When asked “What is the probability of drawing a 13 of diamonds from a deck of cards?” we might respond “zero” because our standard 52-card deck has cards with faces instead of 11, 12, or 13. As always, we must know the sample space in order to determine the probability. The cards in Figure 1 are drawn from a deck for Five Hundred, a card game popular in Australia and New Zealand but copyrighted by the United States Playing Card Company in 1904. The specialized deck contains 63 cards: the standard 52 cards, plus a joker, an 11 and 12 in each suit (spades ♠, hearts ♥, clubs ♣, and diamonds ♦), and a 13 in each of hearts and diamonds.

In this chapter, we explore three questions: “What are the chances?” (probability, Sections 8.1 and 8.2), “How do we count the ways?” (methods of counting, Section 8.3), and “What is an important pattern?” (Pascal’s triangle, Section 8.4).
Probability: Definitions and Single-Event Probability

OBJECTIVES

- Define probability and related terms.
- Find theoretical probabilities for single events.
- Find empirical probabilities for single events.
- Identify the effect of the law of large numbers.

WARM-UP

1. Convert to a decimal and a percent.
   a. $\frac{3}{8}$ 0.375, 37.5%
   b. $\frac{4}{15}$ 0.2666..., 26.7%

2. Convert to a fraction and a percent.
   a. 0.475 $\frac{19}{40}$, 47.5%
   b. 0.1666... $\frac{1}{6}$, 16.7%

3. Convert to a fraction and a decimal.
   a. 87.5% $\frac{7}{8}$, 0.875
   b. 62.5% $\frac{5}{8}$, 0.625

HAVE YOU EVER WONDERED what your chances were of hitting it big in the lottery or been curious about how likely you were to be selected to be a contestant on Big Brother? Probabilities play a role in your life, whether you realize it or not. Probability affects your ability to get a home loan, the cost of your car insurance, and the availability of a new medicine, as well as your chances of becoming an instant millionaire.

You probably have some idea of what probability means but may not have a good idea of how probabilities are calculated or how they are used. Informally, probability is defined as the chance that an event will happen. Probability is studied in an attempt to describe predictable long-term patterns in random occurrences.

BASIC DEFINITIONS

An experiment is an activity with an observable outcome, or result. In a random experiment, the outcome is not known in advance. A sample space is the set of all possible outcomes. An event is a subset of a sample space. An event may include one or more outcomes.

In Example 1, we roll a die (one of a pair of dice). Figure 2 shows the six faces.

EXAMPLE 1

Applying definitions For the random experiment of rolling a single six-sided die, tell which of the following is an event containing two outcomes, which is the sample space, and which is an event containing one outcome.

a. The set {1, 2, 3, 4, 5, 6}
b. Obtaining a 5 on the top face of the die
c. Obtaining either a 5 or a 6 on the top face of the die
Solution

a. The sample space
b. An event containing one outcome
c. An event containing two outcomes

EXAMPLE 2 Applying definitions: Roulette In the popular casino game of Roulette, a ball is thrown onto a spinning wheel (Figure 3). The wheel has 38 pockets. Two pockets, numbered 0 and 00, are colored green. The rest of the pockets are numbered 1 to 36; half are colored red and half are colored black. The ball will land in one of the pockets.

Figure 3 Roulette wheel.

a. If the experiment is to play Roulette and note the color of the pocket in which the ball lands, what is the sample space?
b. If the experiment is to play Roulette and note the number of the pocket in which the ball lands, what set describes the event of getting a number greater than 30?
c. If the experiment is to play Roulette and note the number of the slot in which the ball lands, what set describes the event of the number being a factor of 24?

Solution

a. Because there are only three colors on the Roulette wheel, the sample space is {green, red, black}.
b. \{31, 32, 33, 34, 35, 36\}
c. \{1, 2, 3, 4, 6, 8, 12, 24\}

PRACTICE 1 If the experiment is to roll a single six-sided die, as in Example 1, what set describes the following events?

a. The outcome is an even number. \{2, 4, 6\}
b. The outcome is a prime number. \{2, 3, 5\}
c. The outcome is greater than or equal to 2. \{2, 3, 4, 5, 6\}

THEORETICAL PROBABILITY OF AN EVENT

When we can count the number of different outcomes and the number of outcomes in an event, we can write the theoretical probability of the event.
**Theoretical Probability of an Event**

Assume that an experiment has \( n \) different outcomes, each of which has an equal chance of occurring. If event \( A \) can occur in \( a \) of these ways, then the probability that event \( A \) happens, written \( P(A) \), is

\[
P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}} = \frac{a}{n}
\]

\( P(A) \) is read “probability of \( A \)” and is a function similar to \( f(x) \). \( P(\text{sum of 6}) \) means the probability of obtaining a sum of 6.

**Example 3** Finding probability

*Find the theoretical probability of the following events in the experiment of rolling a single die.*

**a.** The outcome is a 5.

**b.** The outcome is an even number.

**c.** The outcome is an 8.

**d.** The outcome is the number 1, 2, 3, 4, 5, or 6.

**Solution**

\[a. \quad P(5) = \frac{\text{number of ways to roll a 5}}{\text{total number of outcomes}} = \frac{1}{6}\]

\[b. \quad P(\text{even number outcome}) = \frac{\text{number of ways to roll an even number}}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}\]

\[c. \quad P(8) = \frac{\text{number of ways to roll an 8}}{\text{total number of outcomes}} = \frac{0}{6} = 0\]

It is not possible to roll an 8. The probability of this event is zero! An event with probability zero is an impossible event.

\[d. \quad P(1, 2, 3, 4, 5, \text{ or 6}) = \frac{\text{number of ways to roll a 1, 2, 3, 4, 5, or 6}}{\text{total number of outcomes}} = \frac{6}{6} = 1\]

When we roll a single die, we are certain to get a 1, 2, 3, 4, 5, or 6. So the probability of this event is 1, or 100%. An event with probability 1 is a certain event.

Example 3 demonstrates one of the most important rules of probability:

For any event \( A \),

\[0 \leq P(A) \leq 1\]

**Practice 2**

What is the probability of the following events in the experiment of playing one round of Roulette?

**a.** The ball lands in the pocket numbered 17. \( P(17) = \frac{1}{38} \)

**b.** The ball lands in a red pocket. \( P(\text{red}) = \frac{18}{38} = \frac{9}{19} \)

**c.** The ball lands in a green pocket or a black pocket. \( P(\text{green or black}) = \frac{2}{38} + \frac{18}{38} = \frac{10}{19} \)
Critical to the definition of theoretical probability is the assumption that the outcomes are equally likely—each has an equal chance of occurring. If this is not the case, one approach is to write the sample space in such a way that the outcomes are equally likely.

When we do an experiment by rolling two six-sided dice and looking at the sum of the top faces, we might reasonably say that the sample space is \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}. Unfortunately, the outcomes in this sample space are not equally likely. Although there is only one way to get a sum of 2 (roll a 1 on both dice), there is more than one way to get a sum of 7 (roll a 1 and a 6, a 2 and a 5, or a 3 and a 4).

The dice in Figure 4 illustrate how to write the sample space so that the outcomes are equally likely. In the figure, one die is red and one die is green. The red die has six faces (1, 2, 3, 4, 5, 6), and the green die has six faces (1, 2, 3, 4, 5, 6). By showing all the possible pairs of dice, we show the 36 equally likely outcomes as a sample space.

### EXAMPLE 4 Finding probability

a. Find \(P(\text{sum of 6})\) when two dice (one red and one green) are rolled.

b. Find \(P(\text{sum of 4})\) when two dice are rolled.

c. Find \(P(\text{sum of 1})\) when two dice are rolled.

#### Solution

The sample space for two dice contains 36 outcomes (Figure 4).

a. The event “sum of 6” happens five ways, as shown in Figure 5.

![Figure 5: Sum of 6](image)

The probability for a sum of 6 = \(P(\text{sum of 6}) = \frac{5}{36}\).

b. The event “sum of 4” happens three ways, as shown in Figure 6.

![Figure 6: Sum of 4](image)

The probability for a sum of 4 = \(P(\text{sum of 4}) = \frac{3}{36} = \frac{1}{12}\).

c. The event “sum of 1” cannot happen. The probability for a sum of 1 = \(P(\text{sum of 1}) = 0\).
In Example 5, we select a jelly bean at random, so each outcome (candy) is equally likely to be chosen.

**EXAMPLE 5** Finding probability: selecting a jelly bean from a bowl

Suppose we select one jelly bean at random from the bowl in Figure 7. What is the probability of

- a. selecting a white jelly bean?
- b. selecting a green jelly bean?
- c. not selecting a white jelly bean?

![Figure 7: Bowl of jelly beans.](image)

**Solution**

a. There are 12 jelly beans in the bowl, and 5 of them are white. So the probability of selecting a white jelly bean is $P(\text{white}) = \frac{5}{12}$.

b. There are 12 jelly beans in the bowl, and 3 of them are green. So the probability of selecting a green jelly bean is $P(\text{green}) = \frac{3}{12} = \frac{1}{4}$.

c. Not selecting a white jelly bean means selecting either a red jelly bean or a green jelly bean. There are 4 red and 3 green, so the probability is $P(\text{not white}) = \frac{7}{12}$.

In Example 5, for the experiment described, there are three outcomes: selecting a white jelly bean, selecting a red jelly bean, or selecting a green jelly bean. The probability of each event is

- $P(\text{white}) = \frac{5}{12}$
- $P(\text{green}) = \frac{3}{12}$
- $P(\text{red}) = \frac{4}{12}$

The sum of these probabilities is

$$P(\text{white}) + P(\text{green}) + P(\text{red}) = \frac{5}{12} + \frac{3}{12} + \frac{4}{12} = \frac{12}{12} = 1$$

This leads us to another important rule of probability:

The sum of the probabilities of all the outcomes in a sample space is 1.

In Example 5, selecting a white jelly bean and not selecting a white jelly bean are opposite events, or complementary events. We found that

$$P(\text{white}) = \frac{5}{12} \text{ and } P(\text{not white}) = \frac{7}{12}.$$  

This illustrates another important rule and gives a shortcut for finding $P(\text{not white})$ when we know $P(\text{white})$:

The probability of event $A$ not happening is $1 - P(A)$. 
PRACTICE 3 Suppose we select a jelly bean at random from the nine jelly beans remaining in the bowl in Figure 8. What is the probability of

a. not selecting a red jelly bean? \( P(\text{not red}) = \frac{7}{9} \)

b. not selecting a red or green jelly bean? \( P(\text{not red and not green}) = \frac{4}{9} \)

![Figure 8: Bowl of jelly beans, later.](image)

**EMPIRICAL PROBABILITY**

The previous examples deal with theoretical probabilities. A *theoretical probability* is a probability calculated through reasoning. An *empirical probability* is a probability calculated by making repeated observations. Empirical probability is useful when we cannot create a sample space with equally likely outcomes.

**Empirical Approach to Probability**

After conducting or observing an experiment a large number of times and counting the number of times that event \( A \) occurs, estimate the probability of \( A \) as follows:

\[
P(A) \approx \frac{\text{number or times } A \text{ occurred}}{\text{total number of times experiment was run}}
\]

The number of times the experiment was run may be the number of random observations made.

**EXAMPLE 6 Finding empirical probability** Your dog has eyes of two different colors. As a veterinary researcher, you wish to study this phenomenon. To do so, you take a random sample of dogs and record their eye color. Your results are in Table 1. What is an estimate of the probability that a dog has two different eye colors?

**Table 1** Eye Color among Dogs

<table>
<thead>
<tr>
<th>Eye Color</th>
<th>Number of Dogs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brown</td>
<td>26</td>
</tr>
<tr>
<td>Yellow</td>
<td>14</td>
</tr>
<tr>
<td>Blue or blue grey</td>
<td>34</td>
</tr>
<tr>
<td>Two different colors</td>
<td>21</td>
</tr>
</tbody>
</table>

**Solution** To use the empirical method for estimating probability, you need a random sample size, which in this case is 95 dogs. If event \( A \) is dogs having two different eye colors,

\[
P(A) \approx \frac{\text{number of dogs with two different eye colors}}{\text{total number of observations}} = \frac{21}{95} \approx 0.221
\]
EXAMPLE 7  
Finding empirical probability: earthquakes

a. According to the U.S. Geological Service, during the last 150 years there have been four earthquakes of magnitude 6.7 or greater in the San Francisco Bay region. What is the empirical probability that an earthquake of magnitude 6.7 or greater will hit the Bay Area in the next year?

b. Prior to 2010, there had never been a tsunami that disabled a nuclear power plant. 
Is 0 the probability that a tsunami disables a nuclear power plant?

Solution

a. The empirical probability of an earthquake of magnitude 6.7 or greater in the Bay Area is $\frac{4}{150}$.

b. The true probability may be low, but not 0, as evidenced by the tsunami that struck Japan in 2011. Empirical probability should not be used when an event is rare.

PRACTICE 4

Macy’s Thanksgiving Day Parade began in 1924 and has been held each year since, except 1942–1944. According to the National Oceanographic and Atmospheric Administration, from 1924 to 2008 there were 31 Thanksgiving days with measurable precipitation (rain or snow or ice pellets). Of these days, three had measurable snow. For any given year during this period, what was the probability of measurable precipitation on parade day? What was the probability of measurable snow on parade day?

$$P(\text{precipitation}) = \frac{31}{82}; P(\text{snow}) = \frac{3}{82}$$

LAWS OF LARGE NUMBERS

If you ask four different people to predict the outcome when they roll a die, each may give a different guess. Each guess has an equal likelihood of being correct. However, if you ask the question “How many times will a 2 appear if the die is rolled 600 times?” the person answering 100 will be the one most likely to be correct.

The outcome of rolling a die once is completely unpredictable, but over the long run, a pattern emerges. The law of large numbers (also known as Bernoulli’s theorem or Tchebycheff’s theorem) states that the empirical probability of an event is approximately equal to the theoretical probability if the number of trials is very large.

EXAMPLE 8  
Investigating the law of large numbers  
Suppose we conduct the experiment of spinning the spinner in Figure 9.

a. What might we expect for the frequency of each outcome for 10 spins? for 400 spins?

b. What probabilities are suggested by the outcomes in Table 2, summarizing an experiment with 10 spins?

c. What probabilities are suggested by the outcomes in Table 3, summarizing an experiment with 400 spins?

**TABLE 2**  
<table>
<thead>
<tr>
<th>Event</th>
<th>Number of Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
</tr>
</tbody>
</table>

**TABLE 3**  
<table>
<thead>
<tr>
<th>Event</th>
<th>Number of Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>97</td>
</tr>
<tr>
<td>B</td>
<td>110</td>
</tr>
<tr>
<td>C</td>
<td>97</td>
</tr>
<tr>
<td>D</td>
<td>96</td>
</tr>
</tbody>
</table>

Solution

a. Because each region takes up a fourth of a complete turn of the spinner, the four outcomes appear equally likely. Thus, we expect $P(A) = P(B) = P(C) = P(D) = \frac{1}{4}$. Because 10 does not divide evenly by 4, the spins cannot land exactly 10 ÷ 4 times on each outcome. This suggests that each of the 4 outcomes will appear either 2 or 3 times. For 400 spins, we expect each of the 4 outcomes to appear close to 100 times.
Using the data in Table 2, we have \( P(A) = \frac{6}{10} = 0.6 \), \( P(B) = \frac{1}{10} = 0.1 \), \( P(C) = \frac{2}{10} = 0.2 \), and \( P(D) = \frac{1}{10} = 0.1 \). Our experimental results are not close to the theoretical result of 0.25.

c. Using the data in Table 3, we have \( P(A) = \frac{97}{400} \approx 0.243 \), which is very close to the theoretical result of 0.25. Similarly, \( P(B) = \frac{110}{400} \approx 0.275 \), \( P(C) = \frac{97}{400} \approx 0.243 \), and \( P(D) = \frac{96}{400} \approx 0.24 \).

The notion of the law of large numbers will be explored further in the Activity.

The data in Example 8 were generated using Probability Simulation, or Prob Sim, an application on the TI-84 Plus calculator. The application is called a simulation because it uses the random number generator on the calculator to give outcomes similar to those we would expect from an actual experiment.

You may have the probability simulation application in your TI-84 Plus APPS menu. If not, download it at http://education.ti.com. To run the application, press (APPS) and ALPHA P to move to that part of the alphabetical listing (Figure 10). Use the cursor to highlight Prob Sim, in the middle of Figure 10. Press ENTER to obtain the introduction screen (Figure 11).

**Figure 10** APPS menu.  
**Figure 11** Probability Simulation screen.

**Step 1: Select the app.** Press any key to obtain the main menu (Figure 12).

**Figure 12** Prob Sim menu.

The calculator does not permit screen pictures of most apps, so read the following directions carefully.

APPS are run with the keys across the top: Y=, WINDOW, ZOOM, TRACE, and GRAPH. In the screen in Figure 12, Y= selects OK, ZOOM selects OPTION, TRACE selects ABOUT, and GRAPH selects QUIT.

To select Spin Spinner, press 4 or move the cursor to 4 and select OK by pressing Y=.

**Step 2: Run the activity.** You will see a screen with a spinner and an empty graph. Select the Spin option, and press WINDOW to spin the spinner.

(continued)
You will see a random spin result and then a bar representing this result on the bar graph. The keys \( \text{WINDOW}, \text{ZOOM}, \) and \( \text{TRACE} \) will spin the spinner one more time, 10 more times, and 50 more times, respectively. Experiment with these keys.

**Step 3: Summarize the data.** Pressing the cursor (left or right) will reveal the frequency for each outcome. On another 400 spins, one of the authors recorded 93 As, 107 Bs, 99 Cs, and 101 Ds.

**Step 4: Stop the app.** Once you are finished with your simulation, press \( \text{Y} = \) to return to the previous screen and \( \text{Y} = \) again to return to the main menu in Figure 12. At the screen in Figure 12, choose \text{QUIT} by pressing \( \text{GRAPH} \).

---

**Answer Box**

**Warm-up:**
1. a. \( \frac{3}{8} \) = 37.5%  
   b. \( \frac{4}{12} \) = 26.7%  
2. a. \( \frac{475}{40} \) = 47.5%  
   b. \( \frac{166}{6} \) = 16.7%  
3. a. 87.5% = \( \frac{7}{8} \)  
   b. 62.5% = \( \frac{5}{8} \)  
**Practice 1:**
   a. \( \{2, 4, 6\} \)  
   b. \( \{2, 3, 5\} \)  
   c. \( \{2, 3, 4, 5, 6\} \)

**Practice 2:**
   a. \( P(17) = \frac{1}{38} \)  
   b. \( P(\text{red}) = \frac{18}{38} = \frac{9}{19} \)  
   c. \( P(\text{green or black}) = \frac{2}{38} + \frac{18}{38} = \frac{10}{19} \)  
**Practice 3:**
   a. \( P(\text{not red}) = \frac{7}{9} \)  
   b. \( P(\text{not red and not green}) = \frac{4}{9} \)  
**Practice 4:**
   a. \( P(\text{precipitation}) = \frac{31}{82} \)  
   b. \( P(\text{snow}) = \frac{3}{82} \)

---

**Reading Questions**

1. The ___________ is the set of all outcomes for an experiment. sample space
2. If the outcomes are ___________, \( P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}} \) equally likely
3. When two six-sided dice are rolled, the probability of rolling a sum of 6 is ___. \( \frac{5}{36} \)
4. The probability of an event \( A \), \( P(A) \), is always greater than or equal to ____ and less than or equal to _____. 0, 1
5. Is the following statement the result of empirical or theoretical probability? The Red River in North Dakota exceeds flood stage about once every 25 years. empirical probability

---

**ACTIVITY**

**The Shoe Toss Experiment.** When a shoe is tossed in the air, what is the probability it will land on the bottom side? on the top side? on the inner side? on the outer side? (See Figure 13.)

