Appendix F: Systems of Inequalities

F.1 Solving Systems of Inequalities

The Graph of an Inequality

The statements $3x - 2y < 6$ and $2x^2 + 3y^2 \geq 6$ are inequalities in two variables. An ordered pair $(a, b)$ is a solution of an inequality in $x$ and $y$ for which the inequality is true when $a$ and $b$ are substituted for $x$ and $y$, respectively. The graph of an inequality is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the corresponding equation. The graph of the equation will normally separate the plane into two or more regions. In each such region, one of the following must be true.

1. All points in the region are solutions of the inequality.
2. No point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality by simply testing one point in the region.

Example 1 Sketching the Graph of an Inequality

Sketch the graph of $y \geq x^2 - 1$ by hand.

Solution

Begin by graphing the corresponding equation $y = x^2 - 1$, which is a parabola, as shown in Figure F.1. By testing a point above the parabola $(0, 0)$ and a point below the parabola $(0, -2)$, you can see that $(0, 0)$ satisfies the inequality because $0 \geq 0^2 - 1$ and that $(0, -2)$ does not satisfy the inequality because $-2 \not< 0^2 - 1$. So, the points that satisfy the inequality are those lying above and those lying on the parabola.

Now try Exercise 13.

The inequality in Example 1 is a nonlinear inequality in two variables. Most of the following examples involve linear inequalities such as $ax + by < c$ ($a$ and $b$ are not both zero). The graph of a linear inequality is a half-plane lying on one side of the line $ax + by = c$. 

CHECKPOINT Now try Exercise 13.

Figure F.1
Example 2  Sketching the Graphs of Linear Inequalities

Sketch the graph of each linear inequality.

a. \( x > -2 \)   b. \( y \leq 3 \)

Solution

a. The graph of the corresponding equation \( x = -2 \) is a vertical line. The points that satisfy the inequality \( x > -2 \) are those lying to the right of (but not on) this line, as shown in Figure F.2.

b. The graph of the corresponding equation \( y = 3 \) is a horizontal line. The points that satisfy the inequality \( y \leq 3 \) are those lying below (or on) this line, as shown in Figure F.3.

Figure F.2  Figure F.3

Now try Exercise 19.

Example 3  Sketching the Graph of a Linear Inequality

Sketch the graph of \( x - y < 2 \).

Solution

The graph of the corresponding equation \( x - y = 2 \) is a line, as shown in Figure F.4. Because the origin \((0, 0)\) satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line. Regardless of which point below the line you choose, you will see that it does not satisfy the inequality.)

To graph a linear inequality, it can help to write the inequality in slope-intercept form. For instance, by writing \( x - y < 2 \) in Example 3 in the form

\[ y > x - 2 \]

you can see that the solution points lie above the line \( y = x - 2 \) (or \( x - y = 2 \)), as shown in Figure F.4.
Systems of Inequalities

Many practical problems in business, science, and engineering involve systems of linear inequalities. A solution of a system of inequalities in \( x \) and \( y \) is a point \((x, y)\) that satisfies each inequality in the system.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is common to every graph in the system. For systems of linear inequalities, it is helpful to find the vertices of the solution region.

Example 4  Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system.

\[
\begin{align*}
 x - y &< 2 & \text{Inequality 1} \\
 x &> -2 & \text{Inequality 2} \\
 y &\leq 3 & \text{Inequality 3}
\end{align*}
\]

Solution

The graphs of these inequalities are shown in Figures F.4, F.2, and F.3, respectively. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate system, as shown in Figure F.5. To find the vertices of the region, solve the three systems of corresponding equations obtained by taking pairs of equations representing the boundaries of the individual regions and solving these pairs of equations.

**Vertex A:** \((-2, -4)\)

\[
\begin{align*}
 x - y &= 2 \\
 x &= -2
\end{align*}
\]

**Vertex B:** \((5, 3)\)

\[
\begin{align*}
 x - y &= 2 \\
 y &= 3
\end{align*}
\]

**Vertex C:** \((-2, 3)\)

\[
\begin{align*}
 x &= -2 \\
 y &= 3
\end{align*}
\]

Figure F.5

Note in Figure F.5 that the vertices of the region are represented by open dots. This means that the vertices are not solutions of the system of inequalities.

