Modeling Hybrid Selection**

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. At the beginning of your study you find that half the population has characteristic D. After four generations \((t = 4)\), you find that 80% of the population has characteristic D. Use the hybrid selection model above with \(a = 2\) and \(b = 1\) to find the percent of the population that will have characteristic D after 10 generations.

**SOLUTION**  
Using \(a = 2\) and \(b = 1\), the differential equation for the hybrid selection model is
\[
\frac{dy}{dt} = k y (1 - y) (2 - y).
\]
Using separation of variables or a symbolic integration utility, you can find the general solution to be
\[
y \frac{(2 - y)}{(1 - y)^2} = Ce^{kt}.
\]
To solve for the constants \(C\) and \(k\), use the initial conditions. That is, because \(y = 0.5\) when \(t = 0\), you can determine that \(C = 3\). Similarly, because \(y = 0.8\) when \(t = 4\), it follows that
\[
\frac{0.8(1.2)}{(0.2)^2} = 3e^{8k}
\]
which implies that
\[
k = \frac{1}{8} \ln 8 = 0.2599.
\]
So, the particular solution is
\[
y \frac{(2 - y)}{(1 - y)^2} = 3e^{0.5199t}.
\]
Using the model, you can estimate the percent of the population that will have characteristic D after 10 generations to be given by
\[
y = \frac{2 - y}{1 - y} = 3e^{0.5199(10)}.
\]
Using a symbolic algebra utility, you can solve this equation for \(y\) to obtain \(y \approx 0.96\).  

The graph of the model is shown in Figure F.8.

**Checkpoint 4**

Repeat Example 4 given that only 25% of the population has characteristic D when \(t = 0\) and 50% of the population has characteristic D when \(t = 4\).
Example 5  Modeling a Chemical Mixture

A tank contains 40 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 4 gallons per minute. At the same time, the tank is being drained at the rate of 4 gallons per minute, as shown in Figure F.9. Assuming that the solution is stirred constantly, will there be at least 14 gallons of alcohol in the tank after 10 minutes?

SOLUTION  Let $y$ be the number of gallons of alcohol in the tank at any time $t$. The percent of alcohol in the 40-gallon tank at any time is $y/40$. Moreover, because 4 gallons of solution are being drained each minute, the rate of change of $y$ is

$$\frac{dy}{dt} = -4 \left( \frac{y}{40} \right) + 2$$

where 2 represents the number of gallons of alcohol entering each minute in the 50% solution. In standard form, this linear differential equation is

$$y' + \frac{1}{10}y = 2.$$  

Standard form

Using an integrating factor or a symbolic integration utility, you can find the general solution to be

$$y = 20 + Ce^{-t/10}.$$  

General solution

Because $y = 4$ when $t = 0$, you can conclude that $C = -16$. So, the particular solution is

$$y = 20 - 16e^{-t/10}.$$  

Particular solution

Using this model, you can determine that the amount of alcohol in the tank when $t = 10$ is

$$y = 20 - 16e^{-10/10} \approx 14.1 \text{ gallons}.$$  

Yes, there will be at least 14 gallons of alcohol in the tank after 10 minutes.

Checkpoint 5

A tank contains 50 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 5 gallons per minute. At the same time, the tank is being drained at the rate of 5 gallons per minute. Assuming that the solution is stirred constantly, how much alcohol will be in the tank after 10 minutes?

Summarize  (Section F.4)

1. Describe a real-life example of how a differential equation can be used to model a chemical reaction (page F19, Example 1).
2. Describe a real-life example of how a differential equation can be used to model population growth (page F21, Example 3).
3. Describe a real-life example of how a differential equation can be used to model hybrid selection (page F22, Example 4).
In Exercises 1–4, use separation of variables to find the general solution of the differential equation.

1. \( \frac{dy}{dx} = 3x \)
2. \( \frac{dy}{dx} = 3 \)
3. \( \frac{dy}{dx} = 2xy \)
4. \( \frac{dy}{dx} = \frac{x - 4}{4y^3} \)

In Exercises 5–8, use an integrating factor to solve the first-order linear differential equation.

5. \( y' + 2y = 4 \)
6. \( y' + 2y = e^{-2x} \)
7. \( y' + xy = x \)
8. \( xy' + 2y = x^2 \)

In Exercises 9 and 10, write the equation that models the statement.

9. The rate of change of \( y \) with respect to \( x \) is proportional to the square of \( x \).
10. The rate of change of \( x \) with respect to \( t \) is proportional to the difference of \( x \) and \( t \).

Chemical Reaction In Exercises 1 and 2, use the chemical reaction model described in Example 1 to find the amount \( y \) (in grams) as a function of time \( t \) (in hours). Then use a graphing utility to graph the function.