**Team and Equipment:** Work in groups of three. Select a shoe from within your group. You need a shoe that has the possibility of landing on each side described above. Because of their sides (or lack of sides), cowboy boots, workboots, and sandals tend to be limited in outcomes. For consistency across groups, use a right-foot shoe.

**Task:** Toss the shoe in the air 100 times to estimate the probability of its landing on different sides. One person should toss the shoe, another should count the number of tosses, and the third should record the data. Before you toss, make a prediction about which side the shoe will land on most.
1. What is your prediction of which side (bottom, top, inner, outer) the shoe will land on most? least? Why does this make sense?

2. Toss the shoe 100 times, and collect the data in a table like Table 4.

<table>
<thead>
<tr>
<th>Side</th>
<th>Tally</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom</td>
<td></td>
</tr>
<tr>
<td>Top</td>
<td></td>
</tr>
<tr>
<td>Inner</td>
<td></td>
</tr>
<tr>
<td>Outer</td>
<td></td>
</tr>
</tbody>
</table>

3. Based on your tallies in part 2, give the experimental probabilities for $P(\text{bottom})$, $P(\text{top})$, $P(\text{inner})$, and $P(\text{outer})$.

4. Are your results in part 3 good estimates of these probabilities? Why or why not? What could you do to improve your estimates?

Instructor Note:
One of the authors has run this experiment with 40 different classes over the past 20 years. The results for part 2 are about 66% for the bottom side, about 12% each for the inside and outside, and about 9–10% for the top. Of course, individual group results will vary. If shoes are somewhat similar, pooling the data from the groups for a whole-class estimate may be reasonable.

8.1 Exercises

1. List the sample space for birth orders (outcomes) in terms of girls (G) or boys (B) for families with
   a. 2 children: GG, GB, BG, BB
   b. 3 children: GGG, GGB, GBG, BGG, GGB, GBG, BGG, GGB, GBG, BBG, BGB, BGG, BBG, HBB
   c. 4 children: GGGG, GGGB, GGBB, GGBG, GBGG, GBBG, BGGG, BGGB, BGBG, BBGG, BBGB, BBG, HBBB

2. List the sample space in terms of heads (H) or tails (T) for flipping a coin 4 times. Organize your list to group outcomes together by the number of heads. See Additional Answers.

3. List the sample space for possible outcomes (first, second, third) of a golf tournament playoff among Jonathan Byrd, K. J. Choi, and Lucas Glover. BCG, BGC, CBG, CGB, GBC

4. List the sample space for possible outcomes (first, second, third) of a tennis tournament playoff among Kim Clijsters, Serena Williams, and Venus Williams. KSV, KVS, SVK, SKV, VKS, VSK

5. Name the outcomes for the events described. The experiment is rolling a single six-sided die.
   a. A number greater than 3: {4, 5, 6}
   b. A number less than 8: {1, 2, 3, 4, 5, 6}
   c. A perfect square: {1, 4}
   d. A number greater than 1: {2, 3, 4, 5, 6}

6. Identify the events described. The experiment is playing a round of Roulette and noting the pocket in which the ball lands.
   a. The ball lands in a pocket that has a prime number. {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31}
   b. The ball lands in pocket that has a number less than 10. {0, 00, 1, 2, 3, 4, 5, 6, 7, 8, 9}
   c. The ball lands in a pocket that is colored green. {0, 00}

* Circled numbers are core exercises.
7. Identify the events described. The experiment is tossing a fair coin twice in succession.
   a. Getting at least 1 head \([HH, HT, TH]\)
   b. Getting at most 1 head \([TT, TH, HT]\)
   c. Getting exactly 2 heads \([HH]\)
   d. Not getting exactly 2 heads \([TH, HT, TT]\)

8. Name the outcomes for the events described. The experiment is selecting a card from a standard deck of 52 cards and noting the value of the card. Assume that the ace is the highest value card and not a 1.
   a. Selecting a face card or an ace from any suit: spades, hearts, clubs, or diamonds. (Diamonds are shown in the figure.) \([4\text{ As}, 4\text{ Ks}, 4\text{ Qs}, 4\text{ Js}]\)
   b. Selecting a card showing a number less than 9 \([2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A of spades; 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A of clubs}]\)
   c. Not selecting an even-numbered card \([4\text{ 2s}, 4\text{ 3s}, 4\text{ 4s}, 4\text{ 5s}, 4\text{ 6s}, 4\text{ 7s}, 4\text{ 8s}]\)
   d. Not selecting a red card \([\text{spades, hearts, clubs, or diamonds. (Diamonds are not a red card.)}]\)

9. There are 5 handles (3 forks and 2 spoons) showing in the clean dish rack. What is the probability of grabbing a spoon? \(\frac{2}{5}\)

10. In the refrigerator are identical jars containing mayonnaise, salad dressing, caesar dressing, blue cheese dressing, and ranch dressing. What is the probability of grabbing the jar of mayonnaise in the dark? \(\frac{1}{5}\)

11. A number is chosen at random from the numbers 1 to 17. Find
   a. \(P(2\text{-digit number})\)
   b. \(P(\text{prime number})\)
   c. \(P(\text{number divisible by both 2 and 3})\)
   d. \(P(\text{perfect square})\)
   e. \(P(\text{number less than 8})\)

12. A card is drawn at random from ten cards numbered 1 to 10. Find
   a. \(P(2)\)
   b. \(P(\text{number divisible by 3})\)
   c. \(P(\text{multiple of 2})\)
   d. \(P(\text{even number})\)
   e. \(P(\text{number less than 8})\)

13. A letter is selected at random from \textsc{probability}.
   \(P(A)\)
   \(P(B)\)
   \(P(\text{one of the vowels A, E, I, O, U, Y})\)
   \(P(\text{letter also in the word BABAR})\)
   \(P(\text{letter also in the word BROTHER})\)

14. A letter is selected at random from \textsc{multiplication}.
   \(P(I)\)
   \(P(L)\)
   \(P(\text{one of the vowels A, E, I, O, U, Y})\)
   \(P(\text{letter also in the word PLUM})\)
   \(P(\text{letter also in the word TRIPLE})\)

15. Complete the chart.

<table>
<thead>
<tr>
<th>Sample Space</th>
<th>Event A</th>
<th>Event (not A)</th>
<th>(P(A))</th>
<th>(P(\text{not A}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>birth of a child</td>
<td>girl</td>
<td>boy</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>BB, BG, GB, GG</td>
<td>GG</td>
<td>BB, BG, GB</td>
<td>(\frac{1}{4})</td>
<td>(\frac{3}{4})</td>
</tr>
<tr>
<td>BB, BG, GB, GG</td>
<td>at most 1 boy</td>
<td>BB</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>1, 2, 3, 4, 5</td>
<td>2, 4</td>
<td>1, 3, 5</td>
<td>(\frac{2}{5})</td>
<td>(\frac{3}{5})</td>
</tr>
<tr>
<td>x, y, z</td>
<td>z</td>
<td>x, y</td>
<td>(\frac{1}{3})</td>
<td>(\frac{2}{3})</td>
</tr>
<tr>
<td>sum of 2 dice</td>
<td>odd sum</td>
<td>even sum</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
</tbody>
</table>

16. Complete the chart.

<table>
<thead>
<tr>
<th>Sample Space</th>
<th>Event A</th>
<th>Event (not A)</th>
<th>(P(A))</th>
<th>(P(\text{not A}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>position of 1 light switch</td>
<td>on</td>
<td>off</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>position of 2 light switches</td>
<td>(on, on), (off, on)</td>
<td>(off, off), (on, off)</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>1, 2, 3, 4, 5, 6</td>
<td>1, 4, 5, 6</td>
<td>2, 3</td>
<td>(\frac{2}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>HH, HT, TH, TT</td>
<td>at least 1 head</td>
<td>TT</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>p, q, r, t, u, v</td>
<td>p, q, r, t, u, v</td>
<td>s</td>
<td>(\frac{6}{7})</td>
<td>(\frac{1}{7})</td>
</tr>
<tr>
<td>sum of 2 dice</td>
<td>sum of 6 or 8</td>
<td>sum of 2, 3, 4, 5, 7, 9, 10, 11, or 12</td>
<td>(\frac{1}{11})</td>
<td>(\frac{10}{11})</td>
</tr>
</tbody>
</table>
SECTION 8.1  PROBABILITY: DEFINITIONS AND SINGLE-EVENT PROBABILITY

Use the following figure as needed for Exercises 17 and 18.

17. Two dice (red and green) are rolled. Find
   a. \( P(\text{not sum of } 7) \) \( \frac{2}{3} \)
   b. \( P(\text{not sum of } 1) \) 1
   c. \( P(\text{not sum of } 2, 3, 11, \text{ or } 12) \) \( \frac{5}{6} \)
   d. \( P(\text{sum of } 13) \) 0
   e. \( P(\text{sum of } 2 \text{ or sum of } 9) \) \( \frac{5}{36} \)

18. Two dice (red and green) are rolled. Find
   a. \( P(\text{not sum of } 8) \) \( \frac{1}{6} \)
   b. \( P(\text{sum greater than } 6) \) \( \frac{15}{36} = \frac{5}{12} \)
   c. \( P(\text{not sum of } 36) \) 1
   d. \( P(\text{sum of } 4 \text{ or sum of } 10) \) \( \frac{1}{6} \)
   e. \( P(\text{sum of } 1) \) 0

Use the following figure for Exercises 19 and 20. The figure shows a set of 28 dominoes, each marked with 0 to 6 dots on a side. There is only one domino that has zero dots on each side.

19. Suppose a domino is drawn at random. Find
   a. \( P(0 \text{ dots on at least } 1 \text{ side}) \) \( \frac{1}{4} \)
   b. \( P(6 \text{ dots on at least } 1 \text{ side}) \) \( \frac{1}{4} \)
   c. \( P(\text{sum of dots is } 6) \) \( \frac{1}{7} \)
   d. \( P(\text{sum of dots is even}) \) \( \frac{1}{2} \) (zero is an even number)
   e. \( P(\text{sum of dots is not a perfect square}) \) \( \frac{3}{7} \)
   f. \( P(\text{sum of dots is } 4 \text{ or } 5) \) \( \frac{1}{14} \)
   g. \( P(\text{sum of dots is } 13) \) 0

20. Suppose a domino is drawn at random. Find
   a. \( P(1 \text{ dot on at least } 1 \text{ side}) \) \( \frac{1}{4} \)

b. \( P(\text{sum of dots is } 7) \) \( \frac{1}{36} \)

c. \( P(\text{sum of dots is divisible by } 3) \)
   Note: Zero is divisible by 3.
   d. \( P(\text{sum of dots is } 5 \text{ or } 9) \) \( \frac{5}{36} \)
   e. \( P(7 \text{ dots on } 1 \text{ side}) \) 0
   f. \( P(\text{sum of dots is } 4 \text{ or } 5 \text{ or } 6) \) \( \frac{1}{14} \)
   g. \( P(\text{sum of dots is odd}) \) \( \frac{1}{2} \)

21. The experiment is being dealt a card from a standard 52-card deck with 13 cards in each suit: spades, hearts, clubs, and diamonds. Find the probability of
   a. a queen \( \frac{4}{52} \)
   b. a diamond \( \frac{13}{52} \)
   c. a card greater than 3 and less than 7 \( \frac{12}{52} \)
   d. a card that is not a face card \( \frac{40}{52} \)

22. The experiment is selecting one jelly bean at random from the bowl in the figure. Find the probability of
   a. selecting a green jelly bean \( \frac{4}{11} \)
   b. not selecting a red jelly bean \( \frac{8}{11} \)
   c. not selecting a white jelly bean \( \frac{7}{11} \)

Use the following figure for Exercises 19 and 20.

23. The experiment is tossing two six-sided dice, one green and one red. This time the outcome is the maximum of the two dice. For example, if you roll a 2 and 5, you take the 5.
   a. Construct a sample space of equally likely outcomes for this experiment. See Answer Section.
   b. Find the probability that you get a maximum value of 4. \( \frac{1}{6} \)
   c. Find the probability that you get a maximum value greater than 3. \( \frac{7}{36} \) or \( \frac{7}{12} \)
   d. Find the probability that you get a maximum value that is an even number. \( \frac{21}{36} \) or \( \frac{7}{12} \)

24. The experiment is tossing two six-sided dice, one green and one red. This time the outcome is the product of the dice. For example, if you roll a 2 and 3, the outcome is 6.
28. The Tournament of Roses Parade happens on New Year’s Day in Pasadena, California. The Rose Parade began in 1890, and until 2010 measurable rain had occurred on only 9 days. Based on historical data, what is the probability of measurable rain on the Rose Parade? \( \frac{9}{111} \)

29. Punxsutawney Phil is a resident of Punxsutawney, Pennsylvania. On 15 of the last 115 years, this groundhog (or an ancestor) has not seen his shadow on February 2. What is the empirical probability that Phil will see his shadow this coming February 2 and therefore predict six more weeks of winter for Pennsylvania? \( \frac{100}{115} = 87\% \)

30. Sarah tossed a single six-sided die and recorded her results in the following table.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of First 100 Tosses</td>
<td>12</td>
<td>21</td>
<td>18</td>
<td>16</td>
<td>22</td>
<td>11</td>
</tr>
<tr>
<td>Number of Next 900 Tosses</td>
<td>123</td>
<td>141</td>
<td>165</td>
<td>150</td>
<td>174</td>
<td>147</td>
</tr>
</tbody>
</table>

a. According to the table, what is the estimated probability for each of the six different outcomes after the first 100 rolls? \( P(1) = 0.12, P(2) = 0.21, P(3) = 0.18, P(4) = 0.16, P(5) = 0.22, P(6) = 0.11 \)
b. According to the table, what is the estimated probability for each of the six different outcomes after the first 1000 rolls? \( P(1) = 0.135, P(2) = 0.162, P(3) = 0.183, P(4) = 0.166, P(5) = 0.196, P(6) = 0.158 \)
c. Comment on how the law of large numbers relates to these results. Values after 1000 rolls are slightly closer to theoretical value of \( \frac{1}{6} = 0.167 \).

31. The experiment is selecting a marble from a bag of 5 marbles labeled A, B, C, D, and E.

a. What is the probability of selecting a marble labeled C? \( \frac{1}{5} = 0.20 \)
b. The experiment is run 100 times, with the outcomes listed in the table. What is the experimental probability of selecting each marble based on this table?

<table>
<thead>
<tr>
<th>Marble</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count</td>
<td>21</td>
<td>22</td>
<td>16</td>
<td>21</td>
<td>20</td>
</tr>
</tbody>
</table>

\( P(A) = 0.21, P(B) = 0.22, P(C) = 0.16, P(D) = 0.21, P(E) = 0.2 \)
c. The experiment is run 500 times, with the results shown below. What is the experimental probability of selecting each marble based on this table?
Section 8.2

Multiple-Event Probability

Objectives

- Use the addition principle to find probabilities involving overlapping events.
- Use the addition principle to find probabilities for mutually exclusive events.
- Use the multiplication principle to find probabilities for independent events.
- Use the multiplication principle to find probabilities for dependent events.

Warm-Up

1. Simplify the following fractions.
   a. \( \frac{10}{36} \) b. \( \frac{27}{50} \) c. \( \frac{22}{9} \) d. \( \frac{14}{36} \) e. \( \frac{26}{52} \)
   
   2. Perform the following calculations.
   a. \( \frac{2}{3} \) b. \( \frac{1}{2} \) c. \( \frac{1}{3} \)

36. Project: Dice Sums and Differences. Use the sample space for the rolling of two dice from Exercise 17. Organize your answers to parts a and b in one table. Organize your answers to parts c and d in another table.

   a. List the sums of the top faces on the pairs of dice.
   b. How many times does each sum appear?
   c. What are all the possible differences between the top faces on the two dice? Give the absolute values of the differences.
   d. How many times does each difference appear?

   e. Are all sums equally likely (do they each have the same chance of happening)? If not, what sum is most likely? least likely?

   f. Are all differences equally likely? If not, what difference is most likely? least likely?

   g. How are the outcomes (sums and differences) different?
   h. How are the outcomes alike?

   marble A B C D E
   count 105 104 86 96 109

   \( P(A) = 0.21, P(B) = 0.208, P(C) = 0.172, P(D) = 0.192, P(E) = 0.218 \)

   Comment on how the law of large numbers relates to these results. Larger experiment has slightly less variation from theoretical probability of 0.20.

32. Match the word, phrase, or expression to the probability. Choose from even chance, certainty, impossibility, \( 1 - P(A) \)

   a. \( P(A) = 1 \) certainty
   b. \( P(A) = 0 \) impossibility
   c. \( P(\text{not } A) = 1 - P(A) \)
   d. \( P(A) = 0.5 \) even chance

33. What are the domain and range for the probability function?

   - Domain: any event; any real number \( p \), where \( 0 \leq p \leq 1 \)
   - Range: any real number \( p \), where \( 0 \leq p \leq 1 \)

34. The probability of an outcome, \( P(n) \), is between 0 and 1, inclusive. Describe a new probability situation.

35. If \( a \) is one of \( n \) equally likely events, what is \( P(a) \) for \( n = 10,000 ? \)

32. Project: Dice Sums and Differences. Use the sample space for the rolling of two dice from Exercise 17. Organize your answers to parts a and b in one table. Organize your answers to parts c and d in another table.

   a. List the sums of the top faces on the pairs of dice.
   b. How many times does each sum appear?
   c. What are all the possible differences between the top faces on the two dice? Give the absolute values of the differences.
   d. How many times does each difference appear?

   e. Are all sums equally likely (do they each have the same chance of happening)? If not, what sum is most likely? least likely?

   f. Are all differences equally likely? If not, what difference is most likely? least likely?

   g. How are the outcomes (sums and differences) different?
   h. How are the outcomes alike?

In the popular board game of Yahtzee, players roll five six-sided dice. The most desired outcome (a Yahtzee) is for all five dice to show the same number. How likely is this to happen?

In 2010 and 2009, it snowed on December 25 in Nashville, Tennessee. What are the chances of snow on December 25 two years in a row in this southern city in the United States?
To answer these questions, you need to find the probability of two or more events occurring. In the first section of this chapter, we considered the probability of a single event (e.g., rolling a single die or spinning a single spinner). We need additional tools to help us determine probability of multiple events.

**PROBABILITIES INVOLVING A OR B**

**EXAMPLE 1** Finding probability

A standard deck of 52 cards has 4 suits (spades, hearts, diamonds, and clubs) of 13 cards each. The cards are numbered 1 to 10, jack, queen, king. The jack, queen, and king are known as face cards. One card is drawn at random. Find \( P(\text{heart or face card}) \).

**Solution**

In order to do this problem, you might count the number of hearts (13) and the number of face cards (12). It is tempting to add these numbers and then divide by 52 cards to get the probability.

However, the problem with this arithmetic is that we have counted the jack, queen, and king of hearts twice. The drawing in Figure 14 shows the overlap: the jack, queen, and king of hearts are in the set of hearts \( \{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\} \) and in the set of face cards \( \{\text{spades: } J, Q, K; \text{hearts: } J, Q, K; \text{clubs: } J, Q, K; \text{diamonds: } J, Q, K\} \). Counting the face cards twice is illustrated by the overlap in the circles in Figure 14, a drawing called a Venn diagram. Overlapping events are events that have one or more outcomes in common.