**CHECKPOINT**  Now try Exercise 51.

Study Tip

Using different colored pencils to shade the solution of each inequality in a system makes identifying the solution of the system of inequalities easier. The region common to every graph in the system is where all shaded regions overlap. This region represents the solution set of the system.
For the triangular region shown in Figure F.5, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure F.6. To keep track of which points of intersection are actually vertices of the region, you should sketch the region and refer to your sketch as you find each point of intersection.

![Figure F.6](image)

**Example 5  Solving a System of Inequalities**

Sketch the region containing all points that satisfy the system of inequalities.

\[
\begin{align*}
    x^2 - y &\leq 1 \quad \text{Inequality 1} \\
    -x + y &\leq 1 \quad \text{Inequality 2}
\end{align*}
\]

**Solution**

As shown in Figure F.7, the points that satisfy the inequality \(x^2 - y \leq 1\) are the points lying above (or on) the parabola given by

\[ y = x^2 - 1. \quad \text{Parabola} \]

The points that satisfy the inequality \(-x + y \leq 1\) are the points lying below (or on) the line given by

\[ y = x + 1. \quad \text{Line} \]

To find the points of intersection of the parabola and the line, solve the system of corresponding equations.

\[
\begin{align*}
    x^2 - y &= 1 \\
    -x + y &= 1
\end{align*}
\]

Using the method of substitution, you can find the solutions to be \((-1, 0)\) and \((2, 3)\). So, the region containing all points that satisfy the system is indicated by the purple shaded region in Figure F.7.

![Figure F.7](image)

**CHECKPOINT**  Now try Exercise 55.
When solving a system of inequalities, you should be aware that the system might have no solution, or it might be represented by an unbounded region in the plane. These two possibilities are shown in Examples 6 and 7.

**Example 6  A System with No Solution**

Sketch the solution set of the system of inequalities.

\[
\begin{align*}
\begin{cases}
  x + y &> 3 &\text{Inequality 1} \\
  x + y &< -1 &\text{Inequality 2}
\end{cases}
\end{align*}
\]

**Solution**

From the way the system is written, it is clear that the system has no solution, because the quantity \((x + y)\) cannot be both less than \(-1\) and greater than \(3\). Graphically, the inequality \(x + y > 3\) is represented by the half-plane lying above the line \(x + y = 3\), and the inequality \(x + y < -1\) is represented by the half-plane lying below the line \(x + y = -1\), as shown in Figure F.8. These two half-planes have no points in common. So the system of inequalities has no solution.

![Figure F.8 No Solution](image)

**CHECKPOINT**  Now try Exercise 57.

**Example 7  An Unbounded Solution Set**

Sketch the solution set of the system of inequalities.

\[
\begin{align*}
\begin{cases}
  x + y &< 3 &\text{Inequality 1} \\
  x + 2y &> 3 &\text{Inequality 2}
\end{cases}
\end{align*}
\]

**Solution**

The graph of the inequality \(x + y < 3\) is the half-plane that lies below the line \(x + y = 3\), as shown in Figure F.9. The graph of the inequality \(x + 2y > 3\) is the half-plane that lies above the line \(x + 2y = 3\). The intersection of these two half-planes is an *infinite wedge* that has a vertex at \((3, 0)\). This unbounded region represents the solution set.

![Figure F.9 Unbounded Region](image)

**CHECKPOINT**  Now try Exercise 59.
Applications

The next example discusses two concepts that economists call consumer surplus and producer surplus. As shown in Figure F.10, the point of equilibrium is defined by the price $p$ and the number of units $x$ that satisfy both the demand and supply equations. Consumer surplus is defined as the area of the region that lies below the demand curve, above the horizontal line passing through the equilibrium point, and to the right of the $p$-axis. Similarly, the producer surplus is defined as the area of the region that lies above the supply curve, below the horizontal line passing through the equilibrium point, and to the right of the $p$-axis. The consumer surplus is a measure of the amount that consumers would have been willing to pay above what they actually paid, whereas the producer surplus is a measure of the amount that producers would have been willing to receive below what they actually received.

Example 8  Consumer Surplus and Producer Surplus

The demand and supply equations for a new type of personal digital assistant are given by

\[
\begin{align*}
  p &= 150 - 0.00001x \\
  p &= 60 + 0.00002x
\end{align*}
\]

where $p$ is the price (in dollars) and $x$ represents the number of units. Find the consumer surplus and producer surplus for these two equations.