1. \( y = 45 \) grams when \( t = 0 \); \( y = 4 \) grams when \( t = 2 \)
2. \( y = 75 \) grams when \( t = 0 \); \( y = 12 \) grams when \( t = 1 \)

Advertising Awareness In Exercises 3 and 4, use the advertising awareness model described in Example 2 to find the number of people \( y \) (in millions) aware of the product as a function of time \( t \) (in years).

3. \( y = 0 \) when \( t = 0 \); \( y = 0.75 \) when \( t = 1 \)
4. \( y = 0 \) when \( t = 0 \); \( y = 0.9 \) when \( t = 2 \)

Population Growth In Exercises 5 and 6, use the Gompertz growth model described in Example 3 to find the population \( y \) as a function of time \( t \) (in years).

5. \( y = 100 \) when \( t = 0 \); \( y = 150 \) when \( t = 2 \)
6. \( y = 30 \) when \( t = 0 \); \( y = 60 \) when \( t = 4 \)

Hybrid Selection In Exercises 7 and 8, use the hybrid selection model described in Example 4 to find the percent \( y \) (in decimal form) of the population that has the indicated characteristics as a function of time \( t \) (in generations).

7. \( y = 0.1 \) when \( t = 0 \); \( y = 0.4 \) when \( t = 4 \)
8. \( y = 0.6 \) when \( t = 0 \); \( y = 0.75 \) when \( t = 2 \)

Finding a Particular Solution In Exercises 9–14, assume that the rate of change in \( y \) is proportional to \( y \). Solve the resulting differential equation \( \frac{dy}{dx} = ky \) and find the particular solution that passes through the points.

9. \( (0, 1), (3, 2) \)
10. \( (0, 4), (1, 6) \)
11. \( (0, 4), (4, 1) \)
12. \( (0, 60), (5, 30) \)
13. \( (2, 2), (3, 4) \)
14. \( (1, 4), (2, 1) \)

15. Chemical Reaction During a chemical reaction, a compound changes into another compound at a rate proportional to the unchanged amount \( y \). Write the differential equation for the chemical reaction model. Find the particular solution when the initial amount of the original compound is 20 grams and the amount remaining after 1 hour is 16 grams.

16. Chemical Reaction Using the result of Exercise 15, when will 75% of the compound have been changed? When will 95% of the compound have been changed?

17. Population Growth The rate of change of the population of a city is proportional to the population \( P \) at any time \( t \) (in years). In 2000, the population was 200,000, and the constant of proportionality was 0.015. Estimate the population of the city in the year 2020.
18. **Fruit Flies** The rate of change of an experimental population of fruit flies is proportional to the population \( P \) at any time \( t \) (in days). There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

19. **Chemistry** A wet towel hung from a clothesline to dry loses moisture through evaporation at a rate proportional to its moisture content. After 1 hour, the towel has lost 40% of its original moisture content. How long will it take the towel to lose 80% of its original moisture content?

20. **Meteorology** The barometric pressure \( y \) (in inches of mercury) at an altitude of \( x \) miles above sea level decreases at a rate proportional to the current pressure according to the model

\[
\frac{dy}{dx} = -0.2y
\]

where \( y = 29.92 \) inches when \( x = 0 \). Find the barometric pressure (a) at the top of Mt. St. Helens (8364 feet) and (b) at the top of Mt. McKinley (20,320 feet).

21. **Sales Growth** The rate of change in sales \( S \) (in thousands of units) of a new product is proportional to the difference between \( L \) and \( S \) at any time \( t \) (in years), where \( L \) is the maximum number of units of the new product available. When \( t = 0, S = 0 \). Write and solve the differential equation for this sales model.

22. **Sales Growth** Use the result of Exercise 21 to find the particular solutions when (a) \( L = 100 \) and \( S = 25 \) when \( t = 2 \), and (b) \( L = 500 \) and \( S = 50 \) when \( t = 1 \).

23. **Biology** A population of eight beavers has been introduced into a new wetlands area. Biologists estimate that the maximum population the wetlands can sustain is 60 beavers. After 3 years, the population is 15 beavers. The population follows a Gompertz growth model. How many beavers will there be in the wetlands after 10 years?

24. **Biology** A population of 30 rabbits has been introduced into a new region. It is estimated that the maximum population the region can sustain is 400 rabbits. After 1 year, the population is estimated to be 90 rabbits. The population follows a Gompertz growth model. How many rabbits will there be after 3 years?

25. **Biology** At any time \( t \) (in years), the rate of growth of the population \( N \) of deer in a state park is proportional to the product of \( N \) and \( L - N \), where \( L = 500 \) is the maximum number of deer the park can maintain.

(a) Use a symbolic integration utility to find the general solution.