We add the hearts and the face cards and then, to avoid counting the jack, queen, and king twice, we subtract the number in the overlap. The number of hearts or face cards is 13 + 12 − 3 = 22, so

\[ P(\text{heart or face card}) = \frac{22}{52} = \frac{11}{26} \]

We summarize this idea using the addition principle for probability.

**Addition Principle for Probability**

Suppose the probability of event \( A \) is \( P(A) \) and the probability of event \( B \) is \( P(B) \). If \( A \) and \( B \) have outcomes in common, the probability of \( A \) or \( B \) is

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]
The more formal solution to Example 1, using the addition principle of probability, is

\[ P(\text{heart or face card}) = P(\text{heart}) + P(\text{face card}) \]
\[ - P(\text{heart and face card}) \quad \text{Substitute.} \]
\[ = \frac{1}{4} + \frac{12}{52} - \frac{3}{52} \quad \text{Change to common denominator.} \]
\[ = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \quad \text{Simplify.} \]
\[ = \frac{22}{52} \]
\[ = \frac{11}{26} \]

**EXAMPLE 2** Finding probability: student transportation Suppose there are 30 students in your mathematics class. Of these 30 students, 20 are female and 10 are male. Of the 30 students, 18 ride the bus to college. Eleven females ride the bus and 7 males ride the bus. If you select one student at random, what is the probability that you get a female student or a student who rides the bus?

**Solution** Using the addition principle for probability, we have

\[ P(\text{female or ride bus}) = P(\text{female}) + P(\text{ride bus}) - P(\text{female and ride bus}) \]
\[ = \frac{20}{30} + \frac{18}{30} - \frac{11}{30} \]
\[ = \frac{27}{30} \]

Not all events overlap, as illustrated in Example 3.

**EXAMPLE 3** Finding probability You draw one card from a deck of cards. What is the probability of drawing a heart or drawing a spade?

**Solution** Using the addition principle for probability, we have

\[ P(\text{heart or spade}) = P(\text{heart}) + P(\text{spade}) - P(\text{heart and spade}) \]

As we substitute for each part, we realize that it is impossible to draw one card from the deck and have it be both a heart and a spade. \( P(\text{heart and spade}) = 0 \). Thus,

\[ P(\text{heart or spade}) = \frac{13}{52} + \frac{13}{52} - 0 \]
\[ = \frac{1}{4} + \frac{1}{4} \]
\[ = \frac{1}{2} \]

### MUTUALLY EXCLUSIVE EVENTS

*Two events that cannot happen at the same time* are **mutually exclusive**. The events of drawing a heart and drawing a spade in Example 3 are mutually exclusive events. We modify the addition principle for probability to include mutually exclusive events.

**Addition Principle for Probability**

Suppose the probability of event \( A \) is \( P(A) \) and the probability of event \( B \) is \( P(B) \). If \( A \) and \( B \) have outcomes in common, then

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

If \( A \) and \( B \) are mutually exclusive, then

\[ P(A \text{ or } B) = P(A) + P(B). \]
The Venn diagrams in Figure 15 illustrate the addition principle for events that are not mutually exclusive and for events that are mutually exclusive.

\[ P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \]

\[ P(A \text{ or } B) = P(A) + P(B) \]

**FIGURE 15** The addition principle for probability.

**PRACTICE 1** Indicate whether the following pairs of events are mutually exclusive or not mutually exclusive. If they are not mutually exclusive, describe the overlap.

**a.** Experiment: Rolling two six-sided dice and observing the sum of the top faces.
   
   Event 1: Getting a sum of 7.
   
   Event 2: Getting a sum of 2. **mutually exclusive**

**b.** Experiment: Rolling two six-sided dice (one red and one green) and observing the sum of the top faces.
   
   Event 1: Getting a sum of 7.
   
   Event 2: Getting exactly one 2 on the roll of the dice. **not mutually exclusive; the overlap is sums of 7 that contain a 2**

**c.** Experiment: Drawing one card from a standard deck of cards.
   
   Event 1: Getting a red card.
   
   Event 2: Getting a 5. **not mutually exclusive; the overlap is the 5 of hearts and the 5 of diamonds**

**d.** Experiment: Drawing one card from a standard deck of cards.
   
   Event 1: Getting an even-number card (2, 4, 6, 8, 10).
   
   Event 2: Getting a face card. **mutually exclusive**

**PRACTICE 2** Suppose you roll two dice and observe the sum of the top faces. (See Figure 4, p. 577, as needed.) Determine the probability of

**a.** Rolling a sum of 7 or rolling a sum of 5. \( \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9} \)

**b.** Rolling a sum of 7 or getting exactly one 2 on the two dice. \( \frac{6}{36} + \frac{2}{36} - \frac{1}{36} = \frac{7}{36} \)

**PROBABILITIES INVOLVING A AND B**

**EXAMPLE 4** Finding probability You are playing a carnival game that involves spinning the spinners in Figure 16. The top prize is awarded to anyone who gets a 2 on the first spinner and a D on the second spinner, as shown. What is the probability of this happening?

**FIGURE 16** Two spinners.
The tree diagram in Figure 17 shows 12 different outcomes. Only one outcome has a 2 on the first spinner and a D on the second spinner. So the probability of getting a 2 on the first spinner and a D on the second spinner is one of the 12 outcomes shown at the base of the tree:

\[ P(2 \text{ and } D) = \frac{1}{12}. \]

**Independent Events** When the number of different outcomes is small, a tree diagram is a good way to list all possible outcomes for event \( A \) followed by event \( B \). The tree permits us to count outcomes and state the probability, \( P(A \text{ and } B) \). The diagrams are called tree diagrams because of the branching from one step to the next. However, we can find probabilities for these events without listing all the options. The spinners give us a hint. The probability of spinning a 2 on the first spinner is \( \frac{1}{3} \), and the probability of spinning a D on the second spinner is \( \frac{1}{4} \), and, in agreement with the tree diagram,

\[ \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}. \]

This process works only if the events are independent.

Two events are independent if the outcome of the first event does not affect the probability of the second event. Two events are dependent if the outcome of the first event does affect the probability of the second event.

**Dependent Events** The outcome on the first spinner in Example 4 has no bearing on the probability of the outcome on the second spinner. The outcomes on the two spinners are independent events. In Example 5, the probability of the second event depends on the outcome of the first event.

**EXAMPLE 5** Finding probability: dependent events

You draw two cards from a standard deck of cards one at a time, without replacing the first card after you draw it. What is the probability of getting a six on the second card?

**Solution** The probability is dependent on what card you drew first. If you drew a six on the first card, then there are 3 sixes left among the 51 cards remaining in the deck, so \( P(\text{six on second card}) = \frac{3}{51} \). If you did not draw a six on the first card, then there are 4 sixes left among the 51 cards remaining in the deck, so \( P(\text{six on second card}) = \frac{4}{51} \).

Because the outcome of the first event determines the probability of the second event, these events are dependent.
We need special notation to help us differentiate between the two possibilities in Example 5.

**Conditional probability** is the probability of \( B \) given the condition \( A \). The notation for conditional probability is \( P(B|A) \), read “probability of \( B \) given \( A \).” For Example 5, we would write

\[
P(\text{six on second card} \mid \text{six on first card}) = \frac{3}{51} \\
P(\text{six on second card} \mid \text{not six on first card}) = \frac{4}{51}
\]

**Practice 3** Determine if the following pairs of events are dependent or independent.

a. Experiment: Playing two rounds of Roulette.
   Event 1: The ball lands in pocket 17 on the first round.
   Event 2: The ball lands in a black pocket on the second round. independent

b. Experiment: Drawing two marbles one at a time, without replacement, from a bag of marbles with 4 blue marbles, 3 white marbles, and 8 red marbles.
   Event 1: Selecting a red marble on the first draw.
   Event 2: Selecting a red marble on the second draw. dependent

**Multiplication Principle for Probability** Let the probability of event \( A \) be \( P(A) \), the probability of event \( B \) be \( P(B) \), and the probability of \( B \) given \( A \) be \( P(B|A) \).

If \( A \) and \( B \) are independent events, then \( P(A \text{ and } B) = P(A) \times P(B) \).

If \( A \) and \( B \) are dependent events, then \( P(A \text{ and } B) = P(A) \times P(B|A) \).

**Example 6** Finding probability: \( A \) and \( B \) You draw one ball from box 1 and one ball from box 2 in Figure 18. You observe the color of each ball.

a. What is the sample space for this experiment?

b. What is the probability that both balls are blue?

c. What is the probability that both balls are white?

**Solution**

a. The sample space for this experiment is two white balls, two blue balls, and one ball of each color.

b. These events are independent, so

\[
P(\text{two blue balls}) = P(\text{blue ball from box 1 and blue ball from box 2}) \\
= P(\text{blue ball from box 1}) \times P(\text{blue ball from box 2}) \\
= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}
\]
c. Once again, these events are independent, so
\[
P(\text{two white balls}) = P(\text{white ball from box 1 and white ball from box 2})
\]
\[
= P(\text{white ball from box 1}) \times P(\text{white ball from box 2})
\]
\[
= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}
\]

**EXAMPLE 7** Finding probability: A and B  From the box of marbles in Figure 19, we are to select two marbles one at a time, without replacement. What is the probability both marbles will be red?

**Solution** The probability for the second marble is dependent on the color of the first marble drawn, so

\[
P(\text{both marbles are red}) = P(\text{first marble is red}) \times P(\text{second marble is red | first marble is red})
\]
\[
= \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}
\]

**Reading Questions**

1. If two events A and B can occur simultaneously, then the events are __________ events. overlapping
2. If two events A and B cannot occur simultaneously, then the events are __________ events. mutually exclusive
3. If the outcome of event A affects the probability of event B, the two events are known as __________ events. dependent
4. a. Without replacement means that when we draw a marble from a bag or box, we [do, do not] return the marble before drawing a second time.  
   b. Without replacement means that the second event is __________ the first event. dependent
5. If the outcome of event A does not affect the probability of event B, the two events are known as __________ events. independent
6. The symbol | in \(P(A|B)\) is read __________, given

**ACTIVITY**

**Sum and Product of Dice Probability.** For this Activity, you will need three six-sided fair dice (or use the ProbSim dice app on the TI-84). Work in pairs.

1. **Sum of Three Dice.** In your pair, choose who will be player 1 and who will be player 2.
   a. Roll all three dice at the same time. Find the sum of the numbers on the three dice. If the sum is odd, player 1 wins. If the sum is even, player 2 wins. Repeat rolling the dice ten times. Record the results in a table like Table 5. Answers will vary, but results are likely to be evenly distributed.
**TABLE 5** Sums of Rolling Three Dice

<table>
<thead>
<tr>
<th>Roll Number</th>
<th>Sum</th>
<th>Who Wins?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Player 1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
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<tr>
<td>5</td>
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<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
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<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. A game is fair if the probability of winning is the same for all players. Does this game seem fair from your results? Again answers will vary, but game is fair.

c. Find the probability that the sum is odd on the roll of the three dice. Find the probability that the sum is even on the roll of the three dice. *(Hint: An odd number plus an odd number is even. An even number plus an even number is even. An odd plus an even is odd.)* What do you notice? 
\[ P(\text{sum is even}) = \frac{1}{2}, P(\text{sum is odd}) = \frac{1}{2}. \] The probabilities are equal, so the game is fair.

**2. Product of Three Dice.** Now revise the rules of the game. Instead of adding the numbers on the three dice, multiply them. Player 1 wins if the product of the three dice is *odd*. Player 2 wins if the product of the three dice is *even*.

a. Play this game ten times. Record the results in a table like Table 6.

**TABLE 6** Products of Rolling Three Dice

<table>
<thead>
<tr>
<th>Roll Number</th>
<th>Product</th>
<th>Who Wins?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Player 1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Do your results suggest that this game is fair? Again answers will vary, but game is unfair.

c. Find the probability that the product of the three dice is even. Find the probability that the product of the three dice is odd. Is this a fair game? 
\[ P(\text{product is odd}) = P(\text{odd first die}) \cdot P(\text{odd second die}) \cdot P(\text{odd third die}) = \frac{1}{8}, P(\text{product is even}) = \frac{7}{8}. \] The game is not fair.
In Exercises 1 to 4, doubles is having the same number face up on both dice. Avoid counting to find answers from the sample space. Use probability and the multiplication and addition principles whenever possible.

### 1. For the roll of two dice (red and green), find
- a. \( P(3) \) (Hint: This is a 3 on the first die or a 3 on the second.) \( \frac{11}{36} \)
- b. \( P(2) \) \( \frac{11}{36} \)
- c. \( P(2 \text{ and } 3) \) \( \frac{1}{18} \)
- d. \( P(2 \text{ or } 3) \) \( \frac{13}{18} \)
- e. \( P(\text{doubles or } 3) \) \( \frac{7}{12} \)

### 2. For the roll of two dice (red and green), find
- a. \( P(6) \) \( \frac{11}{36} \)
- b. \( P(1) \) \( \frac{11}{36} \)
- c. \( P(1 \text{ and } 6) \) \( \frac{1}{18} \)
- d. \( P(1 \text{ or } 6) \) \( \frac{5}{9} \)
- e. \( P(\text{at least 1 even number}) \) \( \frac{27}{36} \)

### 3. For the roll of two dice (red and green), find
- a. \( P(\text{sum of } 2 \text{ or sum of } 3) \) \( \frac{1}{12} \)
- b. \( P(\text{doubles or sum of } 7) \) \( \frac{1}{6} \)
- c. \( P(\text{doubles or sum of } 6) \) \( \frac{5}{18} \)
- d. \( P(3 \text{ or sum of } 8) \) \( \frac{5}{12} \)

### 4. For the roll of two dice (red and green), find
- a. \( P(\text{sum of } 3 \text{ or sum of } 4) \) \( \frac{5}{18} \)
- b. \( P(\text{sum of } 3 \text{ and sum of } 4) \) \( 0 \)
- c. \( P(\text{doubles or sum of } 8) \) \( \frac{5}{18} \)
- d. \( P(\text{doubles or sum of } 11) \) \( \frac{1}{6} \)

### 5. A card is pulled at random from a standard deck of 52 cards. Find
- a. \( P(\text{diamond or face card}) \) \( \frac{22}{52} = \frac{11}{26} \)
- b. \( P(9 \text{ or club}) \) \( \frac{18}{52} = \frac{9}{26} \)
- c. \( P(\text{diamond or heart}) \) \( \frac{1}{2} \)
- d. \( P(\text{club or } 3, 4, 5, \text{ or } 6) \) \( \frac{25}{52} \)

### 6. A card is pulled at random from a standard deck of 52 cards. Find
- a. \( P(8 \text{ or a heart}) \) \( \frac{4}{13} \)
- b. \( P(\text{face card or heart}) \) \( \frac{11}{26} \)
- c. \( P(5, 4, 3, \text{ or } 2 \text{ or diamond}) \) \( \frac{25}{52} \)
- d. \( P(\text{spade or club}) \) \( \frac{1}{2} \)

### 7. If event \( A \) and event \( B \) are mutually exclusive (no overlap or intersection), what is \( P(A \text{ and } B) \)? \( 0 \)

### 8. Tell whether the multiplication principle for probability (MP), the addition principle for probability (AP), or neither applies, and find the following:
- a. The probability of hot or cold water in turning on a faucet at a working sink \( \text{AP; } \frac{1}{2} \)
- b. The probability of a boy followed by a boy in the birth of 2 children \( \text{MP; } \frac{1}{4} \)
- c. The probability of a 1 or a 2 in rolling a die \( \text{AP; } \frac{1}{3} \)
- d. The probability of a 3 and a head in rolling a die and then flipping a coin \( \text{MP; } \frac{1}{12} \)
- e. The probability of a 3 or a head in rolling a die and then flipping a coin (Hint: Check with a tree diagram.) \( \text{AP; } \frac{7}{12} \)

### 9. Suppose we flip a coin and then spin a spinner with regions A, B, and C as shown in the figure. Regions A and B are each a fourth of the circle, and region C is half the circle.

- a. Draw a tree diagram of the outcomes. **See Answer Section.**
- b. What are the six possible outcomes and their probabilities? HA: \( \frac{1}{8} \); HB: \( \frac{1}{8} \); HC: \( \frac{1}{4} \); TA: \( \frac{1}{8} \); TB: \( \frac{1}{8} \); TC: \( \frac{1}{4} \)
- c. What is the probability of a tail? \( \frac{1}{2} \)
- d. What is the probability of a tail or a B? \( \frac{5}{8} \)
- e. What is the probability of a B or a C? \( \frac{3}{4} \)
- f. What is the probability of a B and a head? \( \frac{1}{8} \)
10. Draw a tree diagram for gender of a child (girl or boy) followed by color of eyes (pretend the chances for blue, green, grey, and brown eyes are equal). Label the probability on each branch and for each outcome. Find See Additional Answers.
   a. \(P(\text{girl with brown eyes})\)  \(\frac{1}{4}\)
   b. \(P(\text{blue eyes})\)  \(\frac{1}{4}\)
   c. \(P(\text{boy with either blue or brown eyes})\)  \(\frac{1}{4}\)

11. A kitchen drawer contains 6 table knives, 4 forks, and 8 spoons. Assume that once an item is removed, it is not replaced.
   a. Draw a tree diagram for randomly selecting two utensils. See Answer Section.
   b. Find \(P(\text{spoon and fork})\). \(\frac{24}{1110}\) or \(\frac{8}{365}\)
   c. Find \(P(\text{knife and fork})\). \(\frac{24}{1110}\) or \(\frac{8}{365}\)
   d. Find \(P(2 \text{ spoons})\). \(\frac{8}{18} \times \frac{7}{17} = \frac{56}{153}\) or \(\frac{8}{365}\)
   e. Find \(P(2 \text{ forks})\). \(\frac{12}{180} \times \frac{11}{168} = \frac{1}{14}\)

12. One Halloween pack of M&Ms contained 9 candies: 5 browns, 3 yellows, and 1 green. One is taken at random and eaten; then another is removed and eaten.
   a. Draw a tree diagram of the outcomes. See Additional Answers.
   b. Find \(P(2 \text{ browns})\). \(\frac{20}{72} = \frac{5}{18}\)
   c. Find \(P(2 \text{ greens})\). \(\frac{10}{72} = \frac{5}{36}\)
   d. Find \(P(\text{at least 1 yellow})\). \(\frac{7}{12}\)
   e. Find \(P(\text{at least 1 brown or a brown and a yellow})\). \(\frac{60}{72} = \frac{5}{6}\)

13. In the following settings, state whether the given event is dependent or independent.
   a. Rolling 4 threes in succession with 4 rolls of fair die independent
   b. Selecting all women for a 6-person jury from a pool of 10 men and 10 women dependent
   c. The next 5 births in a hospital all being boys independent
   d. Selecting 4 aces in succession from a deck of 52 cards dependent
   e. Tossing 4 heads and then 3 tails when tossing a fair coin 7 times independent

14. Suppose Cushman returns to the motor-scooter market with 100,000 new cycles expected to better their 157 miles per gallon standard. They offer five colors (red, black, blue, green, and white), available in equal numbers with a step-through frame and a step-over frame. Of the new scooters, 75,000 have the step-through frame.
   a. What is the probability that a scooter is red?  \(\frac{1}{4}\)
   b. What is the probability that a scooter is either green or blue?  \(\frac{2}{5}\)
   c. What is the probability that a scooter is red or has a step-over frame?  \(\frac{1}{4} + \frac{1}{4} - \frac{1}{5} = \frac{3}{5}\)
   d. What is the probability that a scooter is white or has a step-over frame?  \(\frac{1}{5} + \frac{1}{4} - \frac{1}{5} = \frac{1}{4}\)
   e. Suppose demand is so high that sales are made by random drawing from the names of those submitting requests to buy. The chance of winning an opportunity to buy is estimated at 5%. What is the probability of winning an opportunity to buy a black scooter? a scooter with a step-over frame?

