Sequences and Geometric Series

Introduction

Mathematical puzzles often lead to interesting and important mathematics. One well-known example credited to Leonardo of Pisa or Fibonacci (1170-1230) asks: *Start with one pair of rabbits and assume a pair of rabbits is productive starting in its second month. How many pairs of rabbits are there each month if every pair produces one new pair every month after it becomes productive?*

The answer to this idealized situation is the collection of numbers, called the Fibonacci sequence,

\[ 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \]

The \( n \)-th number in the list is the number of pairs of rabbits in month \( n \).

Fibonacci numbers describe phenomenon in nature, throughout history have appeared in art, and architecture, and have applications in science and mathematics.

Sequences

A sequence is an ordered list of objects. For example, the names listed in an ordinary phone book comprise a finite list or sequence. But a sequence can also have an infinite number of terms. For example, the list of odd natural numbers

\[ 1, 3, 5, 7, 9, \ldots \]

is an infinite sequence. A sequence can be described by listing enough elements so the pattern is clear or using a formula for the general term. The general term, or \( n \)-th term, for the sequence of odd natural numbers is \( 2n - 1 \). If a sequence is called \( a \), then the \( n \)-th term is denoted \( a_n \). The general term gives a relationship between the location of a term in the list and the corresponding term of the sequence.

**EXAMPLE 1**

Write the first five terms of the sequence, \( a \), with general term

\[ a_n = \frac{n}{n+1} \]

If \( n \) grows without bound, does the sequence have a limiting value?

**Solution**

Substituting \( n = 1, 2, 3, 4 \), and \( n = 5 \), the first five terms of the sequence are

\[ a_1 = \frac{1}{1+1} = \frac{1}{2}, a_2 = \frac{2}{2+1} = \frac{2}{3}, a_3 = \frac{3}{3+1} = \frac{3}{4}, a_4 = \frac{4}{4+1} = \frac{4}{5}, \text{and } a_5 = \frac{5}{5+1} = \frac{5}{6} \]
Graphing the terms of a sequence in a coordinate plane will often reveal information that is not obvious from several terms of the sequence, or the formula for the general term. Define the function

\[ f(x) = \frac{x}{x + 1}, \]

so the general term of the sequence is \( f(n) = n/(n + 1) \). The graph of \( y = f(x) \) and the first five terms of the sequence, corresponding to the points \((n, f(n))\), for \( n = 1, 2, 3, 4, 5 \), are shown in Figure 1(a). In Figure 1(b), a viewing window of \([0, 50] \times [0, 1.5]\) is used showing that as \( n \) grows without bound, the terms of the sequence get closer and closer to the value 1. So the limiting value of the sequence is 1, which is also the horizontal asymptote of \( f(x) = x/(x + 1) \).

\[
\begin{array}{cccc}
\text{Example 2} & \text{The first several terms of a sequence are} & 2, 5, 10, 17, 26, \ldots \\
\text{Solution} & \text{Find a formula for the general term of the sequence.} & \\
<table>
<thead>
<tr>
<th>n</th>
<th>n^2</th>
<th>a_n = n^2 + 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>17</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>26</td>
</tr>
</tbody>
</table>
\end{array}
\]

When looking for patterns in the terms of a sequence consider multiples of \( n \), such as \( 2n, -3n, \frac{1}{2}n \), and so on, or multiples of \( n^2, n^3 \), and so on. As shown in Table 1, the general term of the sequence is

\[ a_n = n^2 + 1. \]

\[
\begin{array}{cccc}
\text{Example 3} & \text{The first several terms of a sequence are} & 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \\
\text{Solution} & \text{Find a formula for the general term of the sequence.} & \\
\end{array}
\]
Solution

Since the terms of the sequence alternate in sign, each numerator is 1, and the denominators are successive powers of 2, the general term of the sequence is given by

\[ a_n = (-1)^{n-1} \frac{1}{2^{n-1}} = \frac{(-1)^{n-1}}{2^{n-1}} = \left( \frac{-1}{2} \right)^{n-1}, \quad \text{for } n = 1, 2, 3, \ldots \]

Finding the general term of a sequence can be difficult, but there are special sequences where the formula for the general term is easy to determine.