(b) Find the particular solution given the conditions \( N = 100 \) when \( t = 0 \) and \( N = 200 \) when \( t = 4 \).

(c) Find \( N \) when \( t = 1 \).

(d) Find \( t \) when \( N = 350 \).

26. **Biology** At any time \( t \) (in years), the rate of growth of the population \( N \) of fish in a pond is proportional to the product of \( N \) and \( L - N \), where \( L = 1000 \) is the maximum number of fish the pond can maintain.

(a) Use a symbolic integration utility to find the general solution.

(b) Find the particular solution given the conditions \( N = 200 \) when \( t = 0 \) and \( N = 500 \) when \( t = 2 \).

(c) Find \( N \) when \( t = 1 \).

(d) Find \( t \) when \( N = 700 \).

27. **Chemical Mixture** A 200-gallon tank is half full of a solution containing 25 pounds of a concentrate. Starting at time \( t = 0 \), distilled water is admitted to the tank at the rate of 5 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

(a) Find the amount of the concentrate in the solution as a function of \( t \) by solving the differential equation

\[
\frac{dQ}{dt} = -5 \left( \frac{Q}{100} \right)
\]

(b) Find the time required for the amount of concentrate in the tank to reach 15 pounds.

28. **Chemical Mixture** A 100-gallon tank is full of a chemical mixture containing 5 pounds of a concentrate. Starting at time \( t = 0 \), a well-stirred mixture containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate.

(a) Find the amount of the concentrate in the solution as a function of \( t \) by solving the differential equation

\[
\frac{dQ}{dt} = -5 \left( \frac{Q}{100} \right) + \frac{5}{2}
\]

(b) Find the amount of concentrate in the tank after 30 minutes.

29. **Population Growth** When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let \( P \) be the population at time \( t \) and let \( N \) be the net increase per unit time due to the difference between immigration and emigration. So, the rate of growth of the population is given by

\[
\frac{dP}{dt} = kP + N, \quad N \text{ is constant.}
\]

Solve the differential equation to find \( P \) as a function of \( t \).
30. **HOW DO YOU SEE IT?** In a learning theory project, the rate of change in the percent $P$ (in decimal form) of correct responses after $n$ trials can be modeled by

$$\frac{dP}{dt} = kP(1 - P),$$

![Graph showing the particular solutions for two different groups.](image)

(a) What was the percent of correct responses before any trials for each group?
(b) What is the limit of $P$ as $t$ approaches infinity for each group?
(c) After how many trials are 75% of the responses correct for each group?

31. **Investment** A large corporation starts at time $t = 0$ to invest part of its profit at a rate of $P$ dollars per year in a fund for future expansion. Assume that the fund earns $r$ percent interest per year compounded continuously. The rate of growth of the amount $A$ in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$, and $r$ is in decimal form. Solve this differential equation for $A$ as a function of $t$.

**Investment** In Exercises 32–34, use the result of Exercise 31.

32. Find $A$ for each situation.
   (a) $P = 100,000$, $r = 0.12$, and $t = 5$ years
   (b) $P = 250,000$, $r = 0.15$, and $t = 10$ years

33. Find $P$ if the corporation needs $120,000,000$ in 8 years and the fund earns 8% interest compounded continuously.

34. Find $r$ if the corporation needs $800,000$ and it can invest $75,000$ per year in a fund earning 13% interest compounded continuously.

### Medical Science

In Exercises 35–38, a medical researcher wants to determine the concentration $C$ (in moles per liter) of a tracer drug injected into a moving fluid with flow $R$ (in liters per minute). Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume $V$ (in liters) of fluid in the compartment is constant.

![Diagram of a single-compartment dilution model](image)

35. **Mixture** If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \frac{-R}{V}C, \quad C = C_0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration as a function of time.
(b) Find the limit of $C$ as $t \to \infty$.

36. **Mixture** Use the solution of the differential equation in Exercise 35 to find the concentration as a function of time. Then use a graphing utility to graph the function.
   (a) $V = 2$ liters, $R = 0.5$ L/min, and $C_0 = 0.6$ mol/L.
   (b) $V = 2$ liters, $R = 1.5$ L/min, and $C_0 = 0.6$ mol/L.

37. **Mixture** In Exercises 35 and 36, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at a constant rate of $Q$ mol/min. The concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \frac{R}{V}C, \quad C = 0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration as a function of time.
(b) Find the limit of $C$ as $t \to \infty$.

38. **Mixture** Use the solution of the differential equation in Exercise 37 to find the concentration as a function of time. Then use a graphing utility to graph the function.
   (a) $Q = 2$ mol/min, $V = 2$ liters, and $R = 0.5$ L/min
   (b) $Q = 1$ mol/min, $V = 2$ liters, and $R = 1.0$ L/min