15. The experiment is to roll the octahedral die (an eight-sided die with numbers 1–8) and spin the spinner in the figure.

16. Suppose you roll two dice and make a proper fraction from the outcomes. For example, rolling a 4 and a 2 makes the fraction \(\frac{2}{4}\). State any assumptions.
   a. Make a table for the outcomes, with 1 to 6 across the top and rows labeled 1 to 6. See Additional Answers.
   b. What is the probability of getting a fraction in simplified form? \(\frac{36}{36}\), assume \(\frac{1}{6}\) is not simplified.
   c. What is the probability of getting a fraction equivalent to \(\frac{1}{2}\)? \(\frac{6}{36} = \frac{1}{6}\)
   d. What is the probability of getting a fraction with a numerator of 2 or a denominator of 3? \(\frac{18}{36} = \frac{1}{2}\)
   e. What is the probability of getting a fraction with a denominator of 6 and a numerator of 4? \(\frac{24}{36} = \frac{1}{18}\). Assume a count before simplifying.
17. Consider the two spinners in the figure. You spin the first spinner and then spin the second.

![Spinners](image)

a. Are the events of spinning spinner 1 and then spinning spinner 2 independent or dependent? Why? independent; outcome of spinner 1 does not affect outcome of spinner 2.

b. What is the probability that you get an odd number on both spinners? \( P(\text{odd on first}) \times P(\text{odd on second}) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64} \)

c. What is the probability that you get an even number on both spinners? \( P(\text{even on first}) \times P(\text{even on second}) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64} \)

18. You selected, at random and without replacement, two jelly beans from the bowl in the figure (You ate them!)

![Jelly Beans](image)

a. Are the events of selecting the first jelly bean and then selecting the second jelly bean independent or dependent? Why? dependent; contents of bowl has changed.

b. What is the probability that both jelly beans are green? \( \frac{4}{17} \times \frac{3}{16} = \frac{3}{56} \)

c. What is the probability that both jelly beans are white? \( \frac{3}{17} \times \frac{2}{16} = \frac{1}{56} \)

19. You select, at random and without replacement, three jelly beans from the bowl in the figure.

![Jelly Beans](image)

a. Are the events of selecting the first jelly bean and then selecting the second jelly bean dependent or independent? Why? dependent; contents of bowl has changed.

b. What is the probability that all three jelly beans are white? \( \frac{3}{17} \times \frac{2}{16} \times \frac{1}{15} = \frac{1}{56} \)

c. What is the probability that all three jelly beans are green? \( \frac{4}{17} \times \frac{3}{16} \times \frac{2}{15} = \frac{1}{56} \)

20. Of the 40 students in the bicycle club at school, 22 are female and 18 are male. Every student owns only one bike, either a mountain bike or a road bike. Of the mountain bike owners, 15 of the 27 are female and 12 are male.

a. A student is selected at random from the club. What is the probability that the student either is a female or owns a mountain bike? \( \frac{22}{40} + \frac{15}{40} - \frac{12}{40} = \frac{35}{40} \)

b. Another student is selected at random from the 40 members. What is the probability that the student is male or owns a road bike? \( \frac{18}{40} + \frac{23}{40} - \frac{12}{40} = \frac{49}{40} \)

21. Project: News Story. Find a recent news story in which two or more probabilities were combined in some way. Describe how and why they were combined.

22. Project: Simulations

a. In drawing from a standard deck of cards, what is the probability of getting two face cards in succession? Assume that you do not replace the first card after selection. \( \approx 0.05 \)

b. Simulate this situation using a standard deck or using the probability simulation application on your TI-84. Do the simulation 100 times. Record your results and calculate the experimental probability for getting two face cards in a row. How does your result compare to the theoretical result in part a?

c. What could you do to improve the experimental result? Do experiment more times.
8.3 Using Counting Principles

OBJECTIVES
- Use the fundamental counting principle to solve counting problems.
- Define, evaluate, and apply factorials.
- Apply permutations and combinations in counting.

WARM-UP
Evaluate these expressions.
1. \(10 \cdot 9 \cdot 8 \cdot 7\) \(= 5040\)
2. \(3 \cdot 2 \cdot 1\) \(= 6\)
3. \(4 \cdot 3 \cdot 2 \cdot 1\) \(= 24\)
4. \(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\) \(= 120\)
5. What shortcut might you use to multiply \(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\)?

THE POWERBALL LOTTERY is offered in 42 states, as well as Washington, D.C., and the Virgin Islands. In this lottery, players select five different numbers from 1 to 59 and one special number, the Powerball, from 1 to 39. To win the jackpot, a player must have the first five numbers correct and the Powerball correct. The order of the first five numbers does not have to match the order in which they are selected when the balls are drawn, but the Powerball must be the special number that the player selected. How many different lottery outcomes are there? What is the probability that you will win the jackpot? The techniques of this section will answer these questions.

FUNDAMENTAL COUNTING PRINCIPLE

EXAMPLE 1 Listing outcomes  You are late getting ready to leave for class one morning. Your closet contains two pairs of clean pants (a blue pair and a black pair) and three clean shirts (a white one, a red one, and a yellow one). Being in a hurry, you randomly select one pair of pants and one shirt. On the way to class, you wonder how many different selections you might have worn.

Solution The tree diagram in Figure 20 shows the selections.
By tracing out each branch of the tree in Figure 20, you find six different selections of clothing.

The tree diagram suggests a general way of thinking about the clothing selections. There are 2 different color pants and 3 different color shirts, and $2 \times 3 = 6$. This example illustrates what is known as the fundamental counting principle.

**The Fundamental Counting Principle**

If you can do item 1 in $a$ ways and you can do item 2 in $b$ ways, then you can do item 1 followed by item 2 in $a \times b$ ways. This principle can be extended to three or more items.

**EXAMPLE 2**  **Counting outcomes**  In 15 different states, license plates for automobiles begin with three digits (0 through 9) followed by three letters. Among license plate collectors, this is known as style 123 ABC. How many different license plates are possible?

**Solution**  Because it is permissible to have a license plate of 555 XXY, repetition of numbers and letters is allowed. Because of the number of options involved, a tree diagram is too cumbersome, so we will use the multiplication principle. We draw a line for each position on the license plate:

<table>
<thead>
<tr>
<th>Digit 1</th>
<th>Digit 2</th>
<th>Digit 3</th>
<th>Letter 1</th>
<th>Letter 2</th>
<th>Letter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10</td>
<td>10</td>
<td>26</td>
<td>26</td>
<td>26</td>
</tr>
</tbody>
</table>

There are 10 choices for the first digit, so we place 10 on the first line.

For each of the 10 choices for digit 1, there are 10 choices for digit 2 (repetition is allowed), so we have

$$10 \times 10 = 100$$

We extend this idea to digit 3:

$$10 \times 10 \times 10 = 1,000$$

In the standard American alphabet, there are 26 choices for letter 1:

$$10 \times 10 \times 26 = 2,600$$

We extend this idea to letter 2 and letter 3 (again, repetition is allowed):

$$10 \times 10 \times 26 \times 26 = 676,000$$

The total number of outcomes is $10 \times 10 \times 10 \times 26 \times 26 \times 26$, or 17,576,000 different license plates.

**EXAMPLE 3**  **Counting outcomes**  How many license plates of the style 123 ABC in Example 2 do not have repeated digits or letters?

**Solution**  We still have 10 choices for digit 1, but because we have no repetition and we have used 1 digit as digit 1, we have only 9 choices for digit 2. Similarly, we have 8 choices...
for digit 3. We have 26 choices for letter 1, but only 25 choices for letter 2, and 24 choices for letter 3. The final answer is

\[
\frac{10}{\text{Digit 1}} \times \frac{9}{\text{Digit 2}} \times \frac{8}{\text{Digit 3}} \times \frac{26}{\text{Letter 1}} \times \frac{25}{\text{Letter 2}} \times \frac{24}{\text{Letter 3}}
\]

The total number of outcomes is \(10 \times 9 \times 8 \times 26 \times 25 \times 24\), or 11,232,000 different license plates.

An informal name for a set or sequence of numbers is \textit{combinations}. One source of this word is a combination lock (Figure 21), a lock with number inputs. After Example 8, look for a mathematical definition of \textit{combination}.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{combination_lock.png}
\caption{Locker combination lock.}
\end{figure}

\textbf{PRACTICE 1} Opening the combination lock in Figure 21 takes three numbers from 0 to 49. Assume that the numbers must be different.

\begin{enumerate}
\item How many different “combinations” are there for the lock? \(50 \times 49 \times 48 = 117,600\)
\item What is the probability of someone randomly selecting your combination? \(\frac{1}{117,600}\), or \(0.0000085\)
\end{enumerate}

\textbf{FACTORIALS}

\textbf{EXAMPLE 4} Counting outcomes You have taken 8 books down from a small shelf in your kitchen in order to look for a recipe. Assume that you replace the books in standing position with the bindings facing into the room. In how many ways can you put the books back on the shelf?

\textbf{Solution} You have 8 choices for the first book you place on the shelf. You have only 7 choices for the second book, 6 choices for the third book, and so on. With the fundamental counting principle, you get

\[8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320\] ways

The product \(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1\) can be written as a factorial: \(8!\)

\begin{center}
\textbf{Factorial}
\end{center}

If \(n\) is a positive integer, then \(n\) factorial is defined as

\[n! = n(n - 1)(n - 2)(n - 3) \cdots 3 \cdot 2 \cdot 1\]

If \(n = 1\), then \(1! = 1\). If \(n = 0\), then \(0! = 1\).
Look for factorial on your calculator. On the TI-84, factorial is under the PRB option in the MATH menu. Press (MATH) for the menu in Figure 22, and then use the left cursor arrow to highlight PRB to get the PRB menu in Figure 23. Factorial is option 4 and is placed after a number already on the home screen.

**EXAMPLE 5**  Simplifying a factorial expression  
Evaluate \( \frac{1002!}{1000!} \).

**Solution**  If you attempt this problem on a calculator, you will get an overflow error. There is no error. The calculator has tried to evaluate 1002!, and the result is too large a number to be stored on the calculator. However, if you write some of the factors, the expression simplifies.

\[
\frac{1002!}{1000!} = \frac{1002 \times 1001 \times 1000 \times 999 \times \cdots \times 3 \times 2 \times 1}{1000 \times 999 \times \cdots \times 3 \times 2 \times 1}
\]
\[
= 1002 \times 1001
\]
\[
= 1,003,002
\]

**PRACTICE 2**  Evaluate.

a. \( \frac{576!}{578!} \)  
\[
\frac{1}{578 \times 577} = \frac{1}{333,506}
\]

b. \( \frac{48!}{(48 - 1)!} \)  
\[
\frac{48!}{47!} = 48
\]

**PERMUTATIONS AND COMBINATIONS**

**EXAMPLE 6**  Counting outcomes  In your math class, there are 30 seats and 25 students. How many different seating arrangements are possible?

**Solution**  The fundamental counting principle provides one way to do this problem. The first student in the room has 30 choices for a seat, the second student has 29 choices for a seat, the third has 28 choices, and so on, for each of the 25 students. The choices are listed as a product:

\[
30 \times 29 \times 28 \times 27 \times 26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6
\]

The answer is about \( 2.21 \times 10^{30} \) ways.

**Permutation**  A second way to do Example 6 is with a permutation. This approach is less cumbersome than listing the factors with the fundamental counting principle.
Permutation

If you choose \( r \) items from \( n \) items without replacement and the order of selection is important, you have a permutation. The formula for the number of permutations is

\[
\text{P}_r = \frac{n!}{(n-r)!}
\]

In Example 6, the 25 students selected 25 seats out of the group of 30 seats in a setting where the order of selection was important. We can use the permutation formula to choose 25 items from 30 items:

\[
_{30}P_{25} = \frac{30!}{(30 - 25)!} = \frac{30!}{5!} = 2.21 \times 10^{30}
\]

**EXAMPLE 7** Applying the permutation formula

The math club has 20 members. In how many ways can they choose a president, vice-president, and secretary? Assume each member can hold only one office.

**Solution**

You can use the permutation formula because you are selecting 3 items from the group of 20 and the order of selection is important. (A slate of officers with Joe as president, Carlos as VP, and Ahn as secretary is different from Ahn as president, Joe as VP, and Carlos as secretary.)

\[
_{20}P_3 = \frac{20!}{(20 - 3)!} = \frac{20!}{17!} = 6840 \text{ ways}
\]

**PRACTICE 3**

Twelve students apply for a set of community service awards. The top student selected will receive $10,000, the second $5000, the third $1000, and the fourth $500. In how many ways can the awards be given to 4 of these 12 students?

The order of selection is not always important. If the total prize money in Practice 3, $16,500, were divided equally and given to the top 4 students, then the order of selection would not matter. The total number of outcomes would decrease. In Example 8, we see by how much the number of outcomes decreases when order of selection does not matter.

**EXAMPLE 8** Counting outcomes

The Pizza Palace has 15 different toppings to choose from. You want a large pizza with 3 different toppings. How many different outcomes are there?

**Solution**

Suppose the pizza is to have pepperoni, mushrooms, and green peppers. Whether we start with the fundamental counting principle, \(15 \times 14 \times 13 = 2730\), or a permutation, \( _{15}P_3 = \frac{15!}{(15 - 3)!} = \frac{15!}{12!} = 15 \times 14 \times 13 = 2730\), our result is too large, because both consider as different possibilities all 6 of these outcomes:

- Pepperoni first, mushrooms second, and green peppers third
- Pepperoni first, green peppers second, and mushrooms third
- Mushrooms first, green peppers second, and pepperoni third
- Mushrooms first, pepperoni second, and green peppers third
- Green peppers first, mushrooms second, and pepperoni third
- Green peppers first, pepperoni second, and mushrooms third

However, to the Pizza Palace these six outcomes are the same pizza. Likewise, each distinct pizza has been counted six times. Thus, the fundamental counting principle and permutations give us results that are \(3! = 6\) times too large. Dividing 2730 by 6 gives the correct answer, 455.
The formula based on the results in Example 8 is the formula for combinations.

**Combination**

If you choose \( r \) items from \( n \) items without replacement and the order of selection is not important, you have a combination. The formula for the number of combinations is

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]

**EXAMPLE 9**

**Applying the combination formula**

Apply the combination formula to the situation in Example 8, where you select 3 toppings from the group of 15 toppings and the order in which these toppings are selected (and put on the pizza) is not important.

**Solution**

The combination formula gives

\[
\binom{15}{3} = \frac{15!}{3!(15-3)!} = \frac{15!}{3!12!} = 455 \text{ ways}
\]

Sets of toppings, committee members, playing cards, and unranked job finalists are all examples of combinations because the order does not matter.

**THE FUNDAMENTAL COUNTING PRINCIPLE WITH COMBINATIONS AND PERMUTATIONS**

In some applications, we use a product of two or more combinations or permutations to find the number of outcomes. This involves applying the fundamental counting principle with combinations or permutations. In Example 10, we use combinations. In a later Practice, we use a product of permutations.

**EXAMPLE 10**

**Counting outcomes**

If we deal 5 cards from a standard deck of cards, in how many ways can we get four of a kind?

**Solution**

First, we simplify the problem to finding the number of five-card hands with 4 aces. This is done in steps 1–4 below. Since the number is the same for each of the other 12 cards (2–10, jack, queen, king), we multiply the number of five-card hands with 4 aces by 13 to get the total number of five-card hands containing four of a kind. This is done in step 5 below.

**Step 1:** We are dealt five cards. We want four of them to be aces and one to be another card in the deck. So, using the fundamental counting principle, the problem is

(number of ways to get 4 aces) \( \times \) (number of ways to get 1 other card)

**Step 2:** We need to find the number of ways to get 4 aces. The order in which the cards are dealt is *not* important. (Rearranging the cards does not change anything.) Because order is not important, we use a combination, not a permutation. There are 4 aces in the deck and we want all four. The number of ways this can happen is \( \binom{4}{4} \).

**Step 3:** Now we need to find the number of ways to get 1 other card. There are 48 non-aces in the deck, and we will receive one of them. The number of ways this can happen is \( \binom{48}{1} \).

**Step 4:** We substitute the results of steps 2 and 3 into the fundamental counting principle in step 1:

\[
= \binom{4}{4} \binom{48}{1} = \frac{4!}{4!(0!)} \cdot \frac{48!}{1!(48 - 1)!} = 1 \cdot \frac{48!}{1 \cdot 47!} = 1 \cdot 48 = 48
\]

There are 48 different five-card hands with 4 aces.

**Step 5:** Now we multiply the 48 different five-card hands with 4 aces by 13 to obtain the total number of hands: 48 \( \times \) 13 = 624 ways.
How many ways can you be dealt a “full house” of 3 kings and 2 jacks in a five-card hand? \[
\binom{4}{3} \binom{4}{2} \frac{5!}{3!(4 - 3)!} \cdot \frac{4!}{2!(4 - 2)!} = 24
\]
The number of outcomes from permutations and combinations is large in many settings, and a calculator is useful. Two practice problems will follow the calculator technique.

**Graphing Calculator Technique: Permutations and Combinations**

To calculate \(nPr = nP_r\), enter 6, the value of \(n\). Then press [MATH] and use the left arrow to highlight PRB (Figure 24). Select option 2 and enter 3 for \(r\) (Figure 25). Press [ENTER] to obtain the result, 120.

**Practice 4**

A 4-person committee is to be selected from a group of 20 to fly to Tennessee and do background research on a candidate for the job of local school superintendent. In how many ways can the committee of 4 be selected? \(\binom{20}{4} = 4845\)

**Practice 5**

In Example 2, we found the number of license plates with the style 123 ABC. The order of the letters and numbers was important. Which would be appropriate to find the number of arrangements of numbers (and of letters): a combination or a permutation? Use the fundamental counting principle with the appropriate formula (combination or permutation) to find the total number of license plates. With order being important, we use a permutation; \((nPr)(nPr) = 11,232,000\)

**Answer Box**

- Warm-up: 1. 5040 2. 6 3. 24 4. 120 5. 6 times the answer to Exercise 4
- Practice 1: a. \(\frac{1}{117,600}\) or \(0.0000085\) b. \(\frac{48}{578 \times 577} = 0.0000106\)
- Practice 2: a. \(\frac{1}{117,600}\) b. \(\frac{48}{578 \times 577} = 0.0000106\)
- Practice 3: \(^{12}P_4 = 11,880\) Practice 4: \((\frac{4!}{3!(4 - 3)!}) \cdot \frac{4!}{2!(4 - 2)!} = (4)! = 24\) Practice 5: \(^{20}C_4 = 4845\)
- Practice 6: With order being important, we use a permutation; \((nPr)(nPr) = 11,232,000\)

**Reading Questions**

1. The symbol 5! is read as _________. five factorial
2. If you select \(r\) items from a group of \(n\) items when order is not important, you have a _________. combination
3. If you select \( r \) items from a group of \( n \) items when order is important, you have a \( \text{permutation} \).

4. Finding the number of ways to select 4 members from the 25 members of the math club to be on a publication committee is a \( \text{combination, permutation} \) problem.

5. Finding the number of different passwords starting with a letter followed by five other letters or digits is a \( \text{fundamental counting principle, combination} \) problem. Assume this password is not case sensitive.

**ACTIVITY**

**The Powerball Lottery.** The Powerball lottery requires players to select 5 different numbers from 1 to 59 and another number (the Powerball) from 1 to 39. To win the jackpot, the player needs to match the first five numbers and the Powerball. The order in which the player selects the first five numbers does not matter.

