Vectors

Introduction

Although we live in a three-dimensional world, the functions and concepts we have considered in PreCalculus are generally restricted to two dimensions. This is also true in your first calculus courses. Only later in calculus are these concepts extended to higher dimensions.

To gain an understanding of how much adding a third dimension can complicate matters, consider the problem of describing a line that joins two points. If the points lie in an \(xy\)-plane this is quite easily done. In this situation the direction of the line can be described by the slope, that is, the ratio of the difference in the \(y\)-coordinates to the difference in the \(x\)-coordinates. However, we need three coordinates, generally denoted, \(x\), \(y\), and \(z\) to describe three-dimensional space. So for a line joining two points in space we would expect to have differences in the \(x\), \(y\), and \(z\) coordinates. No single ratio would describe the direction of such a line.

This chapter lays the groundwork for studying multi-dimensional problems. The initial step is to consider how points and directions in three-dimensional space are described. The later sections of the chapter show how vectors are used to represent planes and lines in space.
The Rectangular Coordinate System in Space

A three-dimensional coordinate system is needed to describe objects analytically in space. The natural extension of the Cartesian, or rectangular, \( xy \)-coordinate system in the plane is made by simply adding a third coordinate axis, labeled the \( z \)-axis, which is perpendicular to the \( xy \)-plane.

Theoretically and analytically, this is a satisfactory procedure. Graphically, however, problems can arise. The work of M. C. Escher titled “Waterfall,” shown in Figure 1, illustrates in two dimensions a waterfall that is impossible to have in three dimensions. The water in the channel appears to be at a constant level, yet it must also flow uphill. This is an example of the fallacies that can arise from adding a depth perception to a two-dimensional plane. However, the plane is the only vehicle generally available for graphic representation, so we use it as best we can, while being alert for possible contradictory images.

![Figure 1](image-url)

There are many ways to add a perception of depth to the two-dimensional representation of a three-dimensional coordinate system. In your hand-drawn sketches we suggest using a coordinate system constructed in the following manner: Assume the plane determined by the \( y \)- and \( z \)-axes, the \( yz \)-plane, coincides with the plane in which you draw, with the \( y \)-axis horizontal and the \( z \)-axis vertical. Then the positive \( x \)-axis will be directed toward you. Draw the positive \( x \)-axis as a straight line that has the appearance of making an angle of 135° with both the positive \( y \)- and \( z \)-axes, and draw only the positive portion of each axis unless the negative portion is needed for clarity. As an additional aid to the perception of depth, use a scale with a physical length of \( \sqrt{2}/2 \) (approximately 3/4) units along the \( x \)-axis for each unit along the \( y \)- and \( z \)-axes, as shown in
Figure 2. This is a convenient scaling factor, since the diagonal of a unit square has length $\sqrt{2}$, as shown in Figure 3. This relatively simple technique should enable you to sketch any of the figures you are likely to encounter in calculus.

Some of the figures in the text have been computer generated, using sophisticated scaling and rotation techniques. They are generally oriented so that they have a similar appearance to those you will be constructing by hand, but have the additional advantage that you can “see” down both the positive $x$- and $y$-axes.

A point in space is represented in this rectangular coordinate system by an ordered triple called the coordinates of the point. A point with coordinates
(a, b, c) is drawn by sketching the rectangular parallelepiped shown in Figure 4. On this parallelepiped the point (a, b, c) is the vertex diagonally opposite the vertex at the origin (0, 0, 0).

The coordinates of a point in the xy-plane have the form (a, b, 0); similarly, in the xz-plane, (a, 0, c); and in the yz-plane, (0, b, c). The xy-, yz-, and xz-planes (also known as the coordinate planes) divide space into eight sections called octants. The portion of space determined by those points whose coordinates are all positive is called the first octant. The other seven octants are not given names.

EXAMPLE 1 Sketch the points (2, 3, 2) and (4, 4, 3).

Solution These points lie in the first octant and are shown in Figures 5(a) and (b).
Although it is easy to observe the relative position of the points in Example 1 when the parallelepipeds are drawn, note that when the parallelepipeds are removed from the sketch, the points are indistinguishable, as shown in Figure 6.

In fact, the point in this sketch could have coordinates (0, 2, 1), (−2, 1, 0), or even (−100, −48, −49). Although drawing a parallelepiped involves extra work, this example demonstrates that it is the only way to be confident of the graphic representation.

A parallelepiped is also used to find a formula for the distance between two points in space. Suppose $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points in space. Consider the parallelepiped shown in Figure 7 with diagonally opposite vertices at $P_1$ and $P_2$ and sides parallel to the coordinate planes. Let $Q$ be the point whose first two coordinates agree with those of $P_2$ and whose third coordinate
is \(z_1\) (see Figure 7). The line from \(Q\) to \(P_1\) is perpendicular to the line from \(Q\) to \(P_2\), so the Pythagorean Theorem implies that

\[
[d(P_1, P_2)]^2 = [d(P_1, Q)]^2 + [d(Q, P_2)]^2,
\]

where, as usual, \(d\) denotes the distance between the specified points. However,

\[
d(Q, P_2) = |z_2 - z_1|
\]

and

\[
d(P_1, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},
\]

so

\[
[d(P_1, P_2)]^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.
\]

Thus the distance between \(P_1(x_1, y_1, z_1)\) and \(P_2(x_2, y_2, z_2)\) is

\[
d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
\]

(1)

This is the natural extension of the two-dimensional distance formula.

In a similar manner, the midpoint of the line segment joining \(P_1\) and \(P_2\) has the same form as in the plane (see Figure 8). The midpoint of the line segment joining \(P_1(x_1, y_1, z_1)\) and \(P_2(x_2, y_2, z_2)\) is

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).
\]

(2)

**EXAMPLE 2** Find (a) the distance between the points \(P_1(1, -1, 2)\) and \(P_2(3, 4, -1)\) and (b) the midpoint of the line segment joining \(P_1\) and \(P_2\).
Solution  a) The distance is
\[ d(P_1, P_2) = \sqrt{(3 - 1)^2 + (4 - (-1))^2 + (-1 - 2)^2} = \sqrt{4 + 25 + 9} = \sqrt{38}. \]

b) The midpoint of the line segment joining \( P_1 \) and \( P_2 \) is
\[ \left( \frac{1 + 3}{2}, \frac{-1 + 4}{2}, \frac{2 - 1}{2} \right) = \left( 2, \frac{3}{2}, \frac{1}{2} \right). \]

EXAMPLE 3  Find an equation of the sphere with center \((h, k, l)\) and radius \(r\). This is shown in Figure 9.

Solution  A point is on the sphere if and only if the distance from the point to the center \((h, k, l)\) is the constant \(r\). So \((x, y, z)\) lies on the sphere precisely when
\[ r = d((x, y, z), (h, k, l)) = \sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2}. \]

Squaring both sides gives
\[ r^2 = (x - h)^2 + (y - k)^2 + (z - l)^2, \]
the **standard equation of a sphere of radius** \(r\) **with center** \((h, k, l)\).

EXAMPLE 4  Describe geometrically the set of points \((x, y, z)\) that satisfy the equation \(x^2 + y^2 = 1\).

Solution  For every value of \(z\), the equation \(x^2 + y^2 = 1\) describes a circle with radius 1. There is no restriction on \(z\), so \(z\) can assume any real-number value. The equation describes a right circular cylinder whose center is the \(z\)-axis, as shown in Figure 10. The cylinder extends infinitely above and below the \(xy\)-plane.
Exercise Set: Three-Dimensional Space

1. Plot each of the following points and sketch the associated parallelepiped.
   a. (1, 3, 4)
   b. (1, −3, 4)
   c. (2, 4, −3)

2. Plot each of the following points and sketch the associated parallelepiped.
   a. (2, 3, 3)
   b. (−2, 3, 3)
   c. (−2, −3, 3)

In Exercises 3–8, (a) plot the points A and B, (b) find the distance between them, and (c) find the midpoint of the line segment joining points A and B.

3. A(1, 2, 3), B(−1, 3, 4)
4. A(2, 5, 0), B(3, −3, 1)
5. A(−3, 4, 0), B(−3, 4, 2)
6. A(0, 0, 0), B(4, 2, −4)
7. A(1, 8, 3), B(0, 1, 0)
8. A(3, 5, 0), B(5, 3, 0)

In Exercises 9–12, the points A and B are the endpoints of a diagonal of a parallelepiped having its faces parallel to the coordinate planes.

a) Sketch the parallelepiped.

b) Find the coordinates of the other vertices.

9. A(0, 0, 0), B(2, 4, 4)
10. A(2, 2, 0), B(3, 5, 4)
11. A(3, 0, 0), B(4, 3, 5)
12. $A(0, 2, 0), B(2, -2, 4)$
In Exercises 13–16, find an equation of the sphere with center $C$ and radius $r$.
13. $C(0, 0, 0), r = 2$
14. $C(2, 0, 0), r = 2$
15. $C(2, 3, 4), r = 1$
16. $C(0, 3, 0), r = 3$
In Exercises 17–20, find the center and radius of the sphere whose equation is given.
17. $x^2 + y^2 + z^2 + 2x - 4y + 6z = 0$
18. $3x^2 + 3y^2 + 3z^2 + 6x - 3y + 6z = 1$
19. $2x^2 + 2y^2 + 2z^2 + 8x - 8z = -7$
20. $x^2 + y^2 + z^2 - 10x - 6y - 2z + 31 = 0$
21. Find an equation of the sphere having endpoints of a diameter at $(-2, 1, 1)$ and $(1, 4, 5)$.
22. Find an equation of a sphere having center at $(1, 2, 3)$ and passing through the origin.
23. Sketch the parallelepiped consisting of the points $(x, y, z)$ with $2 \leq x \leq 4, 2 \leq y \leq 3,$ and $0 \leq z \leq 5$. Give the coordinates of the eight corners of the parallelepiped.
24. Consider the points $(4, 4, 1), (1, 1, 1),$ and $(0, 8, 5)$.
   a. Show that they are the vertices of a right triangle.
   b. Find an equation of the sphere passing through the three points and having a diameter along the hypotenuse of the triangle.
25. Sketch the points $(2, 4, 2), (2, 1, 5),$ and $(5, 1, 2)$ and show that they are the vertices of an equilateral triangle.
26. The floor of a room is 12 ft by 8 ft and the height of the ceiling is 7 ft. Make a representative sketch of the room in a three-dimensional coordinate system and label the points corresponding to the corners of the room.
27. A dome tent has a circular floor with diameter 8 ft. The height of the tent in the center is 6 ft. Make a representative sketch of the tent in a three-dimensional coordinate system.
Vectors in Space

Certain physical properties are described by stating a magnitude. These are called scalar properties and include, for example, the distance between points, the temperature of a surface, and the volume and mass of a solid. Other properties can be described only by specifying both a magnitude and a direction. Examples of this type are particularly abundant in physics, and include the concepts of force, velocity, acceleration, and momentum.