Arithmetic Sequences

An arithmetic sequence is a sequence where the difference of any pair of consecutive terms is the same fixed constant. For example, the sequence

3, 8, 13, 18, 23, . . .

is an arithmetic sequence where the difference between consecutive terms is always 5. The sequence can also be described by specifying the first term is 3, and successive terms are obtained by adding 5 to the previous term.

**Arithmetic Sequence**

An arithmetic sequence is a sequence of the form

\[ s, s + c, s + 2c, s + 3c, \ldots \]

The general term of the sequence is

\[ a_n = s + (n - 1)c, \quad \text{for } n = 1, 2, 3, \ldots \]

**EXAMPLE 4**

The first term of an arithmetic sequence is \(-5\) and the common difference between consecutive terms is 7.

a. Find the first five terms of the sequence.

b. What is the general term for the sequence?

c. Find the 105-th term of the sequence.

**Solution**

a. Starting with \(-5\) and adding 7 to obtain each successive term gives

\[-5, -5 + 7 = 2, 2 + 7 = 9, 9 + 7 = 16, \quad \text{and} \quad 16 + 7 = 23.\]

b. The general term of the sequence is

\[ a_n = -5 + (n - 1) \cdot 7 = -12 + 7n, \quad \text{for } n = 1, 2, 3, \ldots \]
c. To find the 105-th term, we substitute \( n = 105 \) in the formula for the general term to obtain
\[
a_{105} = -5 + (105 - 1) \cdot 7 = -5 + 104 \cdot 7 = 723.
\]

EXAMPLE 5
It is estimated that in the current year two million tons of a certain pollutant will be emitted into the atmosphere. Because of an anticipated expansion in the industries that emit the pollutant, and expected changes in the regulations controlling the amount of emissions that will be acceptable, for future planning the industry expects the increase in the amount of emissions per year will remain at 5% of the present level. What will the level of emission be in five years, in ten years, in \( n \) years? Estimate when the level of emissions will reach eight million tons per year.

Solution
Since the change in emissions is fixed at \( 0.05(2) = 0.1 \) million the yearly emissions form an arithmetic sequence. In \( n \) years the level of emission is
\[
2 + 0.1(n - 1) = 1.9 + 0.1n \text{ million tons.}
\]
In five years the level of emission is \( 1.9 + 0.1(5) = 2.4 \) million tons and in ten years the level is \( 1.9 + 0.1(10) = 2.9 \) million tons.
Solving
\[
8 = 1.9 + 0.1n, \text{ gives } n = \frac{6.1}{0.1} = 61,
\]
so it takes 61 years for the level of emission to reach 8 million.

EXAMPLE 6
Find the sum of the first \( n \) terms of the arithmetic sequence with first term \( s \) and common difference \( c \).

Solution
The general term for the sequence is \( s + (n-1)c \), so the sum of the first \( n \) terms is
\[
s + (s + c) + (s + 2c) + \ldots + (s + (n - 1)c)
= s + s + \ldots + s + (c + 2c + 3c + \ldots + (n - 1)c)
= n \cdot s + c \cdot (1 + 2 + 3 + \ldots + (n - 1)).
\]
The sum of the first \( k \) positive integers is given by the formula
\[
1 + 2 + \ldots + k = \frac{k(k + 1)}{2}.
\]
We will verify below that a sum of the first \( k \) positive integers is \( \frac{k(k+1)}{2} \). This allows us to further rewrite the previous expression as
\[
s + (s + c) + (s + 2c) + \ldots + (s + (n - 1)c) = n \cdot s + c \cdot (1 + 2 + 3 + \ldots + (n - 1))
= n \cdot s + c \left( \frac{(n-1)n}{2} \right)
= n \frac{1}{2} (2s + (n - 1)c)
= \frac{n}{2} (s + s + (n - 1)c) = \frac{n}{2} (s + a_n).
\]
Geometric Sequences

A geometric sequence is defined in a similar manner to an arithmetic sequence, except rather than adding a fixed amount to each term to get the successor, a fixed amount is multiplied to a term to get the successor.