Solution

Begin by finding the point of equilibrium by setting the two equations equal to each other and solving for $x$.

\[
\begin{align*}
  60 + 0.00002x &= 150 - 0.00001x \\
  0.00003x &= 90 \\
  x &= 3,000,000
\end{align*}
\]

So, the solution is $x = 3,000,000$, which corresponds to an equilibrium price of $p = 120$. So, the consumer surplus and producer surplus are the areas of the following triangular regions.

\[
\begin{align*}
\text{Consumer Surplus} &\quad \text{Producer Surplus} \\
  p &\leq 150 - 0.00001x \\
  p &\geq 120 \\
  x &\geq 0 & p &\geq 60 + 0.00002x \\
  p &\leq 120 & x &\geq 0
\end{align*}
\]

In Figure F.11, you can see that the consumer and producer surpluses are defined as the areas of the shaded triangles.

- Consumer surplus $= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(3,000,000)(30) = 45,000,000$
- Producer surplus $= \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(3,000,000)(60) = 90,000,000$

Now try Exercise 79.
Example 9 Nutrition

The minimum daily requirements from the liquid portion of a diet are 300 calories, 36 units of vitamin A, and 90 units of vitamin C. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes how many cups of each drink should be consumed each day to meet the minimum daily requirements for calories and vitamins.

Solution

Begin by letting \( x \) and \( y \) represent the following.

\[
\begin{align*}
    x & = \text{number of cups of dietary drink X} \\
    y & = \text{number of cups of dietary drink Y}
\end{align*}
\]

To meet the minimum daily requirements, the following inequalities must be satisfied.

\[
\begin{align*}
    60x + 60y & \geq 300 & \text{Calories} \\
    12x + 6y & \geq 36 & \text{Vitamin A} \\
    10x + 30y & \geq 90 & \text{Vitamin C}
\end{align*}
\]

\[
\begin{align*}
    x & \geq 0 \\
    y & \geq 0
\end{align*}
\]

The last two inequalities are included because \( x \) and \( y \) cannot be negative. The graph of this system of inequalities is shown in Figure F.12. (More is said about this application in Example 6 in Appendix F.2.)

Study Tip

When using a system of inequalities to represent a real-life application in which the variables cannot be negative, remember to include inequalities for this constraint. For instance, in Example 9, \( x \) and \( y \) cannot be negative, so the inequalities \( x \geq 0 \) and \( y \geq 0 \) must be included in the system.

From the graph, you can see that two solutions (other than the vertices) that will meet the minimum daily requirements for calories and vitamins are \((5, 5)\) and \((8, 2)\). There are many other solutions.

Now try Exercise 85.
**Vocabulary and Concept Check**

In Exercises 1–4, fill in the blank(s).

1. An ordered pair \((a, b)\) is a ______ of an inequality in \(x\) and \(y\) for which the inequality is true when \(a\) and \(b\) are substituted for \(x\) and \(y\), respectively.

2. The ______ of an inequality is the collection of all solutions of the inequality.

3. The graph of a ______ inequality is a half-plane lying on one side of the line \(ax + by = c\).

4. The ______ of ______ is defined by the price \(p\) and the number of units \(x\) that satisfy both the demand and supply equations.

**Procedures and Problem Solving**

**Identifying the Graph of an Inequality**  
In Exercises 5–12, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

5. \(x < 2\)

6. \(y \geq 3\)

7. \(2x + 3y \geq 6\)

8. \(2x - y \leq -2\)

9. \(x^2 + y^2 < 9\)

10. \((x - 2)^2 + (y - 3)^2 > 9\)

11. \(xy > 1\)

12. \(y \leq 1 - x^2\)

**Sketching the Graph of an Inequality**  
In Exercises 13–32, sketch the graph of the inequality.

- \(y < 2 - x^2\)
- \(y^2 + 1 \geq x\)
- \(x \geq 4\)
- \(x \leq -5\)
- \(y \geq -1\)
- \(y \leq 3\)
- \(2y - x \geq 4\)
- \(5x + 3y \geq -15\)
- \(2x + 3y < 6\)
- \(5x - 2y > 10\)
- \(4x - 3y \leq 24\)
- \(2x + 7y \leq 28\)
- \(y > 3x^2 + 1\)
- \(y + 9 \geq x^2\)
- \(2x - y^2 > 0\)
- \(4x + y^2 > 1\)
- \((x + 1)^2 + y^2 < 9\)
- \((x - 1)^2 + (y - 4)^2 > 9\)

**Using a Graphing Utility**  
In Exercises 33–44, use a graphing utility to graph the inequality. Use the shade feature to shade the region representing the solution.