In the $xy$-plane the direction from a point in the plane can be described by the slope of the line that passes through the point. The slope is the ratio of the change in the $y$-coordinates of point on the line to the change in the $x$-coordinates.

In three-dimensional space we would need three ratios to describe the direction from a point because there are now three pairs of coordinates, the $xy$, $yz$, and $xz$. Instead of proceeding in this manner we use the concept of a vector, and properties that need to be described using both a magnitude and a direction are consequently called vector properties. Later in the section we will see that this concept could also have been used to describe lines in space.

Definition 1. A vector $\mathbf{a}$ in three-dimensional space is an ordered triple of real numbers, written $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. The numbers $a_1$, $a_2$, and $a_3$ are called the components of $\mathbf{a}$ in the $x$, $y$, and $z$-directions, respectively.

The component of the vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ describe changes in the $x$, $y$, and $z$, direction respectively. The magnitude, or length, of the vector $\mathbf{a}$, written $\|\mathbf{a}\|$, is defined by

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$  

(3)

There are some quite natural definitions associated with vectors.

- Two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ are equal if and only if their components are equal, that is,$$
a_1 = b_1, \quad a_2 = b_2, \quad \text{and} \quad a_3 = b_3.
$$

- Any vector of length 1 is called a unit vector.

- The only vector with length 0, the vector $\mathbf{0} = \langle 0, 0, 0 \rangle$, is called the zero vector.
A nonzero vector $a = \langle a_1, a_2, a_3 \rangle$ can be represented geometrically by a directed line segment of length $\|a\|$. Any line segment that begins at an arbitrary point $P(x, y, z)$ and ends at the point $Q(x + a_1, y + a_2, z + a_3)$ represents $a$. (An instance of this is shown in Figure 1.)

Figure 1

When the vector representation begins at the point $P$ and ends at the point $Q$, the point $P$ is called an initial point for the vector $a$, and $Q$ is the corresponding terminal point. This is expressed by writing

$$\overrightarrow{PQ} = a.$$

Figure 2 gives an example of this representation; the arrowhead on the line segment indicates the direction of the vector $a$. 

Figure 2
Boldface type is used in text to distinguish vector quantities from scalar quantities. It is also common practice to write the vector \( \mathbf{a} \) as \( \mathbf{a} \) when boldface is not available.

**EXAMPLE 1**

a) Find the vector with initial point \((1, 2, -1)\) and terminal point \((2, 3, 4)\).

b) Determine the length of this vector.

**Solution**

a) The vector is \( \langle 2 - 1, 3 - 2, 4 - (-1) \rangle = \langle 1, 1, 5 \rangle \).

b) The length of this vector,

\[
\|\langle 1, 1, 5 \rangle\| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27} = 3\sqrt{3},
\]

is the distance between \((1, 2, -1)\) and \((2, 3, 4)\) (see Figure 3).

![Figure 3](image)

**Vectors in the Plane**

The **vectors in the \(xy\)-plane** are simply those whose third coordinate is zero, that is the collection of all vectors of the form \( \langle a_1, a_2, a_3 \rangle \), where \( a_3 = 0 \).

When considering only vectors in the \(xy\)-plane, we often write the vector \( \langle a_1, a_2, 0 \rangle \) as \( \langle a_1, a_2 \rangle \). Results true for general vectors in space are true as well for vectors in the \(xy\)-plane. In fact, vectors in the \(xy\)-plane are often used to illustrate vector concepts when a two-dimensional representation will suffice.

**EXAMPLE 2**

a) Find the position-vector representation for the vector in the \(xy\)-plane with initial point \((3, 4)\) and terminal point \((-2, 1)\).

b) Determine the length of this vector.
Solution  a) If \( \mathbf{a} = \langle a_1, a_2 \rangle \) denotes this vector, then

\[
a_1 = -2 - 3 = -5 \quad \text{and} \quad a_2 = 1 - 4 = -3,
\]

so \( \mathbf{a} = \langle -5, -3 \rangle \). The position-vector representation is shown in Figure 4.

b) The length of \( \mathbf{a} = \langle -5, -3 \rangle \) is the same as the length of the vector \( \langle -5, -3, 0 \rangle \), that is,

\[
\|\mathbf{a}\| = \sqrt{(-5)^2 + (-3)^2} = \sqrt{34}.
\]

Flexible Positions for Vectors

An infinite number of directed line segments represents each nonzero vector since the choice of initial point for a directed line segment is arbitrary. All directed line segments that represent the same vector are related to one another by parallel translations, as shown in Figure 5.
To establish results concerning vectors, it is generally more convenient to use the analytic representation given in Definition 3. However, in most applications we consider vectors as directed line segments that can be moved using parallel translations.

**EXAMPLE 3** Find the terminal point of the vector \( \langle 3, 0, 1 \rangle \) if the initial point is (a) \((-1, 1, 0)\); (b) \((2, e, \pi)\); (c) \((0, 0, 0)\).

![Diagram of vector and points in space](image)

**Solution** The directed line segments representing \( \langle 3, 0, 1 \rangle \) with these initial points are shown in Figure 6.

- **a)** When the initial point is \((-1, 1, 0)\), the terminal point is \((-1+3, 1+0, 0+1) = (2, 1, 1)\).
- **b)** When the initial point is \((2, e, \pi)\) the terminal point is \((5, e, \pi + 1)\).
- **c)** When the initial point is \((0, 0, 0)\) the terminal point is \((3, 0, 1)\).

Associated with each point \(P(x, y, z)\) in space is a unique vector \(\overrightarrow{OP}\) with initial point at the origin \(O\) and terminal point \(P\). This vector is called the **position vector** for the point \(P\).

Every vector \(a = \langle a_1, a_2, a_3 \rangle\) is the position vector for precisely one point in space, that is, the point with coordinates \((a_1, a_2, a_3)\). The directed line segment from the origin \((0, 0, 0)\) to \((a_1, a_2, a_3)\) is called the **position-vector representation** for \(a\).

**EXAMPLE 4** Find the position-vector representation for the vector with initial point \((1, 3, -1)\) and terminal point \((-1, 0, 4)\).
Solution  This vector, \((-1 - 1, 0 - 3, 4 - (-1)) = (-2, -3, 5)\) is the position vector for the point \((-2, -3, 5)\) The position-vector representation is shown in Figure 7.

![Figure 7](image_url)

**Arithmetic Operations on Vectors**

Arithmetic operations on vectors can be defined in a very natural way.

**Definition 2.** If \(a = \langle a_1, a_2, a_3 \rangle\) and \(b = \langle b_1, b_2, b_3 \rangle\) are vectors and \(\alpha\) is a real number, then

- **i)** \(a + b = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle\);  
- **ii)\) \(a - b = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle\);  
- **iii)\) \(\alpha a = \langle \alpha a_1, \alpha a_2, \alpha a_3 \rangle\).

Definitions 2(i) and 2(ii) are called **vector addition** and **vector subtraction**, respectively, and 2(iii) is known as **scalar multiplication**. The vector sum \(a + b\) is called by some authors the **resultant vector** of \(a\) and \(b\).

Vector addition can be described geometrically as follows. Suppose \(a\) and \(b\) are two vectors drawn so that the terminal point of \(a\) and the initial point of \(b\) coincide. With the vectors in this position, \(a + b\) is the vector from the initial point of \(a\) to the terminal point of \(b\), as illustrated in Figure 8(a). Since \(b + (a - b) = a\), this description of addition enables us to represent vector subtraction geometrically, as shown in Figure 8(b).
Scalar multiplication by a real number $\alpha$ has the effect of compressing, when $|\alpha| < 1$, or expanding, when $|\alpha| > 1$, the length of the vector. In addition, when $\alpha$ is positive, $\alpha \mathbf{a}$ has the same direction as $\mathbf{a}$, but when $\alpha$ is negative, $\alpha \mathbf{a}$ and $\mathbf{a}$ have opposite directions (see Figure 8(c)).

**EXAMPLE 5** Find the sum and difference of the vectors $\mathbf{a} = \langle 3, 1, 0 \rangle$ and $\mathbf{b} = \langle 2, 4, 0 \rangle$ and illustrate this sum geometrically.

**Solution** We have

$$\mathbf{a} + \mathbf{b} = \langle 3, 1, 0 \rangle + \langle 2, 4, 0 \rangle = \langle 5, 5, 0 \rangle$$

and

$$\mathbf{a} - \mathbf{b} = \langle 3, 1, 0 \rangle - \langle 2, 4, 0 \rangle = \langle 1, -3, 0 \rangle.$$

Since these vectors lie in the $xy$-plane, they can be represented as shown in Figure 9.

The arithmetic vector results given on the next page follow from Definition 2.

**Theorem 1.** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and $\alpha$ and $\beta$ are real numbers, then
(i) \( a + b = b + a; \)

(ii) \( a + (b + c) = (a + b) + c; \)

(iii) \( \alpha (\beta a) = \beta (\alpha a); \)

(iv) \( (\alpha + \beta) a = \alpha a + \beta a; \)

(v) \( a + 0 = a; \)

(vi) \( 0(a) = 0; \)

(vii) \( 1(a) = a; \)

(viii) \( \|\alpha a\| = |\alpha| \cdot \|a\|. \)

The proofs of these results follow by applying properties of the real numbers to the components of the vectors. For example, if \( a = \langle a_1, a_2, a_3 \rangle, \ b = \langle b_1, b_2, b_3 \rangle, \) and \( c = \langle c_1, c_2, c_3 \rangle, \) then

\[
a + (b + c) = \langle a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), a_3 + (b_3 + c_3) \rangle
\]

\[
= \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2, (a_3 + b_3) + c_3 \rangle
\]

\[
= (a + b) + c.
\]

Other parts of this theorem are considered in the exercises.

Scalar multiplication provides a precise manner for defining parallel vectors.