**Geometric Sequence**

A geometric sequence is a sequence of the form

\[ s, sr, sr^2, sr^3, \ldots \]

The general term of the sequence is

\[ a_n = sr^{n-1}, \text{ for } n = 1, 2, 3, \ldots \]

The number \( r \) is called the **common ratio**.

**EXAMPLE 7**  Determine the first term and the common ratio for the geometric sequence

\[ 5, \frac{10}{3}, \frac{20}{9}, \frac{40}{27}, \ldots \]

What is the general term of the sequence? What is the 20-th term of the sequence? Does the sequence have a limiting value?

**Solution**  Factoring 5 from each term the sequence can be rewritten in the form

\[ 5, 5 \cdot \left( \frac{2}{3} \right)^1, 5 \cdot \left( \frac{2}{3} \right)^2, 5 \cdot \left( \frac{2}{3} \right)^3, \ldots, \]

The sequence is geometric with common ratio \( 2/3 \) and general term \( 5 \left( \frac{2}{3} \right)^{n-1} \). So the 20-th term is

\[ 5 \left( \frac{2}{3} \right)^{19} = 5 \left( \frac{524288}{1162261467} \right) = \frac{2621440}{1162261467}. \]

The graph of the exponential function \( f(x) = 5 \left( \frac{2}{3} \right)^x \), shown in Figure 2, suggests the limiting value of the sequence is 0.

In Example 5, we considered a scenario where the annual emission of a pollutant formed an arithmetic sequence. In the next example, we change the scenario and the sequence that models the annual emission is a geometric sequence.
EXAMPLE 8  It is estimated that in the current year two million tons of a certain pollutant will be emitted into the atmosphere. The industry estimates the increase in the amount of emissions from one year to the next will be 5% of the current level. What will the level of emission be in five years, in ten years, in \( n \) years? Estimate when the level of emission will reach eight million tons per year.

Solution  Several levels of emission of the pollutant in millions of tons per year are

\[
2, 2 + (0.05)2, \text{ and } 2 + (0.05)2 + 0.05(2 + (0.05)2).
\]

Factoring gives

\[
2 + (0.05)2 = 2(1+0.05), \text{ and } 2(1+0.05)+(0.05)(2(1+0.05)) = 2(1+0.05)(1+0.05) = 2(1+0.05)^2.
\]

This describes a geometric sequence with first term 2 million tons and general term \( 2(1 + 0.05)^{n-1} = 2(1.05)^{n-1} \), for \( n = 1, 2, 3, \ldots \) The level of emission after 5 and 10 years, respectively, is

\[
2(1.05)^4 \approx 2.43 \text{ million tons and } 2(1.05)^9 \approx 3.1 \text{ million tons}.
\]

The level of emission will reach eight million tons when \( 8 = 2(1.05)^{n-1} \). To isolate \( n \), we have

\[
\ln 4 = \ln(1.05)^{n-1} = (n-1)\ln(1.05), \text{ so } n = \frac{\ln 4 + \ln(1.05)}{\ln(1.05)} \approx 29.4.
\]

Under this scenario it takes less than 30 years for the level of emissions to reach 8 million tons per year, whereas in the scenario of Example 5, it required 61 years. ■

To represent an integer in computer memory an integer consists of a fixed number of bits, where a bit is either a 0 or a 1 (base 2 representation). The left most bit determines whether the integer is positive, bit is 0, or negative, bit is 1, and each other bit represents a power of 2, as shown in Figure 3.

<table>
<thead>
<tr>
<th>( 2^5 )</th>
<th>( 2^4 )</th>
<th>( 2^3 )</th>
<th>( 2^2 )</th>
<th>( 2^1 )</th>
<th>( 2^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0/1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 3

The integer represented in Figure 3 is the positive integer

\[
1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 + 1 \cdot 2^5 = 1 + 0 + 4 + 8 + 0 + 32 = 45.
\]

EXAMPLE 9  Find the largest positive integer that can be represented using 16 bits.