1. How many different Powerball choices are possible? \( \binom{59}{5}(39 \choose 1) = 195,249,054 \)

2. How many of the choices will match any given jackpot? \( (5 \choose 5)(1 \choose 1) = 1 \)

3. A prize is usually given to a player who matches the first five numbers but not the Powerball. In how many ways can this happen? \( (5 \choose 5)(38 \choose 1) = 38 \) (Note: The 39th number matches the Powerball and wins the jackpot.)

4. A smaller prize is awarded to a player who matches four of the first five numbers and does not match the Powerball. In how many ways can this happen? \( (5 \choose 4)(55 \choose 1)(38 \choose 1) = 10,450 \)

5. How many different selections will match the Powerball number? \( (5 \choose 5)(39 \choose 1) = 5,006,386 \)

6. How many different 9-digit Social Security numbers are possible? \( 10^9 = 1,000,000,000 \)

7. The first 3 digits of a Social Security number represent the state (or territory) where the application was made. How many different 3-digit numbers are possible? \( 10^3 = 1000 \)

8. A test has 20 multiple-choice questions with choices a, b, c, and d. How many different answer keys are possible? \( 4^{20} \)

9. A test has 15 true/false questions. How many different answer keys are possible? \( 2^{15} = 32,768 \)

**Exercises**

1. Draw a tree diagram showing the outcomes from flipping a coin and then rolling a die. See Answer Section.

2. Draw a tree diagram showing the outcomes from rolling a die and then flipping a coin. See Additional Answers.

3. Draw a tree diagram for the results from spinning a spinner with three equal regions (A, B, C) and then flipping a coin. See Answer Section.

4. Draw a tree diagram for flipping a coin and then spinning a spinner with four equal regions (A, B, C, D). See Additional Answers.

5. How many different 9-digit Social Security numbers are possible? \( 10^9 = 1,000,000,000 \)

6. The first 3 digits of a Social Security number represent the state (or territory) where the application was made. How many different 3-digit numbers are possible? \( 10^3 = 1000 \)

7. A test has 20 multiple-choice questions with choices a, b, c, and d. How many different answer keys are possible? \( 4^{20} \)

8. A test has 15 true/false questions. How many different answer keys are possible? \( 2^{15} = 32,768 \)

9. A round combination padlock has the numbers 0 to 39 (see the figure). How many 3-number combinations are possible.
a. if numbers can be repeated? \(40^5 = 64,000\)
b. if numbers cannot be repeated? \(40 \cdot 39 \cdot 38 = 59,280\)

10. Staples Wordlock combination lock (see the figure) has 4 tumblers containing 10 letters each and 1 tumbler with 9 letters and a blank.

11. A heavy-duty padlock (see the figure) has 4 dials, each with the numbers 0 to 9.

12. A house security number key pad has the digits 0 to 9. How many 4-number combinations are possible

a. if numbers can be repeated? \(10^4 = 10,000\)
b. if numbers cannot be repeated? \(10 \cdot 9 \cdot 8 \cdot 7 = 5040\)

13. How many different radio or television station names can be made if the first letter must be either a K or a W and the name must contain

a. 3 letters (as in KEX or WOR)? \(2 \cdot 26 \cdot 26 = 1352\)
b. 4 letters (as in KLCC or WGBH)? \(2 \cdot 26 \cdot 26 \cdot 26 = 35,152\)
c. How many 3- or 4-letter station names can be made altogether? \(1352 + 35,152 = 36,504\)

14. Suppose a roadside stand has 3 types of cones, 1 type of dish, 33 flavors of ice cream, and 8 flavors of sherbet. How many different outcomes are there for

a. a dish with a scoop of ice cream or sherbet? \(33 \cdot 8 = 264\)
b. a 1-scoop cone? \(3(33 + 8) = 123\)
c. a 2-scoop cone? \(3 \cdot 41 \cdot 41 = 5043\)
d. a dish with a scoop of ice cream and a scoop of sherbet? \(33 \cdot 8 = 264\)
e. a 3-scoop cone with different flavors of sherbet? \(3 \cdot 8 \cdot 7 \cdot 6 = 1008\)

15. A college offers 15 sociology, 5 psychology, 3 economics, and 12 history courses that count toward the Social Sciences lower-division requirement. Courses cannot be repeated. In completing requirements, how many ways can a student take

a. 1 psychology or 1 history course? \(5 + 12 = 17\)
b. 1 sociology and 1 economics course? \(15 \cdot 3 = 45\)
c. 3 history and 1 sociology course? \(12 \cdot 11 \cdot 10 \cdot 15 = 19,800\)
d. 1 history or 1 sociology course? \(12 + 15 = 27\)
e. 3 sociology courses? \(15 \cdot 14 \cdot 13 = 2730\)
f. 1 psychology and 1 history or economics course? \(5(12 + 3) = 75\)

16. The Breakfast Diner has eggs (6 ways), meat (4 choices), toast (4 choices), and potatoes (2 ways). How many ways can Kay order

a. eggs or toast? \(10\)
b. eggs, meat, toast, and potatoes? \(6 \cdot 4 \cdot 4 \cdot 2 = 192\)
c. eggs, meat, and toast? \(6 \cdot 4 \cdot 4 = 96\)
d. eggs, meat, and potatoes? \(6 \cdot 4 \cdot 2 = 48\)
e. eggs and meat with a choice of toast or potato? \(6 \cdot (4 + 2) = 36\)
f. eggs with a choice of meat or toast or potato? \(6(4 + 2) = 60\)

17. Telephone Numbers

a. Until cell phones became popular, telephone area codes were limited to a set of 3 numbers with a 0 or 1 in the center position, such as 201 or 616. No area codes started with a 0 or 1. How many 3-digit codes were possible? \(160\)
b. The center position now can be any number. How many 3-digit area codes are possible? 800

c. The 3-digit area code is followed by a 7-digit local telephone number. If neither the area code nor the local number can start with a 0 or a 1, how many telephone numbers are possible? 8 \times 9 \times 7^6

d. In all area codes, the telephone number 555-1212 is reserved by the phone company for directory assistance. Other 555 numbers are used in television shows and movies. How many 10-digit phone numbers are available if all 555 numbers are excluded? (Think carefully.) 800 \times 99 \times 10^5

18. How many possible 4-digit extensions to the zip code are there? Why is this number multiplied by, not added to, the number of 5-digit zip codes to find the total number of possible 9-digit codes? 10^9; each code can be followed by each extension.

19. Evaluate.
   a. 8! 40,320
   b. 10! 3,628,800
   c. 0! 1
   d. (8 + 2)! 3,628,800
   e. 8! + 2! 40,322
   f. (8 - 2)! 720

20. Simplify.
   a. 5! \cdot 2! 240
   b. 3! \cdot 1! 6
   c. 0! \cdot 4! 24
   d. \frac{6!}{(6 - 2)!2!} 15
   e. \frac{8!}{(8 - 2)!2!} 28
   f. (8 - 2)! 3

In Exercises 21 to 24, tell whether the problem is a factorial and/or a fundamental counting principle (FCP) problem. Answer the question.

21. Brad has tied 6 flies for fly-fishing. How many ways can he use them one at a time without repetition? 6!
   Also FCP.

22. Marie has 4 notebooks. How many ways can she label them for her 4 courses? 4!
   Also FCP.

23. Sally has 7 pairs of space socks. How many ways can she wear them in a week if she wears one pair each day and may repeat wearings? 7^7
   Also FCP.

24. Julia has 9 favorite menus. How many ways can she use them over a period of 9 dinners if she may repeat the menu? 9^9
   Also FCP.

25. Write in permutation notation and then list the permutations: arrangements of 2 letters from the set \{a, b, c, d\}. \sigma P_2; ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc

26. Write in permutation notation and then list the permutations: arrangements of 2 letters from the set \{w, x, y\}. \sigma P_2; wx, wy, wx, xy, yw, yx

27. Write in combination notation and list the outcomes: a combination of 2 letters taken from \{b, c, d\}. \sigma C_2; bc, bd, cd

28. Write in combination notation and list the outcomes: a combination of 2 letters taken from \{w, x, y, z\}. \sigma C_2; wx, wy, wz, xy, xz, yz

In Exercises 29 and 30, evaluate.

29. a. \sigma P_4 120
   b. \sigma P_3 120
   c. \sigma P_4 12

30. a. \sigma P_3 120
   b. \sigma P_2 30
   c. \sigma P_4 479,001,600

In Exercises 31 and 32, evaluate.

31. a. \sigma C_4 15
   b. \sigma C_3 6
   c. \sigma C_4 495

32. a. \gamma C_3 35
   b. \gamma C_2 21
   c. \gamma C_4 220

33. How many ways can 3-letter “words” be made from the set \{L, M, N, O, P\}
   a. if no letter may be repeated? 60
   b. if letters may be repeated? 125

34. How many ways can 2 CDs be selected from a set of 8 CDs
   a. if the first CD is returned to the set after playing? 64
   b. if the first CD is not returned to the set after playing? 56

35. How many ways can a meeting of 10 people be interrupted by 3 personal cell phone calls? State your assumptions. 10^3 = 1000 with repetition

In Exercises 36 to 38, tell what word or phrase indicates that a combination is appropriate and then answer the question.

36. How many different purchases of 3 CDs are possible from an artist’s collection of 25? different purchases; 2300

37. How many different sets of 3 letters can be selected from a set of 10 different letters? different sets; 120

38. How many different 3-person committees can be selected from a department of 12 members? different committees; 220

In Exercises 39 to 46, tell whether the settings require the fundamental counting principle (FCP), permutations (and the FCP), or combinations. Then evaluate.

39. How many ways can you select 4 members of the 25 members of the math club to be on a publication committee? combinations; 12,650

40. If a computer password must begin with a letter and be followed by 5 other letters or digits, how many different passwords are possible? Assume upper case or lower case makes no difference. FCP; 1,572,120,576

41. A college ID has 4 numbers and/or letters drawn randomly from a set of 36 numbers and letters (A to Z, 0 to 9). For example, one ID is ABE6.
   a. How many different 4-place IDs can be made if repetitions are permitted? FCP; 1,679,616
b. How many different 4-place IDs can be made if no repetitions are permitted? \( \text{permutations}; 1,413,720 \)

c. At a school where no repetitions are permitted in IDs, the bookstore decides to give out $10 gift certificates each week for the middle five weeks of the semester. Each week, the bookstore manager draws a set of 4 numbers and/or letters from the set of 36. Students whose ID contains the 4 numbers and/or letters drawn, in any order, may claim a certificate if they stop by the store that week. How many different sets of numbers and/or letters may be drawn? \( \text{combinations}; 58,905 \)

d. For each set of 4 numbers and/or letters drawn by the bookstore, say \( \{4, C, K, 7\} \), what is the largest number of gift certificates that could be claimed? \( \text{permutations}; 4! = 24 \)

42. A basketball team has 14 members.

a. Ignoring positions played, how many different ways can a group of 5 players be chosen to start in a game? \( \text{combinations}; 14C5 = 1,806 \)

b. How many ways can 5 players be chosen if each is assigned a different position? \( \text{permutations}; 14! = 8,718 \times 10^{10} \)

c. How many different committees can be chosen from 3 administrators, 12 deans, and 20 faculty leaders. \( \text{combinations}; 25C5 = 53,130 \)

d. What is the probability that, when you are dealt 5 cards from a deck, you get a full house of 2 kings and 3 tens? \( \text{permutations}; 2 \times 10 \times 10 \times 10 \times 10 = 1,000,000 \)

e. How many different 5-letter “words” can be made without repetition of letters? \( \text{permutations}; 26 \times 25 \times 24 \times 23 \times 22 = 78,936 \)

43. A class of 18 second-graders lines up for a fire drill.

a. How many ways can the students pair up? \( \text{combinations}; 18C2 = 153 \)

b. How many ways can the 9 pairs of students line up? \( \text{permutations}; 9! = 362,880 \)

c. How many ways can the students line up single file? \( \text{permutations}; 18! = 6.402 \times 10^{16} \)

44. The Hawaiian alphabet has 12 letters. How many 5-letter “words” can be made with repetition of letters? \( \text{permutations}; 12^5 = 248,832 \)

b. How many 5-letter “words” can be made without repetition of letters? \( \text{permuations}; 12 \times 11 \times 10 \times 9 \times 8 = 95,040 \)

c. How many groups of 5 different letters can be drawn? \( \text{combinations}; 792 \)

45. An interviewing committee to select a college dean is to be chosen from 3 administrators, 12 deans, and 20 faculty leaders.

a. How many different 6-member committees can be chosen? \( \text{combinations}; 36C6 = 1,623,160 \)

b. How many different committees can be chosen if 1 administrator, 3 deans, and 2 faculty leaders must participate? \( \text{combinations and FCP}; 125,400 \)

c. Prove that \( nC_n = 1 \). (Hint: 1! = 1)

d. Prove that \( nC_1 = n \).

e. Prove that \( nC_n = 1 \).

46. To find how many ways you can get a full house of 2 kings and 3 tens, consider the following questions.

a. How many ways can you select 2 kings? \( \text{combinations}; 2 \times 10 \times 10 = 200 \)

b. How many ways can you select 3 tens? \( \text{combinations}; 3 \times 10 \times 10 \times 10 = 3,000 \)

c. What is the probability that, when you are dealt 5 cards from a deck, you get a full house of 2 kings and 3 tens? \( \frac{2 \times 10 \times 10 \times 10 \times 10 \times 10}{52 \times 51 \times 50 \times 49 \times 48} = \frac{1}{15,548,000} \)

47. Based on the text, what is the probability of receiving a four-of-a-kind hand when dealt 5 cards? Which example did you use? \( \frac{1}{623 \times 15 \times 14 \times 13 \times 12} = \frac{1}{623 \times 15 \times 14 \times 13 \times 12} \)

48. License Plates. The Wyoming license plate starts with a number from 1 to 23 (which represents the county of issue), followed by either 4 digits or 3 digits and 2 letters.

a. How many different 3-digit-and-2-letter license plates are available? \( 23 \times 10 \times 10 \times 10 \times 26 = 15,548,000 \)

b. How many different 4-digit license plates are available? \( 23 \times 10 \times 10 \times 10 \times 10 = 23,000 \)

c. What is the total number of license plates available in Wyoming? \( 15,548,000 \)

d. If we consider only Natrona County (county code 1) and only license plates with 3 digits and 2 letters, what is the probability that you receive a license plate with no repetition of digits or letters? \( \frac{368,000}{676,000} = 0.54 \)

49. A committee of 6 is to be formed to plan a math day for all students enrolled in mathematics courses this semester. The committee is to be selected at random from a group of 13 first-year students, 12 second-year students, and 5 students who are still in high school.

a. The committee be made up of all first-year students? \( 13C6 = 1,716 \)

b. The committee contain no high school students? \( 18C6 = 177,100 \)

c. The committee be made up of 2 first-year students and 4 second-year students? \( 13C2 \times 18C4 = 38,610 \)

d. If a committee is selected at random, what is the probability that it contains only first-year students? \( \frac{1,716}{381,680} = 0.00045 \)

50. Project: Proofs

a. Prove that \( nP_1 = nC_1 \). (Hint: 1! = 1)

b. Prove that \( nP_1 = n \).

c. Prove that \( nC_1 = n \).

d. Does \( nP_n = nC_n \)? no

e. Prove that \( nP_n = n! \).

f. Prove that \( nC_n = 1 \).
g. Prove that \( P_0 = C_0 \).

h. Prove that \( P_0 = 1 \).

i. Prove that \( nC_0 = 1 \).

51. Project: Combinations and Repeated Trials. When we flip a coin, we obtain one of two outcomes: H (head) or T (tail). In 4 trials of flipping a coin, there are 16 possible outcomes.

a. Using the fundamental counting principle, show why the 4 trials give 16 outcomes: \( 2 \cdot 2 \cdot 2 \cdot 2 = 16 \).

b. List the 8 different outcomes for flipping a coin 3 times. Group the outcomes by number of heads. HHH, HHT, HTH, THH, HTT, THT, TTH, TTT.

c. Evaluate \( 3C_0, 3C_1, 3C_2, 3C_3 \). 1, 3, 3, 1

d. If \( r \) is the number of tails in \( nC_r \) in part c, what event is \( 3C_0 + 3C_1 + 3C_2 + 3C_3 \) if \( 0, 1, 2 \), or 3 tails.

e. Why can we use the addition principle in part d? The events cannot all happen at the same time.

f. What is the sum in part d? 8

g. Using the above results, predict the number of outcomes for flipping a coin 5 times. Check your results by calculating the combinations \( 5C_0, 5C_1, 5C_2, 5C_3, 5C_4, \) and \( 5C_5 \) and adding them. 1, 5, 10, 10, 5, 1; 32 = 2^5

h. Suggest the meaning of \( 6C_0, 6C_1, 6C_2, \) and \( 6C_3 \) in terms of coin flipping trials. See Additional Answers.

i. Can your answers to part h be given in terms of either heads or tails? yes

52. Project: Lotteries

a. In the Oregon Lottery, 6 numbers are drawn at random from a set numbered 1 to 48. For $1, a person receives a ticket with 2 sets of 6 numbers. The probability of winning is listed as 1 in 6,135,756. Explain how the probability is found. Find \( 6C_6 \) and then divide by 2.

b. The Michigan Lotto has 6 numbers drawn at random from a set numbered 1 to 49. What is the probability of a match? \( \frac{1}{49C_6} = \frac{1}{13,983,816} \)

c. The multistate Powerball has 5 numbers drawn at random from a set numbered 1 to 59 and 1 number drawn at random from a set numbered 1 to 39. What is the probability of a match? See below.

d. Illinois MegaMillions requires 5 numbers selected from 1 to 52 and then 1 number selected from 1 to 52. What is the probability of a match? Find \( 48C_6 \) and then divide by 2. Ans. \( \frac{1}{48C_6 \cdot 52C_1} \cdot \frac{1}{195,249,054} \)

8.4 Pascal’s Triangle and Combinations

**OBJECTIVES**

- Observe Pascal’s triangle in transportation and combination settings.
- Apply Pascal’s triangle to combination problems.

**WARM-UP**

Find

1. \( 4C_0 \) 1
2. \( 4C_1 \) 4
3. \( 4C_2 \) 6
4. \( 4C_3 \) 4
5. \( 4C_4 \) 1

**WE BEGIN WITH** a question that appears to be a puzzle problem but has some surprising connections to topics studied thus far.

**STREET GRIDS**

**Question 1** The lines in Figure 26 represent city streets. If one can travel **only south or east** on the streets, how many different ways are there to get from point A to point B?
To answer the question, we might consider an easier problem. Solving a simpler but related problem is a good strategy for building understanding of a problem and designing a plan to solve it.

**EXAMPLE 1 Exploring street grids** For Question 1, consider the number of ways to get to each street intersection, one at a time. Use Figure 27.

a. Find the number of ways from $A$ to $B_1$.

b. Find the number of ways from $A$ to $B_2$.

c. Find the number of ways from $A$ to $B$.