**Definition 3.** Nonzero vectors \( a \) and \( b \) are **parallel** if there is a real number \( \alpha \) such that \( a = \alpha b. \) (We also say that the zero vector is parallel to every vector \( a. \))

Any nonzero vector \( a \) can be multiplied by the reciprocal of its length, \( 1/\|a\|, \) to produce the unit vector, \( a/\|a\|, \) which has the same direction as \( a. \) The vector \( -a/\|a\| \) is the unit vector in the direction opposite to \( a. \)

**EXAMPLE 6** Find a unit vector that has the same direction as \( a = \langle 2, -3, 6 \rangle \) and a unit vector in the opposite direction.

**Solution** Since \( \|a\| = \sqrt{4 + 9 + 36} = 7, \) the vector

\[
\frac{a}{\|a\|} = \frac{1}{7} \langle 2, -3, 6 \rangle = \langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \rangle
\]

is the unit vector with the same direction as \( a, \) and \( \langle -\frac{2}{7}, \frac{3}{7}, -\frac{6}{7} \rangle \) is the unit vector in the opposite direction.
Unit vectors of particular interest are those in the positive direction of the coordinate axes (see Figure 10). These vectors are given the following special designations:

\[ \mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle. \] (4)

A vector \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) is expressed uniquely in terms of \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) as

\[ \mathbf{a} = \langle a_1, a_2, a_3 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}. \]

**EXAMPLE 7** Express the vector \( \mathbf{a} = \langle -3, 2, 0 \rangle \) in terms of \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) and illustrate this representation geometrically.

**Solution** We have \( \langle -3, 2, 0 \rangle = -3\mathbf{i} + 2\mathbf{j} + 0\mathbf{k} = -3\mathbf{i} + 2\mathbf{j} \). The representation is shown in the \( xy \)-plane in Figure 11.20(a). Figure 11(b) shows the representation in three-dimensional space.
Applications

EXAMPLE 8  A pilot is flying a plane headed northwest at an indicated airspeed of 140 knots (nautical miles per hour). A wind coming from the southwest is blowing the plane at a steady 30 knots. How far has the plane traveled in one hour?

Solution  Suppose an $xy$-coordinate system is introduced with the positive $x$-axis pointing east and the positive $y$-axis pointing north. Since the plane is heading northwest, the unit vector in the direction of the heading of the plane is

$$a = \cos \frac{3\pi}{4} \mathbf{i} + \sin \frac{3\pi}{4} \mathbf{j} = -\frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j},$$

as shown in Figure 12(a). The unit vector $b$ in the direction of the wind is

\[ b = \cos \frac{\pi}{4} \mathbf{i} + \sin \frac{\pi}{4} \mathbf{j} = \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j}. \]
shown in Figure 12(b), where
\[ b = \cos \frac{\pi}{4} i + \sin \frac{\pi}{4} j = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j. \]

The flight of the plane is described by the resultant vector, shown in Figure 12(c):
\[ 140a + 30b = 140 \left( -\frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j \right) + 30 \left( \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j \right) = -55\sqrt{2}i + 85\sqrt{2}j. \]

The distance traveled by the plane in one hour is
\[ \|140a + 30b\| = \left[ (-55\sqrt{2})^2 + (85\sqrt{2})^2 \right]^{1/2} = \sqrt{20,500} \approx 143.2 \text{ nautical miles.} \]

**Exercise Set: Three-Dimensional Vectors**

In Exercises 1–6, the initial point \( A \) and terminal point \( B \) of a vector are given. Find (a) the position-vector representation, and (b) the length of the vector.

1. \( A(0,0,0), B(1,5,3) \)
2. \( A(0,2,0), B(0,2,7) \)
3. \( A(2,3,4), B(2,-3,4) \)
4. \( A(1,2,0), B(3,5,0) \)
5. \( A(-2,2,1), B(0,5,0) \)
6. \( A(4,1,0), B(1,-2,0) \)

In Exercises 7–10, a vector and its initial point \( A \) are given. Find the terminal point of the vector.

7. \( \mathbf{v} = \langle 1, 3, 4 \rangle, A(0,0,0) \)
8. \( \mathbf{v} = \langle 1, 3, 4 \rangle, A(2,1,0) \)
9. \( \mathbf{v} = \langle -1, 2, 0 \rangle, A(-4,1,1) \)
10. \( \mathbf{v} = \langle 2, -2, 1 \rangle, A(0,3,3) \)

In Exercises 11–18, \( \mathbf{a} = \langle 1, 1, 0 \rangle \), \( \mathbf{b} = \langle -1, 1, 2 \rangle \), and \( \mathbf{c} = \langle 2, 3, 4 \rangle \).

11. Find \( \mathbf{a} + \mathbf{b} \) and illustrate this sum geometrically.
12. Find 3\( \mathbf{a} \). Sketch \( \mathbf{a} \) and 3\( \mathbf{a} \) on the same set of coordinate axes.
13. Find -2\( \mathbf{c} \). Sketch \( \mathbf{c} \) and -2\( \mathbf{c} \) on the same set of coordinate axes.
14. Find \( \mathbf{a} - \mathbf{c} \) and illustrate this difference geometrically.
15. Find \( \mathbf{a} + \mathbf{b} + \mathbf{c} \).
16. Find 2\( \mathbf{a} - 3\mathbf{b} + 4\mathbf{c} \).
17. Find \( \|\mathbf{c}\| \) and \( \|-2\mathbf{c}\| \).
18. Find \( \|\mathbf{a}\|, \|\mathbf{b}\|, \text{ and } \|\mathbf{a} + \mathbf{b}\| \).
19. Find a unit vector that has the same direction as each of the following.
   a. \( \langle 2, 3, 4 \rangle \)
   b. \( \langle -1, 2, 3 \rangle \)
20. Find a unit vector that has direction opposite that of each of the following.
   a. \( \langle 2, 3, 4 \rangle \)
b. \((-1, 2, 3)\)

21. Find two vectors parallel to \(\langle 3, 4, -1 \rangle\) that have length 3.

22. Find two vectors parallel to \(\langle 2, 4, 4 \rangle\) that have length 7.

23. Express each of the following vectors in terms of \(i, j, k\) and illustrate the representation geometrically.
   a. \(\langle 1, 4, 0 \rangle\)
   b. \(\langle 1, 4, 5 \rangle\)
   c. \(\langle 0, 2, 3 \rangle\)

24. Determine which of the following pairs of vectors are parallel.
   a. \(\langle 1, 0, 1 \rangle, \langle -1, 0, -1 \rangle\)
   b. \(\langle 1, 0, 0 \rangle, \langle 1, 1, 0 \rangle\)
   c. \(\langle 2, 3, -1 \rangle, \langle 4, 6, -2 \rangle\)
   d. \(\langle 3, -1, 2 \rangle, \langle -6, 2, 4 \rangle\)

25. Show that if \(a\) and \(b\) are vectors and \(\alpha\) is a real number, then \(\alpha(a+b) = \alpha a + \alpha b\).

26. Show that if \(a\) is a vector and \(\alpha\) is a real number, then \(||\alpha a|| = |\alpha| ||a||\).

27. Consider the accompanying figure, where \(c = (a + b)/2\).
   a. Write \(d\) in terms of \(b\) and \(c\).
   b. Prove that the diagonals of a parallelogram bisect each other.

28. Consider the accompanying figure.

29. Three children are pulling on ropes attached to a common ring (see the figure). The first child pulls in the direction of the positive \(x\)-axis with a force of 20 lb. The second child pulls in the direction of the positive \(y\)-axis with a force of 15 lb. If the ring does not move, in what direction and with what force must the third child be pulling?

30. A steady easterly wind is blowing at the rate of 10 m/sec on a helium-filled balloon attached to a string. In still air, the balloon would rise vertically at the rate of 2 m/sec (see the figure). Where is the balloon and how much string has been released when the balloon is 100 m high?
31. A canoeist crosses a 300-ft stream in 1 min, arriving at a point directly opposite the starting point. The current in the stream is 3 ft/sec (see the figure). In what direction did the canoeist paddle and at what average speed?

32. A private pilot flies for 1 hr and arrives at a point 100 mi due north of his departure point. A steady wind was blowing from the northwest at the rate of 20 mph. In what direction did the pilot fly and at what average speed?

33. Suppose the pilot of the plane in Exercise 32 flies to a point 100 mi due south instead of due north. Assuming that the other conditions remain the same, in what direction and at what average speed did the plane fly?
The Dot Product of Vectors

In the preceding Section we saw that if \( \mathbf{a} \) and \( \mathbf{b} \) are vectors and \( \alpha \) is a real number, then the sum \( \mathbf{a} + \mathbf{b} \), the difference \( \mathbf{a} - \mathbf{b} \), and the scalar product \( \alpha \mathbf{a} \) are all vectors. In this section we introduce an operation on a pair of vectors that results in a real number. This operation is called the dot product of a pair of vectors. It is also known as the inner or scalar product.

**Definition 4.** The dot product of the vectors \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) and \( \mathbf{b} = \langle b_1, b_2, b_3 \rangle \) is

\[
\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.
\]

**EXAMPLE 1** For vectors \( \mathbf{a} = \langle 1, -1, 1 \rangle \) and \( \mathbf{b} = \langle -2, 3, 1 \rangle \), find each of the following.

a) \( \mathbf{a} \cdot \mathbf{b} \)  
b) \( \mathbf{a} \cdot \mathbf{a} \)  
c) \( \mathbf{b} \cdot \mathbf{i} \)  
d) \( \mathbf{b} \cdot \mathbf{j} \)

**Solution**

\[
a) \quad \mathbf{a} \cdot \mathbf{b} = (1, -1, 1) \cdot (-2, 3, 1) = (1)(-2) + (-1)(3) + (1)(1) = -4
\]
\[
b) \quad \mathbf{a} \cdot \mathbf{a} = (1, -1, 1) \cdot (1, -1, 1) = 1 + 1 + 1 = 3
\]
\[
c) \quad \mathbf{b} \cdot \mathbf{i} = (-2, 3, 1) \cdot (1, 0, 0) = 0 + 3 + 0 = -2
\]
\[
d) \quad \mathbf{b} \cdot \mathbf{j} = (-2, 3, 1) \cdot (0, 1, 0) = 0 + 3 + 0 = 3
\]

Some immediate consequences of the definition of the dot product are listed in the following theorem.