Solution  The largest positive integer using 16 bits is a 0 followed by 15 ones. We have

\[
01111111111111 = 2^{14} + 2^{13} + 2^{12} + \cdots + 2^3 + 2^2 + 2^1 + 2^0 = 32767.
\]

■
In Example 9, we summed the powers of 2 to obtain the value 32767 for the largest positive integer that can be represented using only 16 bits. There is a formula for adding consecutive terms of a geometric sequence that can be used in situations like this.

Suppose the common ratio of a geometric sequence is $r$ and the first term is $a$. Let $S_n = a + ar + ar^2 + \ldots + ar^n$ denote the sum of the first $n$ terms. Then

$$S_n - rS_n = a + ar + ar^2 + \ldots + ar^{n-1} + ar^n - r(a + ar + ar^2 + \ldots + ar^{n-1} + ar^n)$$

$$= a + ar + ar^2 + \ldots + ar^{n-1} + ar^n - (ar + ar^2 + \ldots + ar^{n-1} + ar^n + a)$$

$$= a - ar^{n+1}.$$ 

Factoring $S_n$ from both terms on the left gives

$$S_n(1 - r) = a(1 - r^{n+1}), \quad \text{so} \quad S_n = \frac{a(1 - r^{n+1})}{1 - r}.$$ 

In Example 9, the common ratio for the geometric sequence is 2, so

$$1 + 2 + 2^2 + \ldots + 2^{14} = \frac{1 - 2^{15}}{1 - 2} = 2^{15} - 1 = 32768 - 1 = 32767.$$ 

The Fibonacci Sequence and the Golden Ratio

A mathematical puzzle that is credited to Fibonacci (1170-1230) asks: *Start with one pair of rabbits and assume a pair of rabbits is productive starting in its second month. How many pairs of rabbits are there each month if every pair produces one new pair every month after it becomes productive?*

The answer is given by the sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots,$$

as shown in Figure 4. This sequence of numbers is called the Fibonacci sequence. If the $n$-th term is denoted by $F_n$, a recursive definition is given by

$$F_1 = 1, F_2 = 1, \text{ and } F_n = F_{n-2} + F_{n-1}, \quad \text{for } n = 3, 4, 5, \ldots$$ 

Fibonacci numbers describe leaf arrangements, the number of branches in plants, and the number of spirals in sunflower seed pods. The *Fibonacci spiral* describes the spiral shapes in snail shells, the nautilus and other sea shells. The Fibonacci numbers are used in constructing certain types of codes used to compress data. They appear in art and architecture and are connected to the construction of the Egyptian pyramids and the Parthenon in Athens Greece.

The ratio of successive terms of the Fibonacci sequence leads to the number called the *golden ratio*, which is approximately 1.618. See Table 2. The front of
the Parthenon is a rectangle where the ratio of the width divided by the height is 1.62. Throughout history in art and architecture the rectangle with ratio of width to height the golden ratio was considered the most aesthetic.

Assuming the ratio of successive terms of the Fibonacci sequence does approach a number, call the number $R$.

Since $F_n = F_{n-1} + F_{n-2}$, we have $\frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}} = 1 + \frac{1}{F_{n-1}/F_{n-2}}$.

The left hand side approaches $R$ and the right hand side approaches $1 + 1/R$, so $R = 1 + \frac{1}{R}$ implies $R^2 = R + 1$, so $R^2 - R - 1 = 0$.

Using the Quadratic Formula, we have $R = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$.

The golden ratio is the number

$\frac{1 + \sqrt{5}}{2} \approx 1.61803398875$.

Exercise Set

In Exercises 1–8, a formula for the general term of a sequence is given. List the first five terms of the sequence and the one hundredth term of the sequence. Does the sequence have a limiting value?