**Solution**

a. As shown in Figure 28, there are 2 ways to leave position $A$: 1 to the east and 1 to the south. We record a 1 at each adjacent intersection in Figure 28a. Because we are limited to moving east or south, each of the ways gives 1 route to $B_1$. Thus, from $A$ to $B_1$, there are 2 routes. We record a 2 at $B_1$ to indicate the 2 routes.

b. If we count routes to $B_2$, we find there is 1 additional route to $B_2$, beyond our results recorded earlier. Thus, there are 3 ways to get from $A$ to $B_2$. We record a 3 at $B_2$ in Figure 28b.

c. Continuing to record numbers on the intersections confirms that there are 6 routes to $B$ from $A$, as shown in Figure 28c.
EXAMPLE 2  Extending the street grid  Use the questions to add more numbers to the street grid in Figure 29.

a. Will the numbers across the top and down the left side always be 1? Why or why not?
b. What is the pattern of numbers in the second horizontal row of street intersections and the second vertical row of street intersections?
c. As we enter a new intersection, how do we find the number of ways to reach that intersection?

A

1 1
1 2 3
1 3 6

B

FIGURE 29

Solution  a. Yes; there is only 1 way to reach each of these outer intersections—across the top or down the left side.
b. Going from left to right, we add 1 new way to reach the intersection with each block traveled. Going from top to bottom, we add 1 new way to reach the intersection with each block traveled. The second horizontal row and the second vertical row will have the numbers 1, 2, 3, 4, 5, . . .
c. The number at each intersection is the sum of the numbers on the two streets entering the intersection.

The extended street grid appears in Figure 30.

A

1 1 1 1 1
1 2 3 4 5
1 3 6 10
1 4 10
1 5
1

B

FIGURE 30
Historical Note
Blaise Pascal (1623–1662), a French mathematician, was a pioneer in the theory of probability. He is well known for his work on conic sections (Chapter 7), and he was acquainted with Descartes (Cartesian coordinate system, Chapter 1). Pascal studied fluid pressure and hydraulics, designed a public transport system for Paris, and, in the 1640s, designed a machine to carry out addition and subtraction and organized its manufacture and sale. Several of the machines are known to exist today.

PASCAL’S TRIANGLE
The first seven rows of Pascal’s triangle are shown in Figure 31. This array is filled with patterns; it could be a course of study all to itself.

\[
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}
\]

Figure 31 Pascal’s triangle.

EXAMPLE 3 Finding rows in Pascal’s triangle What are the next two rows of Pascal’s triangle? (Hint: The rows start with 1, 7 and 1, 8.) Use a street grid if necessary.

Solution
1, 7, 21, 35, 35, 21, 7, 1
1, 8, 28, 56, 70, 56, 28, 8, 1

COMBINATIONS AND PASCAL’S TRIANGLE
Question 2 What are all the possible birth orders for a family with 4 children? Assume there are no twins.

As with Question 1, we solve simpler but related problems first.

EXAMPLE 4 Exploring birth order Simplify Question 2 in the following way.
a. What are the possible children in a single-child family?
b. What are the possible birth orders for a 2-child family?
c. What are the possible birth orders for a 3-child family?
d. Return to Question 2.

Solution
a. The 1 child is either a boy or a girl:
   Girl Boy

b. With 2 children, we have these birth orders:
   Girl Girl Girl Boy Boy Boy

   Boy Girl

c. With 3 children, we have these birth orders:
   GGG GGB GBB BBB
   GB GBB BB BB
   BG BB B G

d. With 4 children, we have these birth orders:
   GGGG GGBB GGBB B B B B B B B B B B
Thus, we have
1 birth order for 4 girls and 0 boys,
4 birth orders for 3 girls and 1 boy,
6 birth orders for 2 girls and 2 boys,
4 birth orders for 1 girl and 3 boys, and
1 birth order for 0 girls and 4 boys.

The number pattern 1, 4, 6, 4, 1 appears in the birth orders and in Pascal's triangle. The results were matched with combinations.

**EXAMPLE 5** Summarizing birth order and combinations Evaluate these combinations.

- \(_4 \text{C}_0\)
- \(_4 \text{C}_1\)
- \(_4 \text{C}_2\)
- \(_4 \text{C}_3\)
- \(_4 \text{C}_4\)

Compare the results with \(_n \text{C}_r\) for \(n\)-child families. Suggest any choices in the meaning of \(r\).

**Solution**

- \(_4 \text{C}_0 = 1\)
- \(_4 \text{C}_1 = 4\)
- \(_4 \text{C}_2 = 6\)
- \(_4 \text{C}_3 = 4\)
- \(_4 \text{C}_4 = 1\)

The combination \(_n \text{C}_r\) can mean \(n\) children of which \(r\) are girls and \(n - r\) are boys or \(n\) children of which \(r\) are boys and \(n - r\) are girls.

**PRACTICE 1** Write and evaluate the combinations that describe birth orders for 6-child families, and compare your results with the appropriate row of Pascal's triangle.

\(_6 \text{C}_0 = 1, \_6 \text{C}_1 = 6, \_6 \text{C}_2 = 15, \_6 \text{C}_3 = 20, \_6 \text{C}_4 = 15, \_6 \text{C}_5 = 6, \_6 \text{C}_6 = 1\). The combinations match the 1, 6, ... row of Pascal's triangle.

**EXAMPLE 6** Applying combinations and Pascal's triangle Use a combination and Pascal's triangle to find the number of ways a 7-child family can have 5 boys.

**Solution**

\[ \_7 \text{C}_5 = \frac{7!}{5!(7-5)!} = 21 \]

Consider the 1, 7, 21, 35, 35, 21, 7, 1 row. If 1 matches with 0 boys, 7 with 1 boy, 21 with 2 boys, 35 with 3 boys, and 35 with 4 boys, then 21 matches with 5 boys.

There are 21 birth orders of 5 boys in a 7-child family.

**EXAMPLE 7** Finding more patterns

- **a.** Add each row of Pascal’s triangle. What does the sum mean?
- **b.** Write each sum as an expression with the same base but different exponents.

**Solution**

- **a.** The sums of the numbers in each row are 1, 2, 4, 8, 16, 32, 64. The sum is the size of the sample space (number of outcomes) for the \(n\)-child families \((n \geq 0)\).
- **b.** The numbers are powers of 2.

\[ 2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8 \]

\[ 2^4 = 16, \quad 2^5 = 32, \quad 2^6 = 64 \]

**EXAMPLE 8** Finding probability What is the probability that a 6-child family will have 0 boys? 1 boy? 2 boys? 3 boys? 4 boys?

**Solution**

There are 64 birth orders in the sample space. Putting each of the first five numbers in the 1, 6, ... row of Pascal’s triangle over 64 gives the probability:

\[ P(0 \text{ boys}) = \frac{1}{64} \]
\[ P(1 \text{ boy}) = \frac{6}{64} \]
\[ P(2 \text{ boys}) = \frac{15}{64} \]
\[ P(3 \text{ boys}) = \frac{20}{64} \]
\[ P(4 \text{ boys}) = \frac{15}{64} \]
PRACTICE 2  What is the probability that a 7-child family will have 0 girls? 1 girl? 2 girls? 3 girls?
\[ \binom{7}{0} = 1, \quad \binom{7}{1} = 7, \quad \binom{7}{2} = 21, \quad \binom{7}{3} = 35, \quad \binom{7}{4} = 35, \quad \binom{7}{5} = 21, \quad \binom{7}{6} = 7, \quad \binom{7}{7} = 1 \]

Answer Box
Warm-up: 1. 1 2. 4 3. 6 4. 4 5. 1
Practice 1: \[ \binom{6}{0} = 1, \quad \binom{6}{1} = 6, \quad \binom{6}{2} = 15, \quad \binom{6}{3} = 20, \quad \binom{6}{4} = 15, \quad \binom{6}{5} = 6, \quad \binom{6}{6} = 1. \]
The combinations match the 1, 6, . . . row of Pascal’s triangle.
Practice 2: \[ \binom{7}{0} = 1, \quad \binom{7}{1} = 7, \quad \binom{7}{2} = 21, \quad \binom{7}{3} = 35, \quad \binom{7}{4} = 35, \quad \binom{7}{5} = 21, \quad \binom{7}{6} = 7, \quad \binom{7}{7} = 1. \]

Reading Questions
1. With the exception of the number 1, each number in Pascal’s triangle is the sum of ________ numbers in the preceding row.  two
2. Each ________ of Pascal’s triangle has symmetry in the numbers.  row
3. The numbers in each row in Pascal’s triangle can be found with [combinations, permutations].
4. The first number in the row 1, 12, . . . is \( C \) ______.  12, 0
5. The last number in the row 1, 12, . . . is \( C \) ______.  12, 12
6. \( C_3 \) is the value of the ________ entry in its row in Pascal’s triangle.  fourth
7. The sum of each row in Pascal’s triangle is a power of _________.  two
8. The sum of the entries in row 1, 6, . . . is 2 raised to the ________ power.  sixth

ACTIVITY
Pascal’s Triangle and the Binomial Theorem
1. Expand the following polynomials. Write your answers in standard form.
   a. \((x + y)^1\) \( x + y \)
   b. \((x + y)^2\) \( x^2 + 2xy + y^2 \)
   c. \((x + y)^3\) \( x^3 + 3x^2y + 3xy^2 + y^3 \)
   d. \((x + y)^4\) \( x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \)
2. What do you notice about the first term of each polynomial? What do you notice about each successive term and the power of \( x \)? First term is \( x \) raised to the power with which we are expanding. In each successive term, the power of \( x \) decreases by 1.
3. What do you notice about the last term of each polynomial? What do you notice about each previous term and the power of \( y \)? Last term is \( y \) raised to the power with which we are expanding. In each previous term, the power of \( y \) decreases by 1.
4. Place the coefficients of each term of the polynomial expansion in a copy of the chart given below. Start with the first term for the first empty cell in the row and then continue with each successive term until the last term.

| \((x + y)^1\) | \(1\) | \(1\) |
| \((x + y)^2\) | \(1\) | \(2\) | \(1\) |
| \((x + y)^3\) | \(1\) | \(3\) | \(3\) | \(1\) |
| \((x + y)^4\) | \(1\) | \(4\) | \(6\) | \(4\) | \(1\) |
5. What do you notice about the pattern of coefficients in each row? They follow the pattern of Pascal’s triangle. The row in the grid corresponds to the power of the expansion.
6. Using the information from steps 2–5, write out the expansion for \( (x + y)^5 \) and \( (x + y)^6 \).

7. The theorem you are using is called the binomial theorem and was known to the Chinese in 2000 B.C.E. Isaac Newton expanded on these ideas later. The \( n \)th term of the expansion of \( (x + y)^n \) is written \( \binom{n}{r} x^{n-r} y^r \). Verify this formula for the 3rd and 4th terms of the expansion of \( (x + y)^6 \).

32. Possible birth orders for 5 children. They
Describe how to get from one row to the next
in Pascal’s triangle. How is a row of Pascal’s
triangle related to the table method of multiplying
polynomials? Start and end with 1. Add 2 numbers in
previous row; each diagonal has 2 numbers added as like terms.

33. Show that the rows starting 1, 7, . . . and 1, 8, . . . in
Pascal’s triangle add to powers of 2. \( 128 = 2^7; 256 = 2^8 \)

34. Use a calculator to make a list of the powers of 11
from 110 to 116. Compare your list with Pascal’s
triangle. What do you observe? How is the list similar
to Pascal’s triangle? How is it different?
Digits in 10th to 11th match; digits in 11th and 111 do not.

35. List the 32 possible birth orders for 5 children. They
fall into 6 different groups: all girls, 4 girls, 3 girls,
2 girls, 1 girl, and 0 girls. How many birth orders are
in each group? \( 1, 5, 10, 5, 1 \)

In Exercises 6 to 8, use \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \)

6. Evaluate the expressions.
   a. \( \binom{5}{0} \) 1
   b. \( \binom{5}{1} \) 5
   c. \( \binom{5}{2} \) 10
   d. \( \binom{5}{3} \) 10
   e. \( \binom{5}{4} \) 5
   f. \( \binom{5}{5} \) 1

7. Find enough of the 1, 12, . . . row to finish it without
   the formula. \( 1, 12, 66, 220, 495, 792, 924, 792, . \ldots \)

8. Find enough of the 1, 11, . . . row to finish it without
   the formula. \( 1, 11, 55, 165, 330, 462, 462, . \ldots \)

9. Use Pascal’s triangle to evaluate the expression.
   a. \( \binom{6}{1} \) 5
   b. \( \binom{6}{2} \) 15
   c. \( \binom{6}{3} \) 20

10. Use Pascal’s triangle to evaluate the expression.
    a. \( \binom{7}{2} \) 21
    b. \( \binom{7}{3} \) 20
    c. \( \binom{7}{4} \) 15

11. Mentally evaluate the expression.
    a. \( 20 \binom{1}{1} \) 20
    b. \( 15 \binom{15}{1} \) 1
    c. \( 20 \binom{15}{1} \) 20
    d. \( 16 \binom{15}{1} \) 16

12. Mentally evaluate the expression.
    a. \( 20 \binom{10}{1} \) 20
    b. \( 16 \binom{10}{1} \) 1
    c. \( 15 \binom{14}{1} \) 15
    d. \( 16 \binom{16}{1} \) 16

13. Explain why four combinations \( \binom{n}{r} \) in each row are easy
    mental exercises. First two are 1 and \( n \), and last two are \( n \) and 1.

14. Using \( \binom{n}{r} \) notation, write the first two entries and the last
two entries of each row without using the variable \( r \).

15. To find \( \binom{n}{r} \) from Pascal’s triangle, go to row \( 1, n \), . . . and
    find the \( r + 1 \) number in the row. Why is it not the
    \( r \)th number? because the first number represents \( r = 0 \), not \( r = 1 \)

16. Rows have been identified by expressions such as “row
    1, 7, . . . .” Look at Pascal’s triangle and explain why we
    have not said row 7. The row beginning 1, 7, . . . is the 8th row.

In Exercises 17 to 20, we explore outcomes of heads (H)
and tails (T) when coins are flipped.

17. List the head and tail outcomes for flipping 2 coins:
a 1-cent coin and a 5-cent coin. How many outcomes
   for 0 heads? 1 head? 2 heads? HH, HT, TH, TT.

18. List the head and tail outcomes for flipping 3 coins:
a 1-cent coin, a 5-cent coin, and a 10-cent coin.
   How many outcomes for 0 heads? 1 head? 2 head, 3
   heads? HHH, HHT, HTH, THH, TTH, THT, HTT, TTT.

19. Does it appear that outcomes of heads and tails
    on coins are related to Pascal’s triangle? If so, predict the
    head and tail outcomes for tossing 4 different coins,
    yes: 0 H in 1 way, 1 H in 4 ways, 2 H in 6 ways, 3 H in 4 ways, 4 H in 1 way

20. Suppose we toss one coin 4 times. Describe the
    possible head and tail outcomes. 0 H in 1 way, 1 H in
    4 ways, 2 H in 6 ways, 3 H in 4 ways, 4 H in 1 way

21. What is the probability of getting exactly 2 tails
    in flipping 3 coins? at least 2 tails? \( \frac{3}{8} \) because TTT also is included
In Exercises 22 to 28, find the probabilities mentally. (Hint: Use a power of 2 to find the number of outcomes for an n-child family.)

22. Find $P(0 \text{ girls})$ in a 4-child family. $\frac{1}{16}$
23. Find $P(4 \text{ girls})$ in a 5-child family. $\frac{3}{32}$
24. Find $P(4 \text{ girls})$ in a 4-child family. $\frac{1}{16}$
25. Find $P(5 \text{ boys})$ in a 5-child family. $\frac{1}{32}$
26. Find $P(3 \text{ boys})$ in a 4-child family. $\frac{4}{16} = \frac{1}{4}$
27. Find $P(1 \text{ tail})$ in 6 flips of a coin. $\frac{6}{64} = \frac{3}{32}$
28. Find $P(1 \text{ head})$ in 5 flips of a coin. $\frac{5}{32}$

29. How many different birth orders are there for a 12-child family? $2^{12} = 4096$
30. How many different birth orders are there for a 15-child family? $2^{15} = 32,768$
31. Find $P(10 \text{ boys})$ in a 12-child family. $\frac{66}{4096} = \frac{33}{2048}$
32. Find $P(6 \text{ girls})$ in a 15-child family.
33. Find $P(7 \text{ girls})$ in a 12-child family.
34. Find $P(8 \text{ boys})$ in a 15-child family. $\frac{435}{32,768}$
35. Find $P(2 \text{ girls})$ in a 12-child family. $\frac{11}{2048}$
36. Find $P(9 \text{ boys})$ in a 15-child family. $\frac{5005}{32,768}$
37. Why are the answers to Exercises 31 and 35 the same? $12C_{10} = 12C_2$
38. Why are the answers to Exercises 32 and 36 the same? $15C_6 = 15C_9$
39. Use the patterns started in Example 1 to find the number of ways from $A$ to $B$ on the street grids in parts a to d.

a. $10$ ways

40. Project: Extending Ideas. The following question comes from a former student at Winston Churchill High School, Janet Lind: What is the meaning of Pascal’s triangle in the third dimension? Investigate the meaning in terms of geometry similar to street grids, algebra similar to the powers of binomials, and ideas similar to those in this section. Write up your conclusions.

41. Project: Number Patterns. One number pattern in Pascal’s triangle is 1, 3, 6, 10, 15, . . . . Find the algebraic equation that gives this sequence of numbers for integer inputs. Find four other sequences within Pascal’s triangle and their equations. $d_n = \frac{n^2 + n}{2}$

Instructor Note:
Future teachers may note that adding on street grids is great practice for elementary school students.
8.1 Probability: Definitions and Single-Event Probability

The range for probability is \([0, 1]\). The probability of an outcome is between zero and one: \(0 \leq P(A) \leq 1\).

The sum of the probabilities of all the outcomes in a sample space is 1. For any event \(A\), \(P(A) + P(\text{not } A) = 1\).

When a theoretical probability is not available, use empirical methods (experiments or historic research) to determine probability.

8.2 Multiple-Event Probability

To apply the addition principle for probability, look for phrases meaning that event \(A\) and event \(B\) cannot both happen and that the outcome is either event \(A\) or event \(B\).

For activities with replacement, probabilities remain the same from one stage, or trial, to the next.

For activities without replacement, probabilities change from one stage, or trial, to the next.

The probabilities for each branching on a tree diagram add to 1.

The probabilities of the final outcomes on a tree diagram add to 1.

Conditional probability \(P(B|A)\) is the probability of event \(B\) given that event \(A\) has already happened.

Multiplication Principle for Probability: Let the probability of event \(A\) be \(P(A)\), the probability of event \(B\) be \(P(B)\), and the probability of \(B\) given \(A\) be \(P(B|A)\). If \(A\) and \(B\) are independent events, then

\[ P(A \text{ and } B) = P(A) \times P(B) \]

8.3 Using Counting Principles

The Fundamental Counting Principle: If you can do item 1 in \(a\) ways and you can do item 2 in \(b\) ways, then you can do item 1 followed by item 2 in \(a \times b\) ways. This principle can be extended to three or more items.

Both permutations and combinations refer to the selection and arrangement of objects from a set of distinct objects.

The number of permutations of \(n\) objects taken \(r\) at a time \((r \leq n)\) is written \(nPr\) and calculated with the formula

\[ nPr = \frac{n!}{(n-r)!} \]

The number of combinations of \(n\) objects taken \(r\) at a time \((r \leq n)\) is written \(nCr\) and calculated with the formula

\[ nCr = \frac{n!}{(n-r)!r!} \]

In settings where objects chosen from a set may be repeated (or the drawn object is replaced), the fundamental counting principle applies but neither permutations nor combinations do. All permutation problems can be solved with the fundamental counting principle. See Tables 7 and 8 on the next page.

8.4 Pascal’s Triangle and Combinations

Each row of Pascal’s triangle begins and ends with 1. The other terms of the row are found by adding two terms in the row above. The sum of the \(n + 1\) terms in the \(1, n, \ldots\) row is \(2^n\).