**Theorem 2.** If \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) are vectors and \( \alpha \) is a real number, then:

\[
i) \quad \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a};
\]
\[
ii) \quad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c};
\]
\[
iii) \quad (\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha (\mathbf{a} \cdot \mathbf{b});
\]
\[
iv) \quad 0 \cdot \mathbf{a} = 0;
\]
\[
v) \quad \mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.
\]

In addition,

\[
vi) \quad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1;
\]
\[
vii) \quad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.
\]

An alternative representation of the dot product uses the notion of the angle between two vectors. The angle between nonzero vectors \( \mathbf{a} \) and \( \mathbf{b} \) is the angle \( \theta \) in \([0, \pi]\) determined by the position-vector representations of \( \mathbf{a} \) and \( \mathbf{b} \) (see Figure 1). If \( A \) denotes the terminal point of \( \mathbf{a} \) and \( B \) denotes the terminal point of \( \mathbf{b} \), then the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is angle \( AOB \).
If \( \mathbf{a} \) and \( \mathbf{b} \) are parallel vectors and \( \mathbf{b} = \alpha \mathbf{a} \), then we say that the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is zero when \( \mathbf{a} \) and \( \mathbf{b} \) have the same direction, that is, when \( \alpha > 0 \), and the angle is \( \pi \) when \( \mathbf{a} \) and \( \mathbf{b} \) have the opposite direction, that is, when \( \alpha < 0 \).

The following representation is sometimes given as the definition of the dot product.

**Theorem 3.** If \( \theta \) is the angle between the vectors \( \mathbf{a} \) and \( \mathbf{b} \), then

\[
\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta.
\]

**Proof.** Consider the position-vector representation of \( \mathbf{a} \) and \( \mathbf{b} \). Connecting the terminal points of these vectors forms a triangle with sides of length \( \|\mathbf{a}\| \), \( \|\mathbf{b}\| \), and \( \|\mathbf{b} - \mathbf{a}\| \) (see Figure 2).

The Law of Cosines applied to this triangle implies that

\[
\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta.
\]
It follows from Theorem (11.10) that
\[ \| b - a \|^2 = (b - a) \cdot (b - a) = b \cdot b - a \cdot b - b \cdot a + a \cdot a \]
Thus
\[ \| b \|^2 - 2a \cdot b + \| a \|^2 = \| b \|^2 - 2\| a \| \| b \| \cos \theta, \]
so
\[ -2a \cdot b = -2\| a \| \| b \| \cos \theta \quad \text{and} \quad a \cdot b = \| a \| \| b \| \cos \theta. \]
\[ \blacksquare \]

EXAMPLE 2  
Find the dot product of the vectors \( \langle 3, 0, -1 \rangle \) and \( \langle 2, 1, 2 \rangle \) and determine the angle between them.

Solution  
The dot product of these vectors is
\[ \langle 3, 0, -1 \rangle \cdot \langle 2, 1, 2 \rangle = 6 + 0 - 2 = 4. \]
Since
\[ \| \langle 3, 0, -1 \rangle \| = \sqrt{10} \quad \text{and} \quad \| \langle 2, 1, 2 \rangle \| = \sqrt{9} = 3, \]
it follows from Theorem (11.11) that
\[ \cos \theta = \frac{a \cdot b}{\| a \| \| b \|} = \frac{4}{3\sqrt{10}} = \frac{2\sqrt{10}}{15}, \]
so
\[ \theta = \arccos \left( \frac{2\sqrt{10}}{15} \right) \approx 1.14. \]

EXAMPLE 3  
Show that the points (0, 0, 0), (3, 1, 0), and (-2, 6, 0) are the vertices of a right triangle with the right angle at the origin.

Solution  
Let \( \theta \) denote the angle at the origin and \( a = \langle 3, 1, 0 \rangle \) and \( b = \langle -2, 6, 0 \rangle \) be the position vectors for the other two vertices, as shown in Figure 3. Then \( \theta \) is the angle between \( a \) and \( b \).

\[
\text{Figure 3}
\]
Since
\[ a \cdot b = \langle 3, 1, 0 \rangle \cdot \langle -2, 6, 0 \rangle = -6 + 6 + 0 = 0, \]
we have
\[ \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = 0, \]
and \( \theta = \pi/2 \). (An alternative method for solving this problem is to show that \( \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{a} - \mathbf{b}\|^2 \). This is considered in Exercise 34.)

Motivated by the preceding example, we make the following definition.

**Definition 5.** Vectors \( \mathbf{a} \) and \( \mathbf{b} \) are said to be **orthogonal** (or perpendicular) if \( \theta \), the angle between \( \mathbf{a} \) and \( \mathbf{b} \), is \( \pi/2 \). (In addition, we say that \( \mathbf{0} \) is orthogonal to every vector.)

A corollary of Theorem (11.11) follows from this definition.

**Corollary 1.** Vectors \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \).

**Direction Angles and Direction Cosines**

Three angles of particular interest for a vector \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) are the angles \( \alpha \) between \( \mathbf{a} \) and \( \mathbf{i} \), \( \beta \) between \( \mathbf{a} \) and \( \mathbf{j} \), and \( \gamma \) between \( \mathbf{a} \) and \( \mathbf{k} \) (see Figure 4). These are called the **direction angles** of \( \mathbf{a} \). The numbers \( \cos \alpha \), \( \cos \beta \), and \( \cos \gamma \) are called the **direction cosines** of \( \mathbf{a} \).

![Figure 4](image-url)

**Theorem 4.** The direction cosines of a nonzero vector \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) satisfy

\[ \cos \alpha = \frac{a_1}{\|\mathbf{a}\|}, \quad \cos \beta = \frac{a_2}{\|\mathbf{a}\|}, \quad \cos \gamma = \frac{a_3}{\|\mathbf{a}\|} \]

and

\[(\cos \alpha)^2 + (\cos \beta)^2 + (\cos \gamma)^2 = 1.\]
Proof.
\[
\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{\| \mathbf{a} \| \| \mathbf{i} \|} = \frac{a_1}{\| \mathbf{a} \|}, \quad \cos \beta = \frac{\mathbf{a} \cdot \mathbf{j}}{\| \mathbf{a} \| \| \mathbf{j} \|} = \frac{a_2}{\| \mathbf{a} \|},
\]
and
\[
\cos \gamma = \frac{\mathbf{a} \cdot \mathbf{k}}{\| \mathbf{a} \| \| \mathbf{k} \|} = \frac{a_3}{\| \mathbf{a} \|}.
\]
Thus
\[
(\cos \alpha)^2 + (\cos \beta)^2 + (\cos \gamma)^2 = \frac{a_1^2}{\| \mathbf{a} \|^2} + \frac{a_2^2}{\| \mathbf{a} \|^2} + \frac{a_3^2}{\| \mathbf{a} \|^2} = \frac{\| \mathbf{a} \|^2}{\| \mathbf{a} \|^2} = 1.
\]

Since any vector \( \mathbf{a} = \langle a_1, a_2, a_3 \rangle \) can be written as \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \), Theorem 4 implies that
\[
\mathbf{a} = \| \mathbf{a} \| (\cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}). \tag{5}
\]

EXAMPLE 4 Find the direction cosines and direction angles of the vector \( \mathbf{a} = \langle 1, -2, 2 \rangle \).

Solution Since \( \| \mathbf{a} \| = \sqrt{1 + 4 + 4} = 3 \), we have
\[
\cos \alpha = \frac{1}{3}, \quad \cos \beta = -\frac{2}{3}, \quad \text{and} \quad \cos \gamma = \frac{2}{3}.
\]
So
\[
\alpha = \arccos \left( \frac{1}{3} \right) \approx 1.23, \quad \beta = \arccos \left( -\frac{2}{3} \right) \approx 2.30,
\]
and
\[
\gamma = \arccos \left( \frac{2}{3} \right) \approx 0.84.
\]
The position-vector representation of \( \mathbf{a} \) is shown in Figure 5.

Figure 5
The Triangle inequality for Vectors

Theorem 3 implies that for any vectors \( \mathbf{a} \) and \( \mathbf{b} \),

\[
|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\| \|\mathbf{b}\| |\cos \theta|.
\]

Since \( |\cos \theta| \leq 1 \) for all values of \( \theta \),

\[
|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|. \tag{6}
\]

Inequality (6) is called the **Cauchy-Buniakowsky-Schwarz Inequality**. It is used to establish the important result given in Theorem 5. This result is known as the **Triangle Inequality**.

**Theorem 5.** For any vectors \( \mathbf{a} \) and \( \mathbf{b} \), \( \|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \).

**Proof.** Consider \( \|\mathbf{a} + \mathbf{b}\|^2 \) and apply Theorem 2 part v and Inequality (6):

\[
\|\mathbf{a} + \mathbf{b}\|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})
= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}
= \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2
\leq \|\mathbf{a}\|^2 + 2\|\mathbf{a}\| |\mathbf{b}| + \|\mathbf{b}\|^2
\leq \|\mathbf{a}\|^2 + 2\|\mathbf{a}\| \|\mathbf{b}\| + \|\mathbf{b}\|^2 = (\|\mathbf{a}\| + \|\mathbf{b}\|)^2.
\]

Taking the positive square root of each side produces the result. \( \blacksquare \)

If Theorem 5 is interpreted geometrically, it says that the length of one side of a triangle cannot exceed the sum of the lengths of the remaining two sides (see Figure 6).
Orthogonal Projections

For a fixed nonzero vector $a$, any vector $b$ can be written uniquely as the sum of a vector that is parallel to $a$ and another vector that is orthogonal to $a$. The portion of the sum parallel to $a$ is called the orthogonal projection of $b$ onto $a$ and is denoted $\text{proj}_a b$ (see Figure 7).

The triangles in Figure 7 indicate that the length of $\text{proj}_a b$ is given by

$$||\text{proj}_a b|| = ||b|| \cos \theta.$$  

But $a \cdot b = ||a|| \cdot ||b|| \cos \theta$, so

$$||\text{proj}_a b|| = ||b|| \cos \theta = \frac{a \cdot b}{||a||} \frac{||b||}{||b||} = \frac{a \cdot b}{||a||}.$$  

The number $(a \cdot b)/||a||$ is called the component of $b$ in the direction of $a$ and denoted

$$\text{comp}_a b = \frac{a \cdot b}{||a||}.$$  

(7)

Since $a/||a||$ is the unit vector in the direction of $a$,

$$\text{proj}_a b = \text{comp}_a b \left( \frac{a}{||a||} \right) = \frac{a \cdot b}{||a||^2} a.$$  

(8)

The details of the decomposition of a vector $b$ into the sum of two vectors, one parallel to $a$ and the other orthogonal to $a$, are given in the following theorem.