1. $a_n = \frac{n}{n+1}$
2. $a_n = -\frac{3n^3 - 5n^2 - n + 3}{n^2 - 2n + 1}$
3. $a_n = \frac{n^2}{100n + 15}$
4. $a_n = (-1)^n n$
5. $a_n = \frac{n}{2n + 1}$
6. $a_n = \left(\frac{3}{4}\right)^n$
7. $a_n = \left(\frac{n}{3}\right)^n$
8. $a_n = \frac{\ln n}{n}$

In Exercises 9–12, determine a formula for the $n$th term of the sequence.

9. $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{11}, \ldots$
10. $\frac{3}{2}, -\frac{3}{4}, \frac{3}{8}, -\frac{3}{16}, \ldots$
11. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$
12. $3, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \ldots$

In Exercises 13–16, find the common difference of the arithmetic sequence and list the first five terms of the sequence.

13. $a_n = n + 3$
14. $a_n = 3n + 2$
15. $a_n = \frac{1}{4} + \frac{n}{2}$
16. $a_n = -3 - 2n$

In Exercises 17–20, the sequence is a geometric sequence.

a. What is the first term and common ratio of the sequence?

b. List the first five terms of the sequence.

c. Find the sum of the first $n$ terms of the sequence for $n = 1, 2, 3, 4, \text{and } n = 5$. What is the general formula for the sum of the first $n$ terms of the sequence?
d. Does the sequence of partial sums of the geometric sequence have a limiting value?

17. \(a_n = (-2)^n\)  
18. \(a_n = \frac{3^n}{2^n-1}\)

19. \(a_n = 2\left(\frac{5}{8}\right)^{n-1}\)  
20. \(a_n = \frac{4^n}{5^n-1}\)

In Exercises 21–24, determine if the sequence is an arithmetic sequence or a geometric sequence. If it is an arithmetic sequence, find the common difference. If it is a geometric sequence, find the common ratio. Find a formula for the general term of the sequence.

21. 3, 9, 15, 21, 27, ...
22. 1, \(-\frac{2}{5}\), \(-\frac{4}{25}\), \(-\frac{8}{125}\), ...
23. \(\frac{1}{3}, \frac{2}{15}, \frac{4}{45}, \frac{8}{135}, \ldots\)
24. 2, 0, \(-2\), \(-4\), \(-6\), \(-8\), ...

25. The first term of an arithmetic sequence is 12 and the common difference is 3.

a. Write the formula for the general term of the sequence.

b. Is 27431 a term of the sequence?

c. How many terms of the sequence must be added to obtain a sum of 882?

d. Is there an \(n\), so that the sum of the first \(n\) terms of the arithmetic sequence equals 12567? If not, what is the least \(n\), so that the sum of the first \(n\) terms exceeds 12567?

26. The third term of a geometric sequence is \(-\frac{4}{3}\), and the sixth term is \(-\frac{32}{81}\).

a. Find the first term and the common ratio of the sequence. Write the formula for the general term of the sequence.

b. List the first five terms of the sequence.

c. What is the one hundredth term?

27. The first term of a geometric sequence is 3 and the common ratio is 4.

a. Write the formula for the general term of the sequence.

b. Is 196604 a term of the sequence?

c. How many terms of the sequence must be added to obtain a sum of 4194303?

28. On a typical day a landfill operator weighs in approximately 10000 tons of waste that is dumped in the landfill. Write a formula for the number of tons of waste that the operator weighs in after \(n\) days. Estimate the number of years required to add 5 million tons of waste to the landfill.

29. A laboratory research starts a culture with 1000 bacteria and observes that the number of bacteria doubles every hour.

a. Write a formula for the number of bacteria present after \(n\) hours.

b. How many bacteria will be present after 5 hours?

c. If no other factors affect the growth of the bacteria is there a limiting value for the size of the population?

30. A young boy offers to wash the family car every Saturday starting in April through September. The boy says to his father all you have to pay me is 1 penney for the first wash and agree to double the payment on each successive wash. Without thinking the father accepts thinking he is getting a good deal. Who is getting the better deal?