The combinations \(nCr\) are found in the \(1, n, \ldots\) row of Pascal’s triangle. Count terms from left to right, from \(r = 0, r = 1, r = 2, \ldots\) up to \(r = n\).
1. The experiment is selecting a letter at random from the word PROBLEM.

   In parts a and b, name the possible outcomes for the events described.
   
   a. Selecting a vowel {O, E}
   
   b. Not selecting a vowel {P, R, B, L, M}

   In parts c to e, give the probabilities for the events described.
   
   c. \( P(\text{selecting a vowel}) = \frac{2}{7} \)
   
   d. \( P(\text{not selecting a vowel}) = \frac{5}{7} \)
   
   e. \( P(\text{selecting A or E}) = \frac{5}{17} \)

2. The experiment is selecting a letter at random from LABRADOR RETRIEVER. In parts a and b, name the possible outcomes for the events described.

   a. Selecting a vowel {A, E, I, O}
   
   b. Not selecting a vowel {B, D, L, R, T, V}

   In parts c to e, give the probabilities for the events described.
   
   c. \( P(\text{selecting a vowel}) = \frac{7}{17} \)
   
   d. \( P(\text{not selecting a vowel}) = \frac{10}{17} \)
   
   e. \( P(\text{selecting A or E}) = \frac{7}{17} \)
   
   f. \( P(\text{selecting a letter also contained in BEAGLE}) = \frac{7}{17} \)

3. International Morse Code is made up of dashes and dots. For example, - - - represents O and - - - - represents S.

   a. How many codes are possible with 1 of the 2 symbols?
   
   b. How many codes are possible with 2 of the symbols?
   
   c. How many codes are possible with 3 of the symbols?
   
   d. How many codes are possible with 4 of the symbols?

4. How are the sets of outcomes in Exercise 2 different from those in Exercise 1? Outcomes in Exercise 2 are not equally likely.
5. Complete the chart.

<table>
<thead>
<tr>
<th>Sample Space</th>
<th>Event A</th>
<th>Event (not A)</th>
<th>P(A)</th>
<th>P(not A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH, HT, TH, TT</td>
<td>HT, TH</td>
<td>HH, TT</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>HH, HT, TH, TT</td>
<td>TT</td>
<td>HH, HT, TH</td>
<td>1/4</td>
<td>3/4</td>
</tr>
<tr>
<td>1, 2, 3, 4, 5</td>
<td>even numbers</td>
<td>odd numbers</td>
<td>2/5</td>
<td>3/5</td>
</tr>
<tr>
<td>red, white, blue</td>
<td>white, blue</td>
<td>red</td>
<td>2/3</td>
<td>1/3</td>
</tr>
<tr>
<td>a, b, c, d</td>
<td>letters also in cat</td>
<td>b, d</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>a, b, c, d</td>
<td>letters also in dog</td>
<td>a, b, c</td>
<td>2/5</td>
<td>3/5</td>
</tr>
</tbody>
</table>

6. Complete the chart.

<table>
<thead>
<tr>
<th>Sample Space</th>
<th>Event A</th>
<th>Event (not A)</th>
<th>P(A)</th>
<th>P(not A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GG, GB, BG, BB</td>
<td>GB, BG BB</td>
<td>GG</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>GG, GB, BG, BB</td>
<td>GB, BG</td>
<td>GG, BB</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>1, 2, 3, 4, 5</td>
<td>square numbers</td>
<td>2, 3, 5</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>green, yellow, orange</td>
<td>green, yellow</td>
<td>orange</td>
<td>1/3</td>
<td>2/3</td>
</tr>
<tr>
<td>i, j, k, l</td>
<td>letters also in milk</td>
<td>i j</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>i, j, k, l</td>
<td>letters also in milk</td>
<td>i j</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

7. A dentist’s sample Butler G·U·M floss container has octagonal sides, a top where the floss emerges, and a bottom. When tossed 120 times, the container landed top up 32 times, side up 48 times, and bottom up 40 times. What is the probability for each outcome? What kind of probability does this illustrate? Does this seem like a large enough trial for determining probability?

8. On the first day of class, one of the authors asks students to calculate their load factor (3 times their credit hours plus the number of hours of employment). For the 30 members of one Intermediate Algebra class, the load factors were 33, 36, 36, 42, 42, 45, 45, 45, 48, 48, 51, 51, 52, 55, 56, 57, 58, 59, 59, 60, 60, 62, 65, 68, 70, 70, 72, 73, 79. What is the probability that the load factor was

a. 45 or less? \( \frac{6}{30} \)
b. between 46 and 61? \( \frac{13}{30} \)
c. greater than 61? \( \frac{7}{30} \)
d. Discuss the type of probability and whether any conclusions can be drawn. This small sample shows a wide distribution of loads; many more classes should be studied.

In Exercises 9 and 10, tell whether the multiplication principle (MP) or the addition principle (AP) applies and solve the problem.

9. a. Find the probability of a 1, 2, 3, 4, 5, or 6 in rolling a die. AP; 1/6
   b. Find the probability of a girl and then a boy in the births of 2 children. MP; \( \frac{1}{4} \)

10. a. Find the probability of a 3, 4, or 5 in rolling a die. AP; \( \frac{1}{2} \)
    b. Find the probability of a tail followed by a 5 or 6 in flipping a coin and then rolling a die. MP; \( \frac{1}{6} \)

11. Use a tree diagram to find the probability of a tail or a 3 in flipping a coin and then rolling a die. \( \frac{3}{6}; \) see Answer Section.

12. Use a tree diagram to find the probability of a head or a 4 in flipping a coin and then rolling a die. \( \frac{2}{6}; \) see Exercise 11 in Answer Section.

In Exercises 13 and 14, state whether the events have overlapping outcomes (are not mutually exclusive) and solve the problem. (Refer to the figure as needed.)

13. Two dice are rolled.
   a. How many pairs of dice contain a 6 or a sum of 7? no overlap; 15
   b. How many pairs of dice contain a 5 or a sum of 6? overlap; 14
   c. How many pairs of dice contain a 4 or a 6? overlap; 20
   d. Find \( P(6 \text{ or sum of 6}) \). \( \frac{16}{36} = \frac{4}{9} \)
   e. Find \( P(\text{double or sum of 4}) \). overlap; \( \frac{10}{36} = \frac{5}{18} \)
   f. Find \( P(\text{double or odd sum}) \). no overlap; \( \frac{15}{36} = \frac{5}{12} \)
14. Two dice are rolled.
   a. How many pairs of dice contain a 3 or a sum of 5? 
      overlap: 13
   b. How many pairs of dice contain a 2 or a sum of 7? 
      overlap: 15
   c. How many pairs of dice contain a 2 or a 3? 
      overlap: 20
   d. Find \( \Pr(3 \text{ or sum of five}) \). 
      overlap: \( \frac{11}{36} \)
   e. Find \( \Pr(\text{double or sum of 7}) \). 
      no overlap: \( \frac{13}{36} = \frac{1}{3} \)
   f. Find \( \Pr(\text{double or even sum}) \). 
      overlap: \( \frac{11}{36} = \frac{1}{3} \)

15. Two drawings are made from a bag containing 2 red, 3 white, and 4 blue marbles. Marbles are drawn at random, and the first marble is replaced after drawing.
   a. Draw a tree diagram for the two draws. 
      See Answer Section.
   b. Find \( \Pr(\text{red and white}) \). 
      \( \frac{2}{21} \)
   c. Find \( \Pr(\text{red or white on either draw}) \). 
      \( \frac{25}{81} \)
   d. Find \( \Pr(\text{not BB}) \). 
      \( \frac{5}{6} \)
   e. Find \( \Pr(\text{not blue on either draw}) \). 
      \( \frac{25}{81} \)

16. Two drawings are made from a bag containing 4 red, 2 white, and 1 blue marble. Marbles are drawn at random, and the first marble is not replaced after drawing.
   a. Draw a tree diagram for the two draws. 
      See Additional Answers.
   b. Find \( \Pr(\text{blue and white}) \). 
      \( \frac{1}{12} = \frac{1}{12} \)
   c. Find \( \Pr(\text{red or white on either draw}) \). 
      \( \frac{7}{12} = 1 \)
   d. Find \( \Pr(\text{not BB}) \). 
      \( \frac{7}{12} = 1 \)
   e. Find \( \Pr(\text{red on either draw or BR}) \). 
      \( \frac{7}{12} = \frac{7}{12} \)

17. Suppose the marble in Exercise 15 is not replaced after drawing.
   a. Draw a tree diagram for the two draws using the data in Exercise 15. 
      See Answer Section.
   b. Find \( \Pr(\text{red and white}) \). 
      \( \frac{1}{9} \)
   c. Find \( \Pr(\text{red or white on either draw}) \). 
      \( \frac{5}{9} \)
   d. Find \( \Pr(\text{not BB}) \). 
      \( \frac{5}{9} \)
   e. Find \( \Pr(\text{not blue on either draw}) \). 
      \( \frac{5}{9} \)

18. Suppose the marble in Exercise 16 is replaced after drawing.
   a. Draw a tree diagram for the two draws using the data in Exercise 16. 
      See Additional Answers.
   b. Find \( \Pr(\text{blue and white}) \). 
      \( \frac{1}{8} \)
   c. Find \( \Pr(\text{red or white on either draw}) \). 
      \( \frac{2}{3} \)
   d. Find \( \Pr(\text{not BB}) \). 
      \( \frac{2}{3} \)
   e. Find \( \Pr(\text{red on either draw or BR}) \). 
      \( \frac{2}{3} \)

19. Final grade and relevant hobby for the 30 students in the Winter Quarter class in Exercise 8 are listed below by load factor:
    - 33: drop (snowboard); 36: A, D+, drop;
    - 42: A, no credit;
    - 45: D, A, drop (ski);
    - 48: drop, B;
    - 51: B, no credit (snowboard);
    - 52: B+, 55: C;
    - 56: B;
    - 57: A; 58: C;
    - 59: C, D+;
    - 60: A+, drop;
    - 62: drop;
    - 65: drop; 68: drop; 70: D+, drop (snowboard);
    - 72: no credit;
    - 73: drop;
    - 79: C

   Final grade and load factor:
   - 42: A, no credit; 45: D, A, drop (ski); 48: drop, B;
   - 51: B, no credit (snowboard); 52: B+; 55: C; 56: B;
   - 57: A; 58: C; 59: C, D+; 60: A+, drop; 62: drop;
   - 65: drop; 68: drop; 70: D+, drop (snowboard);
   - 72: no credit; 73: drop; 79: C+

   a. What is the probability that a student passed the course with a C or better given that the student’s load factor was 61 or higher? 
      \( \frac{1}{3} \)
   b. What is the probability that a student passed the course with a C or better given that the student’s load factor was between 46 and 61? 
      \( \frac{13}{15} \)
   c. What is the probability that a student passed the course given that the student had a hobby of skiing or snowboarding? 
      \( \frac{5}{6} \) or 0
   d. Given that a student got an A in the course, what is the probability that the student’s load factor was 61 or less? 
      \( \frac{1}{3} \)
   e. What is the probability that a student got a C or better in the course? 
      \( \frac{10}{10} \)

20. A telephone answering system has 8 incoming lines, which are equally likely to be reached. Three are tech lines, answered by technical support personnel, and 5 are message lines. The system is programmed incorrectly so that after the message, the caller re-enters the system as if from the beginning (3 tech and 5 message lines).
   a. Draw a tree diagram showing up to three stages. 
   b. What is the probability that the caller will hear the message three times? 
      \( \frac{15}{125} \)
   c. What is the probability of reaching tech support at least by the third stage? 
      \( \frac{9}{125} \)

21. Thirteen-digit International Standard Book Numbers (ISBNs) are assigned to published books. Using the digits 0 to 9, how many ISBNs are possible if numbers can be repeated? 
   \( 10^{13} \)

22. What is the value of each of these expressions?
   a. \( 9! \) 362,880
   b. \( \gamma C_3 \) 35
   c. \( \gamma C_5 \) 6
   d. \( \gamma P_4 \) 840
   e. \( 20C_{12} \) 125,970
   f. \( \gamma P_6 \) 20,160

23. The character Yoda in Star Wars Episode I: The Phantom Menace exemplifies wisdom. His sentence structure, like that of Shakespeare, is not in today’s ordinary form.
   a. List 6 permutations of the seven words in this statement by Yoda: “Hard to see the dark side is.” 
      See Answer Section.
   b. How many 7-word permutations are possible? 
      \( \gamma P_7 = 5040 \)
24. Another sentence by Yoda (in the same scene as above) is “Revealed your opinion is.”
   a. List 6 permutations of the 4 words. See Additional Answers.
   b. How many 4-word permutations are possible? $P_4 = 24$

In Exercises 25 and 26, tell whether problem involves combinations (C) or permutations (P) and answer the questions.
25. GRCC, a college with a 15-member mathematics department, has a meeting chair, a resource center director, a faculty council member, and a union representative.
   a. How many ways can these positions be selected? C; 5!
   b. How many 3-person committees can be made up from the 15 members? C; 455

26. The HBO channel has proposed 15 new programs for the fall.
   a. How many ways can they be ranked by focus groups? P; 15!
   b. How many ways can they be assigned to groups of 3 for showing to prospective advertisers? C; 1001

In Exercises 27 and 28, tell whether the problem involves the fundamental counting principle (FCP) only, permutations (P), or combinations (C), and solve the problem.
27. A delivery company has a fleet of 10 cars.
   a. How many different ways can a group of 3 cars be selected for washing? C; 120
   b. How many ways can the cars be driven once to the gas station? P; 10!
   c. How many ways can 3 cars be selected to appear on pages 1 to 3 of a sales catalogue? P; 720
   d. How many different groups of 4 cars can be smashed by a compactor? C; 210
   e. Five deliveries are needed. How many ways can the cars be selected for a delivery if the cars can be driven more than once? FCP; $10^5 = 100,000$

28. Eight cats are taken in separate cages to a cat show.
   a. How many ways can the cat cages be placed on a viewing stand? P; $P_8 = 40,320$
   b. How many different groups can be selected as “best group of three” in the show? C; $C_8 = 56$
   c. How many ways can 3 cats be petted if the cats can be petted more than once? FCP; $8^3 = 512$
   d. How many ways can the cats be awarded first, second, and third place? P; $P_3 = 336$

29. A Pick 3 lottery requires picking three numbers from 0 to 9 in any combination.
   a. What is the probability of a match? $\frac{1}{1000}$
   b. What is the probability of the same numbers but in a different order? $\frac{1}{120}$
   c. Is the word combination in this setting the same as $nC_r$? no

30. An antique toy bank has 3 dials with the numbers 1 to 8. The numbers that open the safe may be set to any digit on any dial.
   a. What is the probability of guessing the correct combination? $\frac{1}{8^3} = \frac{1}{512}$
   b. What is the probability of guessing the correct combination if none of the numbers are repeated? See below
   c. Is the word combination in this setting the same as $nC_r$? no

31. Suppose one political party has 218 members of the U.S. House of Representatives; 218 is one more than half the total members and permits the party to elect the Speaker of the House and to have a majority on every committee in the House. The Speaker appoints 8 members of his or her party to the House Ways and Means Committee. In how many ways can the 8 members be appointed from the 218 members of that political party? Assume that order of selection does not matter. $\binom{218}{8} = 1.11 \times 10^{14}$

32. Twelve students apply for a set of community service awards. The top four students selected will each receive $4125.
   a. In how many ways can the awards be given? $nC_r = 495$
   b. In how many ways can the students be ranked first through fourth? Why is the answer different from the one in part a? 11,880; combination has a factor of 4! in the denominator.

33. Suppose the numbers below represent a row of Pascal’s triangle (the middle three numbers are fake).
   1 6 14 22 14 6 1
   a. Copy the row and write the next row below it.
   1, 7, 20, 36, 36, 20, 7, 1
   b. Does the new row you made add up to the correct number for the corresponding row in Pascal’s triangle? yes; $128 = 2^7$
   c. Use the original fake row to find $\binom{n}{4}$
   d. Use the original fake row to find $P(3$ boys and $3$ girls in a family of $6$ children). $\frac{1}{11}$
   e. Explain why the first number in the row is a 1.

34. Suppose the numbers below represent a row of Pascal’s triangle (the middle five numbers are fake).
   1 8 30 53 72 53 30 8 1
   a. Copy the row and write the next row below it.
   1, 9, 38, 83, 125, 125, 83, 38, 9, 1
   Ans. 30. $\frac{1}{8 \cdot 7 \cdot 6} = \frac{1}{336}$
b. Does the new row you made add up to the correct number for the corresponding row in Pascal’s triangle? Yes; $512 = 2^9$

c. Use the original fake row to find $3C_1$. 53

d. Use the original fake row to find $P(4$ boys and 4 girls in a family of 8 children). $\frac{35}{\binom{8}{2}}$

e. Explain why the numbers are the same on either side of 72. Value of $(n-r)!$ in formula for $C_r$ is same for 3 and 5.

35. Write a formula showing the value of $C_r$. Show the first three and last three entries in the corresponding row of Pascal’s triangle. See below.

36. Write a formula showing the value of $C_y$. Show the first three and last three entries in the corresponding row of Pascal’s triangle. See below.

37. Project: Shifty Marbles. Two boxes containing marbles are shown in the figure. The experiment is to draw a marble from box 1 and place it in box 2, and then draw a marble from box 2.

\begin{center}
\begin{tabular}{cc}
\text{Box 1} & \text{Box 2} \\
\begin{tabular}{c}
\end{tabular} & \begin{tabular}{c}
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\end{tabular}
\end{center}

\begin{enumerate}
\item a. Draw a tree diagram for this situation.
\item b. What is the probability that the last ball drawn is white? $\frac{11}{34}$
\item c. What is the probability that the last ball drawn is black? $\frac{13}{34}$
\end{enumerate}

38. Project: Which Way Will It Land? Design an empirical probability experiment involving a cap from a tube of toothpaste or from a bottle of hotel shampoo or hand lotion. Suggest an appropriate number of trials to establish the probability of the object landing on its top, side, or bottom. Discuss the relevance of the law of large numbers. Carry out your experiment and report your findings.

\begin{enumerate}
\item Ans. 35. $\binom{n}{r} = \frac{a!}{(a-r)!b!}; 1, a, \frac{a!}{(a-2)!} \ldots \frac{a!}{(a-2)!}, a, 1$
\item Ans. 36. $\binom{x}{y} = \frac{x!}{(x-y)!y!}; 1, x, \frac{x!}{(x-2)!} \ldots \frac{x!}{(x-2)!}, x, 1$
\end{enumerate}

39. Project: Lunch Spinner. A number of years ago, a probability investigation resulted when students at an elementary school in New York state decided to change their lunch menu schedule. This school had a set lunch for each day of the week. Every Monday the students had sub sandwiches; Tuesday they had hamburgers; Wednesday, a chicken patty; Thursday, pizza; and Friday, hot dogs. The students decided they would like to have pizza more often and devised a scheme that they believed would give them more pizza. They divided a spinner into 5 equal areas, with each area representing one of the 5 menu items. Instead of the predictable schedule above, they got to spin for the lunch to be served the next day!