**Theorem 6.** Suppose $a$ is a fixed nonzero vector. Then any vector $b$ can be written uniquely in the form $b = b_1 + b_2$, where $b_1$ is parallel to $a$ and $b_2$ is orthogonal to $a$. Moreover,
\[ b_1 = \text{proj}_a b = (\text{comp}_a b) \frac{a}{\|a\|} = \left( \frac{a \cdot b}{\|a\|^2} \right) a \quad \text{and} \quad b_2 = b - \text{proj}_a b. \]

Proof. By definition of \( \text{proj}_a b \), the vector \( b_1 \) is parallel to \( a \). With \( b_2 \) defined by \( b_2 = b - b_1 \), it is clear that \( b = b_1 + b_2 \). It remains to show that \( b_2 \) is orthogonal to \( a \), that is, that \( a \cdot b_2 = 0 \):

\[
a \cdot b_2 = a \cdot (b - b_1) = a \cdot b - a \cdot \left( \frac{a \cdot b}{\|a\|^2} \right) a = a \cdot b - \left( \frac{a \cdot b}{\|a\|^2} \right) \|a\|^2 = 0.
\]

\[ \square \]

**EXAMPLE 5** Decompose \( b = \langle 2, 3, -1 \rangle \) into a sum \( b_1 + b_2 \), where \( b_1 \) is parallel to \( a = \langle 0, 4, 2 \rangle \) and \( b_2 \) is orthogonal to \( a \).

Solution We have

\[
\text{comp}_a b = \frac{a \cdot b}{\|a\|} = \frac{\langle 0, 4, 2 \rangle \cdot \langle 2, 3, -1 \rangle}{\sqrt{0 + 16 + 4}} = \frac{10}{2\sqrt{5}} = \sqrt{5}
\]

and

\[
\text{proj}_a b = \text{comp}_a b \left( \frac{a}{\|a\|} \right) = \sqrt{5} \frac{\langle 0, 4, 2 \rangle}{2\sqrt{5}} = \langle 0, 2, 1 \rangle.
\]

So \( b_1 = \langle 0, 2, 1 \rangle \) and \( b_2 = b - b_1 = \langle 2, 3, -1 \rangle - \langle 0, 2, 1 \rangle = \langle 2, 1, -2 \rangle \). Figure 8 shows the decomposition.

**Applications**

One application of the orthogonal projection of one vector onto another arises in the definition of the work done by a force in moving an object. If a constant
force $F$ is applied in the direction of motion, then the work $W$ done by $F$ in moving an object a distance $D$ is

$$W = (\text{force})(\text{distance}) = F \cdot D.$$  

![Figure 9](image)

If a vector force $F$ is constant but is applied at an angle $\theta$ to the direction of motion (as in Figure 9), then the work done by $F$ in moving an object from $P$ to $Q$ is defined by

$$W = (\text{component of force in the direction of motion})(\text{distance}) = (\text{comp}_{\overrightarrow{PQ}} F)||\overrightarrow{PQ}||.$$  

Thus

$$W = (||F|| \cos \theta)||\overrightarrow{PQ}|| = F \cdot \overrightarrow{PQ}. \quad (9)$$

**EXAMPLE 6** A toboggan loaded with camping gear is pulled 500 ft across a frozen lake with a rope that makes an angle of $\pi/4$ with the ground. Find the work done if a force of 20 lb is exerted in pulling the toboggan.

**Solution** The situation is illustrated in Figure 10. The unit vector in the direction of the force is

$$\cos \frac{\pi}{4} \hat{i} + \sin \frac{\pi}{4} \hat{j} = \frac{\sqrt{2}}{2}(\hat{i} + \hat{j}),$$

so the force is

$$F = 20 \left[ \frac{\sqrt{2}}{2}(\hat{i} + \hat{j}) \right] = 10\sqrt{2}(\hat{i} + \hat{j}) \text{ lb.}$$

![Figure 10](image)
The vector describing the path of the toboggan is $500\text{ ft}$, so the work done is

$$W = F \cdot 500\text{ ft} = 10\sqrt{2}(i + j) \cdot 500\text{ ft} = 5000\sqrt{2} \text{ ft-lb}.$$

**Exercise Set: Dot Products**

In Exercises 1–8, (a) find $\mathbf{a} \cdot \mathbf{b}$ for the pair of vectors, (b) find the angle between the pairs of vectors, (c) determine in each case if the vectors are orthogonal, (d) find the component of $\mathbf{b}$ in the direction of $\mathbf{a}$, and (e) determine the orthogonal projection of $\mathbf{b}$ onto $\mathbf{a}$.

1. $\mathbf{a} = \langle 1, 3, 3 \rangle$, $\mathbf{b} = \langle 2, 2, 4 \rangle$
2. $\mathbf{a} = \langle -1, -3, 2 \rangle$, $\mathbf{b} = \langle 1, 3, 2 \rangle$
3. $\mathbf{a} = \langle 2, 4, 0 \rangle$, $\mathbf{b} = \langle 3, 7, 0 \rangle$
4. $\mathbf{a} = \langle 1, -3, 2 \rangle$, $\mathbf{b} = \langle 2, 2, 2 \rangle$
5. $\mathbf{a} = \langle 0, 4, 6 \rangle$, $\mathbf{b} = \langle 0, -3, 2 \rangle$
6. $\mathbf{a} = \langle 1, 5, -3 \rangle$, $\mathbf{b} = \langle 3, 1, 1 \rangle$
7. $\mathbf{a} = \langle 3, 4, \pi \rangle$, $\mathbf{b} = \langle 1, e, 0 \rangle$
8. $\mathbf{a} = \langle -\sqrt{2}, \sqrt{3}, 1 \rangle$, $\mathbf{b} = \langle \sqrt{3}, 0, \sqrt{2} \rangle$

In Exercises 9–12, find the direction cosines of the vector.

9. $\langle 2, 3, 4 \rangle$
10. $\langle -1, 3, -3 \rangle$
11. $\langle 0, 3, 4 \rangle$
12. $\langle 2, 3, 0 \rangle$

13. Find a unit vector that has the same direction cosines as $\langle 0, 3, 4 \rangle$.
14. Find a vector lying in the xy-plane orthogonal to $\langle 2, 4, 1 \rangle$. Is there more than one such vector?
15. Find a unit vector lying in the xy-plane orthogonal to $\langle 2, 4, 0 \rangle$. Is there more than one such vector?
16. Let $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

   a. Find the component of $\mathbf{b}$ in the direction of each of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

   b. Find the orthogonal projection of $\mathbf{b}$ onto each of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

17. Express $\mathbf{b} = \langle -4, 1, -2 \rangle$ as $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$, where $\mathbf{b}_1$ is parallel to $\mathbf{a} = \langle 1, 3, -3 \rangle$ and $\mathbf{b}_2$ is orthogonal to $\mathbf{a}$.
18. Show that $(0, 0, 0)$, $(3, 1, 0)$, and $(-2, 6, 0)$ are vertices of a right triangle by showing that $||\mathbf{a}||^2 + ||\mathbf{b}||^2 = ||\mathbf{a} - \mathbf{b}||^2$ for appropriate vectors $\mathbf{a}$ and $\mathbf{b}$.
19. Show that $(2, 2, 2)$, $(2, 0, 1)$, and $(4, 1, -1)$ are vertices of a right triangle.
20. Write an equation to describe all points $(x, y, z)$ with the property that $(x - 1, y - 2, z - 3)$ is orthogonal to $(1, -1, 2)$.
21. Find a vector orthogonal to both $\mathbf{a} = \langle 1, 2, 4 \rangle$ and $\mathbf{b} = \langle 2, -2, 5 \rangle$. Is there more than one such vector?
22. Show that for any vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
23. Show that for any vector $\mathbf{a}$, $\mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2$.
24. Show that for any vectors $\mathbf{a}$ and $\mathbf{b}$, $||\mathbf{a} - \mathbf{b}|| \geq ||\mathbf{a}|| - ||\mathbf{b}||$.
25. Show that if $\mathbf{v}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{v}$ is orthogonal to $c_1 \mathbf{a} + c_2 \mathbf{b}$ for all real numbers $c_1$ and $c_2$.
26. Find nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ with $\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$ and $\mathbf{a} \neq \mathbf{b}$. 


27. Determine conditions on the vectors $a$ and $b$ necessary for equality to hold in the inequality in Theorem (11.17):

$$\|a + b\| \leq \|a\| + \|b\|.$$ 

28. 

a. Prove the Parallelogram Law: For any vectors $a$ and $b$,

$$\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2).$$

b. Interpret the Parallelogram Law geometrically.

29. Show that vectors $a$ and $b$ are orthogonal if and only if

$$\|a + b\|^2 = \|a\|^2 + \|b\|^2.$$ 

30. Explain how the accompanying figure shows a geometric proof of the distributive law:

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$ 

31. Show that the decomposition of $b$ described in Theorem (11.20) is unique. [Hint: Assume that $b = \alpha a + c$, where $a \cdot c = 0$. Then determine $\alpha$ and $c$.] 

32. Find the work done by the force $F = j + 2k$ (in pounds) applied to an object that moves 2 ft along the $y$-axis.

33. A wagon loaded with groceries is pulled horizontally a half mile by a handle that makes an angle of $\pi/3$ with the horizontal (see the figure). Find the work done if a force of 20 lb is exerted on the handle.

34. A block is to be moved 6 ft up a ramp that makes an angle of $\pi/4$ with the horizontal (see the figure).

a. Find the work done if 20 lb of force is applied in the direction of the motion.

b. Find the work done if 20 lb of force is applied at an angle of $\pi/6$ with the horizontal.

35. A 200-lb piece of slate for the top of a pool table is moved along a horizontal surface by a pulling force applied from above (see the figure). If the force is applied at an angle of $\pi/3$ to the slate, what force must be applied to move it?
Planes

The basic geometric object in three-dimensional space is a surface and the most elementary surfaces are planes. To describe a plane completely, it suffices to specify one point on the plane and the direction orthogonal to all line segments lying on the plane. A vector describing this direction is called a normal vector to the plane. Since only the direction of a normal vector is important, any nonzero vector parallel to a normal vector is also a normal vector to the plane.