31. The Lucas numbers are defined recursively by

\[
L_0 = 2 \\
L_1 = 1 \\
L_n = L_{n-2} + L_{n-1}, \text{ for } n \geq 2.
\]

a. List the first 10 Lucas numbers.

b. Find a recursive definition for the Lucas numbers in terms of the Fi-
bonacci numbers. Lucas numbers.
c. Find a recursive definition for the Fibonacci numbers in terms of the Lu-

Annuities

An annuity is an amount paid at regular intervals. For example, contributing the same amount every month in a retirement fund or the monthly payment on a 30 year home mortgage loan. In the case of the retirement fund, the current amount receives interest also at regular time intervals. In the case of a loan, interest on the amount borrowed is added to the unpaid balance.

EXAMPLE 1 In planning for retirement you decide to contribute $500.00 per month in an individual retirement account (IRA) that returns 3% interest per year compounded monthly. Each payment is made on the 15-th of the month and a deposit must be in the account for a month before it receives interest. What is the value of your account at the end of 30 years?

Solution The interest earned each month is 0.03/12 = 0.0025 and there are 12·30 = 360 deposits. When interest is computed, the value of the account is the current value times $(1 + 0.0025) = 1.0025$. Since the interest on the first deposit is computed February 15 (assuming the deposits start in January) at the end of the first month, the value of the account is 500. Similarly the second deposit does not receive interest until March 15, so at the end of the second month the value is $500(1.0025) + 500$. At the end of the third month the value is

\[ (500(1.0025) + 500)(1.0025) + 500 = 500 + 500(1.0025) + 500(1.0025)^2. \]

After 30 years the value of the account is

\[ 500 + 500(1.0025) + \ldots + 500(1.0025)^{359} \]

\[ = 500 \left( 1 + 1.0025 + (1.0025)^2 + \cdots + (1.0025)^{359} \right) \]

\[ = 500 \frac{1 - (1.0025)^{360}}{1 - 1.0025} \approx \$291,368.44. \]

We used the formula for the sum of the first \( n \) terms of a geometric sequence.
The Value of an Annuity

If \( P \) dollars is deposited \( n \) times per year at an annual interest rate \( r \), compounded \( n \) times per year, then the value of the annuity at the end of \( t \) years is

\[
P + \left(1 + \frac{r}{n}\right)P + \left(1 + \frac{r}{n}\right)^2P + \ldots + \left(1 + \frac{r}{n}\right)^{nt-1}P
\]

\[
= P\frac{1 - \left(1 + \frac{r}{n}\right)^{nt}}{1 - \left(1 + \frac{r}{n}\right)}
\]

\[
= P\frac{(1 + \frac{r}{n})^{nt} - 1}{\frac{r}{n}}.
\]

EXAMPLE 2

How much money will you have to invest every month at 5% interest compounded monthly in order to have $500,000.00 after 25 years?

Solution

Let \( r = 0.05 \), \( n = 12 \), \( t = 25 \), and the value of the annuity after the 25 years is $500000.00. Then

\[
500000 = P\frac{1 + \frac{0.05}{12}}{0.05}^{25(12)} - 1,
\]

so \( P = \frac{500000}{(1 + \frac{0.05}{12})^{300} - 1} \approx 839.62 \).

So you will have to invest approximately $840.00 per month.

- \( A = B_0 \) is the loan amount
- \( B_n \) is the balance after \( n \) payments
- \( i \) is the interest rate
- Interest is compounded when payments are made
- \( P \) is the amount of each payment

We now turn to determining the balance on an installment loan after \( n \) payments have been made. Let \( A = B_0 \) denote the amount of the loan and \( B_n \) the balance of the loan after \( n \) payments have been made. All loans come with an interest rate that is compounded over the life of the loan. Let \( i \) denote the interest rate (for example, if the interest is 8% annually and is compounded monthly, then \( i = 0.08/12 \)), which is compounded at the same time that payments are made. Let \( P \) denote the amount of each payment. At the end of first period (first month for most installment loans) the interest owed is computed on the initial amount and the first payment is deducted, so

\[
B_1 = A(1+i) - P.
\]