- \(y \geq \frac{3}{2}x - 1\)
- \(y < -3.8x + 1.1\)
- \(x^2 + 5y - 10 \leq 0\)
- \(y \leq \frac{1}{1 + x^2}\)
- \(y \geq 6 - \frac{1}{2}x\)
- \(y \leq -20.74 + 2.66x\)
- \(2x^2 - y - 3 > 0\)
- \(y > \frac{-10}{x^2 + x + 4}\)
- \(y < \ln x\)
- \(y \geq 4 - \ln(x + 5)\)
- \(y > 3 - x - 4\)
- \(y \leq 2^{2x-1} - 3\)
Writing an Inequality  In Exercises 45–48, write an inequality for the shaded region shown in the graph.

45.  

46.  

47.  

48.  

Checking Solutions  In Exercises 49 and 50, determine whether each ordered pair is a solution of the system of inequalities.

49.  

50.  

Solving a System of Inequalities  In Exercises 51–68, sketch the graph of the solution of the system of inequalities.

65.  

66.  

67.  

68.  

Writing a System of Inequalities  In Exercises 69–78, find a set of inequalities to describe the region.

69.  

70.  

71.  

72.  

73.  

74.  

75. Rectangle: Vertices at (2, 1), (5, 1), (5, 7), (2, 7)  

76. Parallelogram: Vertices at (0, 0), (4, 0), (1, 4), (5, 4)  

77. Triangle: Vertices at (0, 0), (5, 0), (2, 3)  

78. Triangle: Vertices at (−1, 0), (1, 0), (0, 1)  

Consumer Surplus and Producer Surplus  In Exercises 79–82, (a) graph the regions representing the consumer surplus and producer surplus for the demand and supply equations, and (b) find the consumer surplus and the producer surplus.

\[ \text{Demand} \quad \text{Supply} \]

79. \[ p = 50 - 0.5x \quad p = 0.125x \]

80. \[ p = 100 - 0.05x \quad p = 25 + 0.1x \]

81. \[ p = 300 - 0.0002x \quad p = 225 + 0.0005x \]

82. \[ p = 140 - 0.00002x \quad p = 80 + 0.00001x \]
Solving a System of Inequalities  In Exercises 83–86, (a) find a system of inequalities that models the problem and (b) graph the system, shading the region that represents the solution of the system.

83. Finance  A person plans to invest some or all of $30,000 in two different interest-bearing accounts. Each account is to contain at least $7500, and one account should have at least twice the amount that is in the other account.

84. Arts Management  For a summer concert event, one type of ticket costs $20 and another costs $35. The promoter of the concert must sell at least 20,000 tickets, including at least 10,000 of the $20 tickets and at least 5000 of the $35 tickets, and the gross receipts must total at least $300,000 in order for the concert to be held.

85. Why you should learn it  (p. F1) A dietitian is asked to design a special dietary supplement using two different foods. The minimum daily requirements of the new supplement are 280 units of calcium, 160 units of iron, and 180 units of vitamin B. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B.

86. Retail Management  A store sells two models of computers. Because of the demand, the store stocks at least twice as many units of model A as units of model B. The costs to the store for models A and B are $800 and $1200, respectively. The management does not want more than $20,000 in computer inventory at any one time, and it wants at least four model A computers and two model B computers in inventory at all times.

87. Architectural Design  You design an exercise facility that has an indoor running track with an exercise floor inside the track (see figure). The track must be at least 125 meters long, and the exercise floor must have an area of at least 500 square meters.

88. Geometry  Two concentric circles have radii of $x$ and $y$ meters, where $y > x$ (see figure). The area of the region between the circles must be at least 10 square meters.

(a) Find a system of inequalities describing the constraints on the circles.
(b) Graph the inequality in part (a).
(c) Identify the graph of the line $y = x$ in relation to the boundary of the inequality. Explain its meaning in the context of the problem.

Conclusions

True or False?  In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.

89. The area of the region defined by the system below is 99 square units.
\[
\begin{align*}
x &\geq -3 \\
x &\leq 6 \\
y &\leq 5 \\
y &\geq -6
\end{align*}
\]

90. The graph below shows the solution of the system
\[
\begin{align*}
y &\leq 6 \\
-4x - 9y &> 6. \\
3x + y^2 &\geq 2
\end{align*}
\]

91. Think About It  After graphing the boundary of an inequality in $x$ and $y$, how do you decide on which side of the boundary the solution set of the inequality lies?

92. Writing  Describe the difference between the solution set of a system of equations and the solution set of a system of inequalities.