![Diagram of a plane with a normal vector](image)

Figure 1

Suppose \( P_0(x_0, y_0, z_0) \) is a point on a plane \( \mathcal{P} \) and \( \mathbf{n} = \langle a, b, c \rangle \) is a normal vector to \( \mathcal{P} \), as illustrated in Figure 1. A point \( P(x, y, z) \) lies on the plane \( \mathcal{P} \) precisely when \( \mathbf{n} \) is orthogonal to the vector \( \overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle \), that is, when

\[
\mathbf{n} \cdot \overrightarrow{P_0P} = 0, \tag{10}
\]

so

\[
\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.
\]

Thus the plane with normal vector \( \mathbf{n} = \langle a, b, c \rangle \) that contains the point \( P_0(x_0, y_0, z_0) \) has equation

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \tag{11}
\]

or

\[
x + by + cz = d, \quad \text{where} \quad d = ax_0 + by_0 + cz_0.
\]

**EXAMPLE 1** Find an equation of the plane containing the point \((2, -1, 3)\) that has normal vector \(\langle 3, 6, 2 \rangle\). Sketch the graph of this equation.

**Solution** If \((x, y, z)\) is a point on this plane, then the vector \(\langle x - 2, y + 1, z - 3 \rangle\) is orthogonal to \(\langle 3, 6, 2 \rangle\). So the plane has equation

\[
3(x - 2) + 6(y + 1) + 2(z - 3) = 0,
\]
which simplifies to
\[ 3x + 6y + 2z = 6. \]

To sketch the graph of this equation, we first find the points at which the plane intersects the coordinate axes:

- when \( y = 0 \) and \( z = 0 \), \( 3x = 6 \), so \( x = 2 \);
- when \( x = 0 \) and \( z = 0 \), \( 6y = 6 \), so \( y = 1 \);
- when \( x = 0 \) and \( y = 0 \), \( 2z = 6 \), so \( z = 3 \);

Thus the points \((2, 0, 0), (0, 1, 0), \) and \((0, 0, 3)\) lie on the plane. The line segments joining these points also lie on the plane, so the graph in the first octant is as shown in Figure 2. The plane extends linearly in all directions from line segments in this triangular segment.

![Figure 2](image)

We have seen that a plane with normal vector \( \mathbf{n} = \langle a, b, c \rangle \) and passing through \((x_0, y_0, z_0)\) has equation \( ax + by + cz = d \), where \( d = ax_0 + by_0 + cz_0 \).

On the other hand, if \( a, b, \) and \( c \) are not all zero, then the graph of an equation of the form
\[ ax + by + cz = d \]

must be a plane with normal vector \( \mathbf{n} = \langle a, b, c \rangle \). To see this, suppose \( a \neq 0 \). Equation (11.35) can be written as

\[ 0 = a \left( x - \frac{d}{a} \right) + b(y - 0) + c(z - 0) = \mathbf{n} \cdot \left( x - \frac{d}{a}, y, z - 0 \right), \]
which is an equation of the plane that contains the point \( \left( \frac{d}{a}, 0, 0 \right) \) and has normal vector \( \mathbf{n} = \langle a, b, c \rangle \).

An equation of the form (11.35) is called a linear equation in three variables. The two-dimensional analogue to (11.35) is the equation of a straight line given in Section 1.2.

The close connection between planes and their normal vectors permits us to make the following definition.

**Definition 6.** Suppose \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are planes with normal vectors \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \), respectively.

i) \( \mathcal{P}_1 \) is said to be **orthogonal** to \( \mathcal{P}_2 \) precisely when \( \mathbf{n}_1 \) is orthogonal to \( \mathbf{n}_2 \).

ii) \( \mathcal{P}_1 \) is said to be **parallel** to \( \mathcal{P}_2 \) precisely when \( \mathbf{n}_1 \) is parallel to \( \mathbf{n}_2 \).

Theorem (11.37) follows directly from Corollaries (11.13) and (11.29), which give results about orthogonal and parallel vectors.

**Theorem 7.** Suppose \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are planes with normal vectors \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \), respectively.

i) \( \mathcal{P}_1 \) is orthogonal to \( \mathcal{P}_2 \) if and only if \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \).

ii) \( \mathcal{P}_1 \) is parallel to \( \mathcal{P}_2 \) if and only if \( \mathbf{n}_1 = \alpha \mathbf{n}_2 \), for some constant \( \alpha \).

Since \( \mathbf{k} = \langle 0, 0, 1 \rangle \) is normal to the \( xy \)-plane, part (i) of Theorem (11.37) implies that any plane orthogonal to the \( xy \)-plane has a normal vector \( \mathbf{n} = \langle a, b, c \rangle \) with \( \mathbf{n} \cdot \mathbf{k} = 0 \). Thus \( c = 0 \) and \( \mathbf{n} = \langle a, b, 0 \rangle \). Therefore, any plane orthogonal to the \( xy \)-plane has equation of the form

\[
ax + by = d.
\]
Figure 3

For example, Figure 3(a) shows the graph of the plane with equation \( x + y = 2 \).

Similarly, a plane orthogonal to the \( yz \)-plane has equation of the form

\[ by + cz = d, \]

and a plane orthogonal to the \( xz \)-plane has equation of the form

\[ ax + cz = d. \]

Planes parallel to the coordinate planes are described by an equation in just one variable. For example, the equation of the \( yz \)-plane is \( x = 0 \), and the equation \( z = 3 \) describes the plane parallel to and three units above the \( xy \)-plane shown in Figure 3(b).

Any three points in space that are not collinear (do not lie on the same straight line) determine a unique plane. An equation of this plane can be determined by solving a system of three linear equations. An easier method uses the fact that the line segments joining these points also lie in the plane and that these segments describe vectors.

To this point in the section we have discussed how to determine the equation of a plane when a normal vector to the plane is known. We now need to see how to find such a normal vector.

In calculus your study of vectors will include the definition of the cross product for vectors in three-dimensional space. The cross product gives a convenient means for determining a normal vector to a plane when given two vectors in the plane. The main result from this application is given in the following theorem.

**Theorem 8.** Suppose \( P_1(a_1, b_1, c_1), P_2(a_2, b_2, c_2), \) and \( P_3(a_3, b_3, c_3) \) are points that determine the plane \( P \). Then the vector

\[ \mathbf{n} = \left( (b_2 - b_1)(c_3 - c_1) - (b_3 - b_1)(c_2 - c_1), \right. \]
\[ \left. (a_3 - a_1)(c_2 - c_1) - (a_2 - a_1)(c_3 - c_1), \right. \]
\[ \left. (a_2 - a_1)(b_3 - b_1) - (a_3 - a_1)(b_2 - b_1) \right) \]

is normal to \( P \).

To show the truth of the theorem is true we need to verify that the vector \( \mathbf{n} \) is perpendicular to vectors in the plane \( P \). It suffices to show that the dot product of \( \mathbf{n} \) with both \( P_2P_1 \) and \( P_3P_1 \) is 0. The algebra involved is a bit tedious but not difficult. It is considered in Exercise 39.

**EXAMPLE 2** Find an equation of the plane containing the points \((2,0,1), (0,6,-2), \) and \((-2,3,1)\).

**Solution** Let

\[ (a_1, b_1, c_1) = (2, 0, 1), \quad (a_2, b_2, c_2) = (0, 6, -2), \quad \text{and} \quad (a_3, b_3, c_3) = (-2, 3, 1).\]
By Theorem 8, a normal to the plane is given by
\[ \mathbf{n} = \langle (6 - 0)(1 - 1) - (3 - 0)(-2 - 1),
\quad (-2 - 2)(-2 - 1) - (0 - 2)(1 - 1),
\quad (0 - 2)(3 - 0) - (-2 - 2)(6 - 0) \rangle = \langle 9, 12, 18 \rangle, \]
an the plane has an equation of the form
\[ 9x + 12y + 18z = d. \]
To determine \( d \), we can use the fact that the point \((2, 0, 1)\) lies on the plane. Thus
\[ d = 9 \cdot 2 + 12 \cdot 0 + 18 \cdot 1 = 36, \]
and an equation is
\[ 9x + 12y + 18z = 36 \quad \text{or, if simplified,} \quad 3x + 4y + 6z = 12. \]
The graph of this plane is shown in Figure 4.

The final result in this section shows how the dot product is used to find the distance from a plane to a point not on the plane.

**Theorem 9.** Suppose \( \mathcal{P} \) is a plane with equation \( ax + by + cz = d \) and \( P_0(x_0, y_0, z_0) \) is a point that does not lie on \( \mathcal{P} \). The shortest distance from \( P_0 \) to the plane \( \mathcal{P} \) is given by
\[ d(P_0, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}. \]
Proof. Let \( P(x, y, z) \) be an arbitrary point on the plane and consider the vector \( \overrightarrow{PP_0} \) from \( P \) to \( P_0 \):

\[
\overrightarrow{PP_0} = (x_0 - x, y_0 - y, z_0 - z).
\]

Figure 5

The shortest distance from \( P_0 \) to \( P \) is the absolute value of the component of \( \overrightarrow{PP_0} \) in the direction of the normal \( n = (a, b, c) \) to the plane \( P \) (see Figure 5). This distance is

\[
|\text{comp}_n \overrightarrow{PP_0}| = \frac{|(x_0 - x, y_0 - y, z_0 - z) \cdot (a, b, c)|}{\| (a, b, c) \|}
= \frac{|a(x_0 - x) + b(y_0 - y) + c(z_0 - z)|}{\sqrt{a^2 + b^2 + c^2}}
= \frac{|ax_0 + by_0 + cz_0 - (ax + by + cz)|}{\sqrt{a^2 + b^2 + c^2}}.
\]

But \( P(x, y, z) \) is on the plane \( P \), so \( ax + by + cz = d \) and

\[
d(P_0, P) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.
\]

\[\blacksquare\]

**Example 3** Find the distance from the point \( P(1, 2, 0) \) to the plane with equation \( x + 2y - z = -2 \).
Solution

By Theorem (11.38), this distance, shown in Figure 6, is

\[ \frac{|1(1) + 2(2) - 1(0) - (-2)|}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{7}{\sqrt{6}} = \frac{7\sqrt{6}}{6} \approx 2.86. \]

Exercise Set: Planes

In Exercises 1–6, find an equation of the plane that contains the given point and has normal vector \( \mathbf{n} \).