At the end of the second period, we have

\[
B_2 = (1+i)[A(1+i) - P] - P = A(1+i)^2 - (1+i)P - P = A(1+i)^2 - P[1+(1+i)].
\]

Using this result at the end of the third period, we have

\[
B_3 = A(1+i)^3 - P[1 + (1+i)](1+i) - P = A(1+i)^3 - P[1 + (1+i) + (1+i)^2].
\]
Continuing this process at the end of period \( n \), we have

\[
B_n = A(1 + i)^n - P\left[1 + (1 + i) + (1 + i)^2 + \ldots + (1 + i)^{n-1}\right] = A(1 + i)^n - P\left(\frac{1 - (1 + i)^n}{1 - (1 + i)}\right) = A(1 + i)^n - P\left(\frac{(1 + i)^n - 1}{i}\right).
\]

So the balance of the loan after \( n \) payments is the value of the loan after the interest has been compounded \( n \) times, minus the value of the annuity after \( n \) payments of amount \( P \) at the same interest rate.

**EXAMPLE 3** What is the monthly payment on a car loan of $20,000.00 at 4% interest for 5 years? After 25 payments have been made what is the balance of the loan? How much interest is paid on the life of the loan?

**Solution** The interest rate each month is \( 0.04/12 \), and since the loan is for 5 years, the total number of payments is 60. When the last payment is made the balance of the loan is 0, so

\[
0 = B_{60} = 20000 \left(1 + \frac{0.04}{12}\right)^{60} - P\left(\frac{\left(1 + \frac{0.04}{12}\right)^{60} - 1}{\frac{0.04}{12}}\right).
\]

Solving for \( P \) gives

\[
P = \frac{20000 \left(1 + \frac{0.04}{12}\right)^{60}}{\left(\frac{\left(1 + \frac{0.04}{12}\right)^{60} - 1}{\frac{0.04}{12}}\right)} \approx 368.33047.
\]

So the monthly payment is $368.33. After 25 payments have been made the balance of the loan is

\[
B_{25} = 20000 \left(1 + \frac{0.04}{12}\right)^{25} - 368.33 \left(\frac{\left(1 + \frac{0.04}{12}\right)^{25} - 1}{\frac{0.04}{12}}\right) \approx 19584.0178.
\]

So the balance after 25 payments is $11,748.62. The amount paid over the life of the loan is 60 \cdot 368.33 = 22099.80, so the interest paid is $2,099.80.

**EXAMPLE 4** The price of a new home is $300,000.00. If the down payment is 10% and the mortgage is for 30 years at 5.5% interest, what is the monthly payment?
Solution  Set \( A = 300000 - 0.1(300000) = 270000 \), \( i = \frac{0.055}{12} \), and \( N = 12 \cdot 30 = 360 \). Then

\[
P = \frac{iA}{1 - (1 + i)^{-N}} = \frac{\left(\frac{0.055}{12}\right) \cdot 270000}{1 - \left(1 + \frac{0.055}{12}\right)^{-360}} \approx 1533.03.
\]

So the monthly payment is $1,533.03.

Notice that at the end of the first year the total amount paid is \( 12 \cdot 1533.03 = $18,396.36 \) and yet the remaining balance is \( B_{12} = $269,412.21 \), so only $587.79 of the principal has been paid and the rest has gone towards the interest. ■

The equation for each payment

\[
P = \frac{iA}{1 - (1 + i)^{-N}}
\]

can be used to determine the number of payments required by solving for \( N \) in the equation. Isolating the term \((1 + i)^{-N}\) gives

\[
(1 + i)^{-N} = \frac{P - iA}{P} \quad \text{so} \quad -N \ln(1 + i) = \ln \frac{P - iA}{P}, \quad \text{hence} \quad N = -\frac{\ln \frac{P - iA}{P}}{\ln(1 + i)}.
\]

EXAMPLE 5  An account contains $30,000.00 and receives 5% interest per year compounded monthly. If $1,500.00 is withdrawn every month, how long will it take to exhaust the balance?