The question is, Did they get more pizza over 100 days than they would have if they had just kept the boring old predictable schedule? To find out, answer these questions.

\begin{enumerate}
\item a. Which would you choose: pizza once a week for sure or taking a chance and spinning for more? once a week
\item b. Why did you choose what you did? definitely get pizza at least once a week
\item c. What is your prediction for how many times the spinner will land on pizza (out of 5 spins)? Why? 1 out of 5
\item d. What is the probability that the spinner will land on pizza? $\frac{1}{5}$
\item e. Model the problem with the spinner in the app Prob Sim. Choose Spin Spinner, press Set, and change Sections to 5. Press OK to save your changes. Spin for a total of 100 days, noting that pressing SPIN gives 1 spin. You will need to use a combination of 1, 10, and 50 spins to obtain exactly 100 spins. Answers will vary. In 100 spins, one author found 15 sub days, 20 hamburger days, 22 chicken days, 22 pizza days, and 21 hot dog days.
\item f. How many times out of 100 days will the regular schedule have pizza? about 20, depending on holidays
\item g. From your results calculate the experimental probability of getting pizza. How does this compare with the probability calculated in part d? Why is there a difference? Results will vary. Without a very large number of spins, pizza could end up less frequent, as with subs on the author’s trial.
\item h. What events in a school calendar make pizza more likely than subs and hot dogs with the regular schedule? Many school holidays occur on Monday and Friday, and Thanksgiving is on Thursday.
1. A letter is drawn at random from ILLINOIS. Find these probabilities.
   a. \( P(I) \) \( \frac{3}{8} \)
   b. \( P(L \text{ or } I) \) \( \frac{5}{8} \)
   c. \( P(\text{letter is also in SNOW}) \) \( \frac{3}{8} \)

2. All other things being equal, are the numbers 1 to 31 equally likely to occur as birthdates? Explain.

3. A box contains 3 white, 1 red, and 2 green candies.
   a. \( P(\text{red}) \) \( \frac{1}{6} \)
   b. \( P(\text{green}) \) \( \frac{1}{3} \)
   c. \( P(\text{not white}) \) \( \frac{1}{2} \)
   d. \( P(\text{red or green}) \) \( \frac{1}{2} \)
   e. \( P(\text{blue}) \) \( 0 \)

4. Suppose a second candy is drawn from the box in Exercise 3. (Of course the first candy was eaten!)
   a. Draw a tree diagram for the two drawings. See Answer Section.
   b. \( P(\text{red on either draw}) \) \( \frac{1}{3} \)
   c. \( P(\text{red on both draws}) \) \( 0 \)
   d. \( P(\text{green on either draw or the outcome WG}) \) \( \frac{1}{2} \)

5. Nicole packs 3 blouses (A, B, C) and 2 skirts (D, E).
   a. Draw a tree diagram showing the possible outfits she can wear using one of each. See Answer Section.
   b. List the sample space for the outfits. AD, AE, BD, BE, CD, CE
   c. If event \( M \) is wearing blouse C or skirt E, what is event (not \( M \))? AD or BD
   d. If event \( N \) is wearing either blouse A or blouse C, what is event (not \( N \))? BD or BE
   e. What is the probability of wearing blouse A or blouse C and skirt E? \( \frac{2}{5} \)
   f. What is the probability of wearing blouse A and skirt E? \( \frac{1}{5} \)

6. A bookshelf in a summer cabin holds 15 novels and 10 mysteries. How many ways could a visitor choose
   a. a mystery or a novel? \( 10 + 15 = 25 \)
   b. a mystery and then a novel? \( 10 \cdot 15 = 150 \)
   c. a novel and then another novel? \( 15 \cdot 14 = 210 \)

If a book is chosen at random, what is
   d. \( P(\text{novel}) \) \( \frac{15}{25} = \frac{3}{5} \)
   e. \( P(\text{either a novel or a mystery}) \) \( 1 \)
   f. If one book is chosen at random and then another, what is \( P(\text{novel and then another novel})? \) \( \frac{3}{10} \)

7. Private aircraft registration numbers in the United States start with N.
   a. How many numbers can be made with N followed by 1 number and 2 letters (omitting the letter O)? 6250
   b. How many numbers can be made with N followed by 5 digits? \( 10^5 = 100,000 \)

8. Which values can be calculated mentally? Why? Set up the formula for the remaining expressions.
   \( \binom{n}{0}, \binom{n}{n} \) equal 1; \( \binom{n}{1} \) and \( \binom{n}{n-1} \) equal \( n \).
   a. \( \binom{52}{1} \) 52
   b. \( \binom{52}{52} \) \( \frac{52!}{0!} \)
   c. \( \binom{52}{1} \) 1
   d. \( \binom{52}{0} \) 1
   e. \( \binom{52}{51} \) 52

9. State the permutation, combination, or fundamental counting principle required for each setting.
   a. A load of laundry holds 7 shirts. How many different loads are possible with 10 shirts? \( _{10}C_7 \)
   b. How many ways can 10 shirts be worn during a 5-day work week without repetition? Assume 1 shirt per day. \( _{10}P_5 \)
   c. How many ways can the shirts be worn during a 5-day work week with repetition? \( _{10}P_5 \)
   d. How many ways are there to hang 7 of the 10 shirts in a closet? \( _{10}P_7 \)
   e. How many ways are there to select 3 of the 10 shirts to toss into a drawer? \( _{10}C_3 \)

10. Illinois Little Lotto requires 5 numbers selected from 1 to 30. What is the probability of a match?
    \( \frac{1}{\binom{30}{5}} = \frac{1}{142,506} \)

11. Suppose the numbers below represent a row of Pascal’s triangle (the middle four numbers are fake).
    1 7 20 36 26 10 7 1
    a. Copy the row and write the next row below it.
       1, 8, 27, 56, 72, 56, 27, 8, 1
    b. Does the new row you made add up to the correct number for the corresponding row in Pascal’s triangle? Yes; \( 2^9 = 512 \)

c. Use the original fake row to find \( \binom{5}{2} \).

d. Use the original fake row to find \( P(3 \text{ boys and } 4 \text{ girls in a family of } 7 \text{ children}) \).

e. Explain how you chose the numerator in part d.
Row represents 0, 1, 2, 3, … boys, so fourth number is 36.
f. Explain how you found the denominator in part d.
The total of the row is \( 2^7 = 128 \).

12. The richness of Shakespeare’s language may be due to the structure of his phrases. Sonnet 48 begins “How careful was I, when I took my way, ….”

a. List the permutations of the three words “careful was I.”
“careful was I, careful I was, I was careful, I careful was, was I careful, was careful I.”
b. Which is likely to be used today? Explain.
“I was careful” is a likely answer, but answer could vary with explanation.
Answers to Selected Odd-Numbered Exercises and Test

CHAPTER 8

Exercises 8.1

1. a. GG, GB, BG, BB
   b. GGG, GGB, GBB, BGG, BGB, BBG, BGG, GBB, GBG, BGBG, BBGB, BGBG, GBBG, BGBB, BBGB, BGBB, BBBG, BBGG, BBGB, BBGG
   c. GG, GB, BB, GGGB, GGBB, GBBB, BBGG, BBGB, BGBB, GBGB, BBGGB, BGBGB, GBGBB, BGBBB, GBGBB, BGBGB, GBGBB, BGBGB, GBGBB, BGBGB
   3. BCG, BGC, CGB, GBC, GCB
   5. a. [4, 5, 6] b. [1, 2, 3, 4, 5, 6] c. [1, 4] d. [2, 3, 4, 5, 6]
   7. a. (HH, HT, TH) b. (TT, HT, HT) c. (HH)
   d. (TH, HT, TT)
   9. \( \frac{5}{7} \) 11. a. \( \frac{8}{17} \) b. \( \frac{7}{17} \) c. \( \frac{11}{17} \) d. \( \frac{4}{17} \) e. \( \frac{7}{17} \)
   13. a. \( \frac{1}{11} \) b. \( \frac{2}{11} \) c. \( \frac{5}{11} \) d. \( \frac{11}{11} \) e. \( \frac{5}{11} \)

15. Answers in order from left to right, top to bottom: boy, \( \frac{1}{2} \), \( \frac{1}{2} \); GG, \( \frac{1}{4} \), \( \frac{3}{4} \); BB, \( \frac{3}{4} \), \( \frac{1}{4} \); \{2, 4\}, \( \frac{3}{5} \); \{TH, HT, TT\} \( \frac{4}{6} \), \( \frac{5}{6} \), \( \frac{3}{6} \), \( \frac{1}{6} \); \{HH, HT, TH\} \( \frac{2}{6} \), \( \frac{4}{6} \), \( \frac{2}{6} \); \{4, 5, 6\} \( \frac{1}{6} \), \( \frac{2}{6} \), \( \frac{3}{6} \), \( \frac{1}{6} \);

17. a. \( \frac{5}{6} \) b. \( \frac{1}{2} \) c. \( \frac{5}{6} \) d. \( \frac{0}{6} \) e. \( \frac{5}{6} \) f. \( \frac{3}{4} \) g. \( \frac{0}{6} \)

21. a. \( \frac{4}{32} \) b. \( \frac{13}{32} \) c. \( \frac{12}{32} \) d. \( \frac{40}{32} \) e. \( \frac{50}{32} \) f. \( \frac{0}{6} \)

23. a.  

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b. \( \frac{7}{36} \); c. \( \frac{27}{36} \) or \( \frac{3}{4} \); d. \( \frac{21}{36} \) or \( \frac{7}{12} \)

25. \( P(\text{measurable precipitation on parade day}) = \frac{49}{139} \approx 35.3\% \);

\( P(\text{measurable snow on parade day}) = \frac{5}{139} = 5.8\% \)

27. a. \( \frac{16}{29.3} = 0.573 \); b. \( \frac{4.2}{29.3} = 0.143 \); c. \( \frac{5.3}{29.3} = 0.181 \); d. \( \frac{1.3}{29.3} = 0.044 \)

29. \( \frac{100}{115} = 87\% \)

31. a. \( \frac{1}{2} = 0.20 \); b. \( P(A) = 0.21 \), \( P(B) = 0.22 \), \( P(C) = 0.16 \), \( P(D) = 0.21 \), \( P(E) = 0.2 \); c. \( P(A) = 0.21 \), \( P(B) = 0.208 \), \( P(C) = 0.172 \), \( P(D) = 0.192 \), \( P(E) = 0.218 \)

d. Larger experiment has slightly less variation from the theoretical probability of 0.20.

33. domain: any event; range: any real number \( 0 \leq p \leq 1 \)

35. \( \frac{3}{10,000} \)

Exercises 8.2

1. a. \( \frac{11}{36} \) b. \( \frac{11}{36} \) c. \( \frac{1}{18} \) d. \( \frac{5}{9} \) e. \( \frac{4}{9} \)

3. a. \( \frac{1}{2} \) b. \( \frac{3}{2} \) c. \( \frac{1}{2} \) d. \( \frac{7}{18} \) e. \( \frac{25}{52} \)

5. a. \( \frac{1}{2} \) b. \( \frac{3}{4} \) c. \( \frac{1}{6} \) d. \( \frac{1}{2} \) e. \( \frac{1}{2} \)

7. 0

9. a. 

b. HA: \( \frac{1}{8} \) HB: \( \frac{1}{2} \) HC: \( \frac{1}{8} \) TA: \( \frac{1}{8} \) TB: \( \frac{1}{8} \) TC: \( \frac{1}{8} \)

d. \( \frac{5}{8} \) e. \( \frac{3}{4} \) f. \( \frac{1}{8} \)

11. a. 

b. \( \frac{32}{153} \) c. \( \frac{8}{51} \) d. \( \frac{28}{153} \) e. \( \frac{2}{51} \)

13. a. independent b. dependent c. independent d. dependent e. independent

15. a. independent b. \( \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} \) c. \( \frac{1}{4} \times \frac{1}{8} = \frac{1}{32} \)

17. a. independent; the outcome of spinner 1 does not affect the outcome of spinner 2. b. \( P(\text{odd on both}) = P(\text{odd on first}) \times P(\text{odd on second}) = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64} \) c. \( P(\text{even on both}) = P(\text{even on first}) \times P(\text{even on second}) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{32} \)

19. a. dependent; contents of bowl has changed. b. \( \frac{9}{32} \times \frac{9}{32} = \frac{9}{32} \) c. \( \frac{1}{2} \times \frac{1}{8} \times \frac{1}{8} = \frac{1}{32} \)

A-1
Exercises 8.3

1.

3.

5. \(10^3 = 1,000,000,000\) 7. \(4^{20}\)
9. a. \(40^3 = 64,000\) b. 59,280
11. a. \(\frac{1}{1000}\) b. no; although order counts, numbers are repeated.
13. a. 1352 b. 35,152 c. 36,504
15. a. 17 b. 45 c. 19,800 d. 27 e. 2730 f. 75
17. a. 160 b. 800 c. \(8^3 \times 10^3\) d. 800 \(\times 799 \times 10^4\)
19. a. 40,320 b. 3,628,800 c. 1 d. 3,628,800
e. 40,322 f. 720
21. \(6! = 720\); also FCP 23. \(7^7 = 823,543\); FCP
25. \(P_2; ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc\)
27. \(C_2; bc, bd, cd\) 29. a. 120 b. 120 c. 12
31. a. 15 b. 6 c. 495 33. a. 60 b. 125
35. 1000 with repetition 37. different sets; 120
39. combinations; 12,650
41. a. FCP; 1,679,616 b. permutations; 1,413,720
c. combinations; 58,905 d. permutations; 24
43. a. combinations; 153 b. permutations; 362,880
c. permutations; \(6.402 \times 10^{13}\)
45. a. combinations; 1,623,160 b. combinations and FCP;
125,400
47. \(624_{C_5}\); Example 10
49. a. 1716 b. 177,100 c. 38,610 d. 0.00289

Exercises 8.4

1. 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1
3. \(128 = 2^7\); 256 = \(2^8\)
5. 1, 5, 10, 10, 5, 1
7. 1, 12, 66, 220, 495, 792, 924, 792, . . .
9. a. 35 b. 15 c. 6 11. a. 20 b. 1 c. 1 d. 16
13. The first two are 1 and \(n\), and the last two are \(n\) and 1.
15. because the first number represents \(r = 0\), not \(r = 1\)
17. HH, HT, TH, TT; 1, 2, 1
19. yes; 0 head in 1 way, 1 head in 4 ways, 2 heads in 6 ways,
3 heads in 4 ways, 4 heads in 1 way
21. \(\frac{3}{5}\) because TTT also is included.
23. \(\frac{5}{32}\) 25. \(\frac{1}{32}\)
27. \(\frac{3}{32}\) 29. \(2^{12} = 4096\) 31. \(\frac{33}{2048}\) 33. \(\frac{99}{3125}\) 35. \(\frac{33}{2048}\)
37. \(\frac{12C_{10}}{12C_2}\) 39. a. 10 ways b. 20 ways c. 21 ways d. 66 ways

Chapter 8 Review Exercises

1. a. \{O, E\} b. \{P, R, B, L, M\} c. \(\frac{3}{5}\) d. \(\frac{4}{11}\) e. \(\frac{5}{11}\)
3. a. 2 b. 4 c. 8 d. 16
5. Answers from left to right, top to bottom: \{HT, TH\}, \(\frac{1}{3}\) \(\frac{1}{2}\);
\{HH, HT, TH\}, \(\frac{1}{2}\) \(\frac{1}{6}\); odd numbers, \(\frac{1}{2}\) \(\frac{1}{3}\); white, blue,
\(\frac{2}{5}\) \(\frac{3}{5}\); \{b, d\}, \(\frac{1}{2}\) \(\frac{1}{3}\); \{a, b, c\}, \(\frac{1}{2}\) \(\frac{1}{6}\)
7. 0.267, 0.4, 0.333; empirical; might want to try 1000 times
9. a. AP; 1 b. MP; \(\frac{1}{4}\)
11. \(\frac{7}{12}\)
13. a. overlap; 15 b. overlap; 14 c. overlap; 20
d. no overlap; \(\frac{2}{3}\) e. overlap; \(\frac{3}{5}\) f. no overlap; \(\frac{2}{5}\)
15. a. 

b. \(\frac{4}{27}\) c. \(\frac{65}{81}\) d. \(\frac{65}{81}\) e. \(\frac{25}{81}\)
17. a. 

b. \(\frac{4}{27}\) c. \(\frac{65}{81}\) d. \(\frac{65}{81}\) e. \(\frac{25}{81}\)
23. a. Hard to see the dark side is. To see the dark side is hard. The dark side is hard to see. Hard the dark side to see is. Hard the dark side is to see. Hard to see is the dark side.
   b. 5040
25. a. P: 32,760   b. C: 455
   e. FCP: 100,000
29. a. 1/100   b. 1/120   c. no
31. a. 210C6 = 1.11 x 10^14
33. a. 1, 7, 20, 36, 20, 7, 1   b. yes; 128 = 2^7
   c. 14   d. 11/12
   e. number of ways of obtaining 0 items from a set of n items
35. \( C_b = \frac{a!}{(a-b)!b!}; 1, a, \frac{a!}{(a-2)!2!}; \ldots \frac{a!}{(a-2)!2!}, a, 1 \)

Chapter 8 Test
1. a. \( \frac{1}{6} \)   b. \( \frac{1}{5} \)   c. \( \frac{3}{8} \)
2. No; not all months have 31 days.
3. a. \( \frac{1}{6} \)   b. \( \frac{1}{5} \)   c. \( \frac{3}{8} \)   d. \( \frac{1}{2} \)   e. 0
4. a.

5. a.

   b. AD, AE, BD, BE, CD, CE  c. AD or BD
   d. BD or BE  e. \( \frac{2}{5} \)  f. \( \frac{1}{10} \)
6. a. 25   b. 150   c. 210   d. \( \frac{3}{5} \)  e. 1   f. \( \frac{1}{10} \)
7. a. 6250   b. 100,000
8. \( \sum_{i=0}^{n} C_a \) and \( \sum_{i=0}^{n} C_a \) equal 1; \( \sum_{i=1}^{C_a} \) and \( \sum_{i=1}^{C_a} \) equal n.
9. a. \( \sum_{i=7}^{10} C_i \)  b. \( \sum_{i=3}^{10} C_i \)  c. 10^5  d. \( \sum_{i=7}^{10} C_i \)  e. \( \sum_{i=7}^{10} C_i \)
10. \( \frac{1}{30C_3} = \frac{1}{142,506} \)
11. a. 1, 8, 27, 56, 56, 27, 8, 1   b. yes; 256 = 2^8  c. 20
   d. \( \frac{36}{128} = \frac{9}{12} \)
   e. Row represents 0, 1, 2, 3, \ldots boys, so fourth number is 36.
   f. Total of the row is 2^7 = 128.
12. a. careful was I, careful I was, I was careful, I careful was, was I careful, was careful I
   b. “I was careful” is a likely answer, but answer could vary with explanation.
Exercises 8.1
2. HHHH  HHHT  HHTT  HTTT  TTTT
   HHTH  HTHT  THTT
   HTHH  HTTH  TTHT
   THHH  TTTH  THTH
24. a.

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36. g. There are more different sums. There are up to ten ways
to get a difference of 1. The most for any given sum is
six ways. The number of ways to get sums is symmet-
ic; the differences are not symmetric.
h. (1, 6) and (6, 1) have the same outcome. The sample
space for looking at both sums and differences contains
36 sets of numbers.

Exercises 8.2
10.

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Exercises 8.3
2.
51. h. There would be one way to get 6 heads and 0 tails in 6 trials. There would be 6 ways to get 5 heads and 1 tail in 6 trials. There would be $6!/[6-r](r!)$ ways to get $6-r$ heads and $r$ tails in 6 trials. There would be $n!/[n-r](r!)$ ways to get $n-r$ heads and $r$ tails in $n$ trials.

Chapter 8 Review Exercises

16. a.

18. a.

20. a. 

24. a. Revealed your opinion is (original). Revealed is your opinion. Your opinion is revealed. Your opinion revealed is. Is revealed your opinion. Is your opinion revealed. (The latter sounds like a question.)

37. a.