1. \((3, 3, 2), \mathbf{n} = (1, -1, 1)\)
2. \((-1, 2, 1), \mathbf{n} = (2, 3, 4)\)
3. \((2, 3, 4), \mathbf{n} = (0, 0, 3)\)
4. \((2, -3, 1), \mathbf{n} = (1, 1, 0)\)
5. \((0, 0, 0), \mathbf{n} = (2, 3, 4)\)
6. \((0, 0, 2), \mathbf{n} = (0, 1, 1)\)

In Exercises 7–14, sketch the graph of the equation.

7. \(2x + 3y + 4z = 12\)
8. \(2x + y - z - 4 = 0\)
9. \(x = 2\)
10. \(z = 3\)
11. \(2x + 3z = 4\)
12. \(2y - 3z = 4\)
13. \(z = 0\)
14. \(x + y - 4 = 0\)

In Exercises 15–28, find an equation of a plane that satisfies the conditions stated.

15. Is parallel to and a distance of 3 units above the \(xy\)-plane
16. Is parallel to and a distance of 2 units from the \(yz\)-plane
17. Contains the point \((1, 0, -1)\) and is parallel to the plane \(x + y - z = 4\)
18. Contains the point \((2, 3, 4)\) and is parallel to the plane \(x + y + z = 1\)
19. Contains the point \((1, 2, 3)\) and is parallel to the \(xy\)-plane
20. Contains the point \((2, 3, 4)\) and is parallel to the \(yz\)-plane
21. Contains the points (1, 2, 3) and (0, 1, 1) and is orthogonal to the xy-plane.
22. Contains the points (−1, 2, −3) and (5, 0, 4) and is orthogonal to the xz-plane.
23. Contains the points (1, −1, 4), (0, 2, 3), and (2, 1, 0).
24. Contains the points (3, 2, −1), (2, 3, 5), and (−1, −3, 4).
25. Contains the points (1, 0, −1) and (2, 1, 3) and is orthogonal to 2x − y + 3z = 6.
26. Contains the points (4, −2, 0) and (2, −1, 2) and is orthogonal to 6x + y + z = 15.
27. Contains the point (1, 2, 1) and is orthogonal to both x + y + z = 1 and x + 2y + 3z = 6.
28. Contains the point (1, −1, 4) and is orthogonal to x − 2y + z = 2 and 2x + 2y + z = 1.

In Exercises 29–32, find the distance from the point to the plane.
29. (0, 0, 0), 2x + 3y + 2z = 6
30. (1, −1, 3), z = 0
31. (1, −2, 3), x + z = 1
32. (1, 5, 4), x + y + 2z = 2

In Exercises 33–36, find the distance between the parallel planes.
33. x − y + 2z = 2, x − y + 2z = −2
34. 2x + y + 3z = 6, 2x + y + 3z = 1
35. 2x − 3y + z = 3, 4x − 6y + 2z = 9
36. x − z = 3, x − z = 5

37. Find an equation that describes the set of all points equidistant from (3, 1, 1) and (7, 5, 6).
38. Show that the plane that intersects the coordinate axes at (a, 0, 0), (0, b, 0), and (0, 0, c) has equation
\[ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \]
provided a, b, and c are all nonzero.
39. Show that the vector \( \mathbf{n} \) given in Theorem 8 is normal to the plane \( \mathcal{P} \) by demonstrating that both \( \mathbf{n} \cdot \overrightarrow{P_2P_1} = 0 \) and \( \mathbf{n} \cdot \overrightarrow{P_3P_1} = 0 \).
Lines in Space

Lines are described by specifying a point and a direction. Suppose the line \( l \) passes through the point \( P_0(x_0, y_0, z_0) \) and has direction given by the vector \( \mathbf{v} = (v_1, v_2, v_3) \), as shown in Figure 1. A point \( P(x, y, z) \) lies on \( l \) if and only if the vector \( \overrightarrow{P_0P} \) is parallel to \( \mathbf{v} \), that is, if and only if a real number \( t \) exists with

\[
\overrightarrow{P_0P} = t\mathbf{v}.
\] (13)

![Figure 1](image)

This vector equation for the line \( l \) is rewritten in terms of the components of the vectors as

\[
(x - x_0, y - y_0, z - z_0) = (tv_1, tv_2, tv_3).
\] (14)

Consequently, all points \((x, y, z)\) on the line passing through \((x_0, y_0, z_0)\) in the direction of \( \mathbf{v} = (v_1, v_2, v_3) \) are given by

\[
x - x_0 = tv_1, \quad y - y_0 = tv_2, \quad z - z_0 = tv_3
\]
or

\[
x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3
\] (15)

for some real number \( t \).

The numbers \( v_1, v_2, v_3 \) are called direction numbers of \( l \) for the natural reason that they indicate its direction. The set of equations in 15 is a set of parametric equations for \( l \) in the parameter \( t \).

**EXAMPLE 1** Find a set of parametric equations for the line passing through the point \((-1, 1, 3)\) and having direction given by \( \mathbf{v} = (4, 4, -2) \).
Solution  Since \((x, y, z)\) lies on this line if and only if

\[
\langle x - (-1), y - 1, z - 3 \rangle = t \langle 4, 4, -2 \rangle
\]

for some real number \(t\), a set of parametric equations for the line is

\[
x = -1 + 4t, \quad y = 1 + 4t, \quad z = 3 - 2t.
\]

This line is sketched in Figure 2.

![Figure 2](image)

EXAMPLE 2  Find parametric equations for the line passing through the points \((1, 2, 3)\) and \((0, 1, 3)\).

Solution  A vector \(v\) that describes the direction of this line is a vector determined by the given points:

\[
v = \langle 1 - 0, 2 - 1, 3 - 3 \rangle = \langle 1, 1, 0 \rangle.
\]

Using \(v\) and the point \((1, 2, 3)\), we obtain the parametric equations

\[
x = 1 + t, \quad y = 2 + t, \quad z = 3.
\]

Since \(z = 3\) regardless of the value of \(t\), this line lies in the plane \(z = 3\) (see Figure 3).
In Example 2, the line lies on a plane parallel to the $xy$-plane because the direction number of the line in the $z$-direction is zero. In general, when one of the direction numbers of a line is zero, the line is parallel to a coordinate plane. When two of the direction numbers of the line are zero, the line is parallel to a coordinate axis.

When the direction numbers $v_1, v_2, v_3$ of a line $l$ are all nonzero, each of the parametric equations

$$x = x_0 + v_1 t, \quad y = y_0 + v_2 t, \quad z = z_0 + v_3 t$$

can be solved for $t$:

$$\frac{x - x_0}{v_1} = t, \quad \frac{y - y_0}{v_2} = t, \quad \frac{z - z_0}{v_3} = t.$$

Equating these expressions for $t$ gives symmetric equations for the line $l$:

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}. \quad (16)$$

**EXAMPLE 3** Find symmetric equations for the line passing through the point $(1, 0, 2)$ and parallel to the line with parametric equations

$$x = 2 + t, \quad y = 1 + 3t, \quad \text{and} \quad z = 1 + 4t.$$

**Solution** The direction of this line is given by $v = (1, 3, 4)$, which also gives the direction of any line parallel to the line. Symmetric equations for the line parallel to the given line and passing through the point $(1, 0, 2)$ are

$$\frac{x - 1}{1} = \frac{y - 0}{3} = \frac{z - 2}{4}.$$
EXAMPLE 4  Find the point of intersection of the $xy$-plane and the line with symmetric equations

$$\frac{x - 1}{2} = \frac{y + 1}{3} = -\frac{z - 2}{1}.$$

Solution  The $z$-coordinate of this point of intersection is zero, so

$$\frac{x - 1}{2} = \frac{y + 1}{3} = -\frac{0 - 2}{1} = 2.$$

Solving for $x$ and $y$, we have

$$\frac{x - 1}{2} = 2, \text{ so } x = 5 \quad \text{and} \quad \frac{y + 1}{3} = 2, \text{ so } y = 5.$$

Consequently, the point of intersection of the line and the $xy$-plane is $(5, 5, 0)$, as shown in Figure 4.

Figure 4

Suppose a line has symmetric equations

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}.$$

By taking the equations in pairs, say

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}, \quad \text{that is,} \quad v_2x - v_1y = v_2x_0 - v_1y_0,$$

and

$$\frac{x - x_0}{v_1} = \frac{z - z_0}{v_3}, \quad \text{that is,} \quad v_3x - v_1z = v_3x_0 - v_1z_0,$$

we see that the line is the intersection of the planes with these equations.
If one of the direction numbers of the line is zero, say $v_3 = 0$, then $z = z_0 + v_3 t$ becomes $z = z_0$, so equations for the line are

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} \quad \text{and} \quad z = z_0,$$

and the line is again expressed as the intersection of two planes.

We now have three ways to describe a line in space:

i) by the vector equation (14);

ii) by the parametric equations in (15);

iii) by the symmetric equations in (16).

Lines can also be described as the intersection of two planes.

**Intersections of Lines in Space**

Two arbitrary lines in space do not generally intersect. The final example gives a technique for determining if a pair of lines intersect and, if so, finding a point of intersection.

**EXAMPLE 5** Determine if the lines $l_1$ and $l_2$ intersect and, if so, find a point of intersection.

$$l_1 : \quad x - 2 = \frac{y - 5}{3} = \frac{z - 7}{4}$$

and

$$l_2 : \quad \frac{x + 1}{2} = y - 1 = -\frac{z - 6}{3}.$$

**Solution** If the lines intersect, then a point $(x, y, z)$ exists that satisfies both sets of equations. In particular, then, $x$ and $y$ must satisfy

$$x - 2 = \frac{y - 5}{3} \quad \text{from the equations for } l_1 \tag{17}$$

and

$$\frac{x + 1}{2} = y - 1, \quad \text{from the equations for } l_2. \tag{18}$$

Solving for $x$ in Equation (16) gives

$$x = 2 + \frac{y}{3} - \frac{5}{3} = \frac{y}{3} + \frac{1}{3}, \tag{19}$$

and substituting into Equation (17) gives

$$\frac{(y/3 + 1/3) + 1}{2} = y - 1,$$
which implies that \( y = 2 \). So to satisfy Equations (16) and (17) we must have

\[
y = 2 \quad \text{and from Equation (18)} \quad x = \frac{2}{3} + \frac{1}{3} = 1.
\]

Thus any point on the intersection of the lines \( l_1 \) and \( l_2 \) must have the coordinates \((1, 2, z)\) for some number \( z \). An intersection will occur precisely when the same value of \( z \) satisfies the equations for both \( l_1 \) and \( l_2 \).

When \( x = 1 \) and \( y = 2 \), the equations for \( l_1 \) imply that

\[
z = 4(x - 2) + 7 = 4(1 - 2) + 7 = 3;
\]

and the equations for \( l_2 \) imply that

\[
z = -3(y - 1) + 6 = -3(2 - 1) + 6 = 3.
\]

Thus the lines intersect at the point with coordinates \((1, 2, 3)\), as shown in Figure 5.

![Figure 5](image)

**Exercise Set: Lines in Space**

In Exercises 1–4, find (a) parametric equations, and (b) symmetric equations for the line passing through the point and having the direction given by \( \mathbf{v} \).

1. \((2, -1, 2), \ \mathbf{v} = (1, 1, 1)\)
2. \((0, 4, 3), \ \mathbf{v} = (2, 0, 1)\)
3. \((0, 0, 0), \ \mathbf{v} = (2, 4, 3)\)
4. \((2,3,4), \mathbf{v} = (0,-3,4)\)

In Exercises 5–8, find (a) parametric equations, and (b) symmetric equations for the line passing through the given points.

5. \((1,2,0), (1,-1,4)\)

6. \((2,0,-1), (-1,2,1)\)

7. \((3,4,4), (2,-3,5)\)

8. \((0,0,2), (3,0,0)\)

Find parametric equations of a line that satisfies the conditions stated in Exercises 9–14.

9. Passes through the point \((1,2,3)\) and is parallel to the line with parametric equations \(x = t + 1, y = -t, z = 2 - t\).

10. Passes through the origin and is parallel to the line with parametric equations \(x = 1 - 2t, y = 3t + 2, z = t - 4\).

11. Passes through the point \((1,-2,3)\) and is orthogonal to \(x + y + z = 1\).

12. Passes through the point \((1,1,1)\) and is orthogonal to \(x - 2y - 3z = 6\).

13. Passes through \((1,7,0)\) and is parallel to the \(x\)-axis.

14. Passes through \((2,-2,5)\) and is parallel to the \(y\)-axis.

In Exercises 15–20, determine if the pair of lines \(l_1\) and \(l_2\) is (a) parallel or (b) orthogonal. (c) Find any points of intersection of \(l_1\) and \(l_2\).

15. \(l_1: \frac{x-1}{2} = y + 2 = \frac{z-3}{2}; l_2: \frac{x}{2} = 1 - y = -\frac{z+2}{2}\)

16. \(l_1: x - 1 = \frac{y+1}{3} = \frac{z-2}{3}; l_2: x - 1 = -\frac{y+2}{3} = \frac{z-5}{2}\)

17. \(l_1: x = 1 + t, y = 1 + 3t, z = 2 - t; l_2: x = 1 + t, y = 2 + t, z = 1 + t\)

18. \(l_1: x = 5t, y = 1+2t, z = 3t-2; l_2: x = 3 + 3t, y = 6t - 1, z = 3 - 3t\)

19. \(l_1: \langle x, y, z \rangle = (-3,0,1) + t(1,2,1); l_2: \langle x, y, z \rangle = (1, -4, 5) + t(2,1,2)\)

20. \(l_1: \langle x, y, z \rangle = (-1, -1, 6) + t(2,1,5); l_2: \langle x, y, z \rangle = (-7,2,3) + t(4,-2,2)\)

21. Find the point of intersection of the line with parametric equations \(x = t, y = 2 - t, z = 2t - 3\) and each of the following.

   a. The \(yz\)-plane
   b. The \(xz\)-plane
   c. The \(xy\)-plane

22. Find the point of intersection of the line with parametric equations \(x = t + 3, y = 2t - 1, z = t - 4\) and each of the following.

   a. The \(yz\)-plane
   b. The \(xz\)-plane
   c. The \(xy\)-plane

23. Show that \(x + 2y + z = 4\) and \(x + z = 2\) intersect and find parametric equations of the line of intersection.

24. Find parametric equations of the line that passes through the point \((1,-1,1)\), is orthogonal to the line \(3x = 2y = z\), and is parallel to the plane \(x + y - z = 0\).

25. Show that any line in space must intersect at least one of the coordinate planes.

26. Show that if two planes intersect, then their intersection contains a point in one of the coordinate planes.
Answers to Odd Exercises

Exercise Set 1

1. a.

b.

c.

3. a.

b. √6

c. \( \left( \frac{5}{2}, \frac{5}{2} \right) \)

5. a.

b. 2

c. (−3, 4, 1)

7. a.
ANSWERS TO ODD EXERCISES

b. \( \sqrt{99} \)
c. \( \left( \frac{1}{2}, \frac{9}{2}, \frac{3}{2} \right) \)

9. 

11. 

13. 

9.

Exercise Set 2

1. a. \((1, 5, 3)\); b. \(\sqrt{35}\)
3. a. \((0, -6, 0)\); b. 6
5. a. \((2, 3, -1)\); b. \(\sqrt{14}\)
7. \((1, 3, 4)\)
9. \((-5, 3, 1)\)

11.

13. \(x^2 + y^2 + z^2 = 4\)
15. \((x - 2)^2 + (y - 3)^2 + (z - 4)^2 = 1\)
17. \((-1, 2, -3), \sqrt{14}\)
19. \((-2, 0, 2), \frac{3\sqrt{2}}{2}\)
21. \(\left( x + \frac{1}{2} \right)^2 + \left( y - \frac{5}{2} \right)^2 + (z - 3)^2 = \frac{17}{2}\)

23.
13. \((-4, -6, -8)\)

15. \(\langle 2, 5, 6 \rangle\)

17. \(\|c\| = \sqrt{29}, \| -2c\| = 2\sqrt{29}\)

19. a. \(\langle \frac{2\sqrt{29}}{29}, \frac{3\sqrt{29}}{29}, \frac{4\sqrt{29}}{29} \rangle\)

b. \(\langle -\frac{\sqrt{14}}{14}, \frac{2\sqrt{14}}{14}, \frac{3\sqrt{14}}{14} \rangle\)

21. \(\pm \frac{3(3, 4, -1)}{\| (3, 4, -1) \|} =
\pm \langle \frac{9\sqrt{26}}{26}, \frac{6\sqrt{26}}{13}, -\frac{3\sqrt{26}}{26} \rangle\)

23. a. \(i + 4j\)

b. \(i + 4j + 5k\)

c. \(2j + 3k\)

27. a. \(d = c - b\)

29. \(F = -20i - 15j\)

31. If \(j\) points upstream and \(i\) points across the stream, the direction is \(300i - 180j\). The average speed is 5.8 ft/sec.

33. In the direction of \(-10\sqrt{2}i + \{(10\sqrt{2} - 100)\}j\). The average speed is 87 mi/hr.

Exercise Set 3

1. a. \(\angle \left(\frac{5\sqrt{14}}{57}\right) \approx 0.358\)

c. no

d. \(\frac{20}{19}\)

e. \(\frac{20}{19}(1, 3, 3)\)

3. a. 34

b. \(\arccos \frac{17\sqrt{290}}{290} \approx 0.059\)

c. no

d. \(\frac{3}{5}\sqrt{5}\)

e. \(\frac{17}{5}, \frac{34}{5}, 0\)

5. a. 0

c. yes

d. 0

e. 0

7. a. \(3 + 4e\)

b. \(\arccos \frac{3 + 4e}{\sqrt{(25 + \pi^2)(1 + e^2)}} \approx 0.625\)

c. no

d. \(\frac{3 + 4e}{\sqrt{25 + \pi^2}}\)

e. \(\frac{3 + 4e}{25 + \pi^2}(3, 4, \pi)\)

9. \(\langle \frac{2\sqrt{29}}{29}, \frac{3\sqrt{29}}{29}, \frac{4\sqrt{29}}{29} \rangle\)

11. 0

13. \(\langle 0, \frac{3}{5}, \frac{4}{5} \rangle\)

15. There are two: \(\pm \frac{\sqrt{5}}{5}(2, -1, 0)\)
17. \( \mathbf{b}_1 = \frac{1}{19} (5, 15, -15) \),
\( \mathbf{b}_2 = \frac{1}{19} (-81, 4, -23) \)
21. \((6c, c, -2c)\) for any constant \(c\)
27. \(\mathbf{a}\) and \(\mathbf{b}\) must have the same direction.
33. \(26,400\) ft-lb
35. \(200i + 200\sqrt{3}j\)

**Exercise Set 4**

1. \(x - y + z = 2\)
2. \(z = 4\)
3. \(2x + 3y + 4z = 0\)

**Exercise Set 5**

1. a. \(x = 2 + t, y = -1 + t,\)
   \(z = 2 + t\)
   b. \(x - 2 = y + 1 = z - 2\)
3. a. \(x = 2t, y = 4t, z = 3t\)
   b. \(\frac{x}{2} = \frac{y}{4} = \frac{z}{3}\)
5. a. \(x = 1, y = 2 - 3t, z = 4t\)
   b. \(x = 1, -\frac{y - 2}{3} = \frac{z}{4}\)
7. a. \(x = 3 + t, y = 4 + 7t, z = 4 - t\)
   b. \(x - 3 = \frac{y - 4}{7} = -\frac{z - 4}{1}\)
9. \(x = t + 1, y = 2 - t, z = 3 - t\)
11. \(x = 1 + t, y = -2 + t, z = 3 + t\)
13. \(x = 1 + t, y = 7, z = 0\)
15. \(z = 3\)
17. \(x + y - z = 2\)
19. \(z = 3\)
21. \(y - x = 1\)
23. \(2x + y + z = 5\)
25. \(7x + 5y - 3z = 10\)
27. \(x - 2y + z = -2\)
29. \(\frac{6\sqrt{17}}{17}\)
31. \(\frac{3\sqrt{2}}{2}\)
33. \(\frac{2\sqrt{6}}{3}\)
35. \(\frac{3\sqrt{14}}{28}\)
37. \(8x + 8y + 10z = 99\)

15. No; b. no; c. no intersection
17. No; b. no; c. \(\begin{pmatrix} 3 & 5 & 3 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}\)
19. No; b. no; c. \((-7, -8, -3)\)
21. a. \((0, 2, -3)\); b. \((2, 0, 1)\); c. \(\begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}\)