Solution  Set \( A = 30000, P = 1500, \) and \( i = \frac{0.05}{12}, \) so

\[
N = -\frac{\ln \frac{P - iA}{P}}{\ln(1 + i)} = -\frac{\ln \frac{1500 - \left(\frac{0.05}{12}\right)30000}{1500}}{\ln \left(1 + \frac{0.05}{12}\right)} \approx 20.92620433.
\]

Consequently, the $1,500.00 can be withdrawn for only 20 months. ■

Exercise Set

1. Find the amount of an annuity that has a life of 30 years, requires monthly contributions of $350.00, and returns 5% interest per year compounded monthly.

2. At 25 retirement seems a long way off but you have decided to plan now and decide that if you want to retire at 55 you will need $700,000.00 in a retirement account. If you can afford to contribute $625.00 a month and the return is 5.25% per year compounded monthly, can you reach your goal?
3. Determine the monthly payment on an annuity that has a life of 25 years, and returns 7% interest per year compounded monthly, in order to have a final value of $300,000.00.

4. To plan for retirement you decide to consider different scenarios for investment. The primary constraint is on the amount you can contribute per month and still maintain a comfortable life style for you and your family. You also want to determine reasonable options on the age you can retire. Assume the analysis is done when you are 25 years old and you can conservatively get a return of 7.5% interest per year compounded monthly.
   a. If you invest $400.00 per month and you retire at 55, what is the value of your investment fund?
   b. If you invest $500.00 per month and you retire at 55, what is the value of your investment fund?
   c. If you invest $500.00 per month and you retire at 65, what is the value of your investment fund?
   d. If you anticipate needing $750,000.00 when you retire at 55, what is the monthly contribution?
   e. If you anticipate needing $750,000.00 when you retire at 65, what is the monthly contribution?
   
5. What is the monthly payment on a car loan of $25,000.00 at 4.5% interest for 5 years?

6. A $20,000.00 car loan is taken for 5 years at 4% interest.
   a. What is the monthly payment?
   b. After 2 years of payments what is the balance of the loan?

7. If you are willing to pay $325.00 per month for a car loan and the car company is offering 3% loans for 5 years, what price car can you afford?

8. If the car you want costs $17,500.00 and you are willing to pay $300.00 per month for 5 years, what interest rate will allow you to purchase the car?

9. The price of a new home is $185,000.00. If the buyers qualify for a 30 year mortgage at 4.5% interest and are willing to make monthly payments of $850.00 what down payment will they have to make?

10. The price of a new home is $135,000.00 and the buyers can obtain an interest rate of 5.5%.
   a. If the loan is for 30 years what is the monthly payment?
   b. If the loan is for 15 years what is the monthly payment?
   c. If the loan is for 30 years what is the total interest paid?
   d. If the loan is for 15 years rather than 30 years how much less interest is paid?
   e. If the loan is for 30 years what is the balance of the loan after five years of payments?
   f. If the loan is for 30 years how much of the first five years of payments is interest?

11. An account contains $500,000.00 and receives 4% interest compounded monthly. If $4500.00 is withdrawn each month, how long will it take to exhaust the account?
Geometric Series

A sequence is a list of numbers and a series is a sum of numbers. In base ten the decimal number 0.2131 has the meaning

\[ 0.2131 = \frac{2}{10} + \frac{1}{10^2} + \frac{3}{10^3} + \frac{1}{10^4} = \frac{2}{10} + \frac{1}{100} + \frac{3}{1000} + \frac{1}{10000}. \]

With respect to a coordinate line, the digits describe the location of the number on the line, as shown in Figure 5. But what happens when we try to represent the fraction 1/3 as a decimal?

The fraction 1/3 has the repeating decimal expansion

\[ \frac{1}{3} = 0.3333 \ldots = 0,3, \]

where the 3 in the decimal expansion continues indefinitely. This implies that no finite number of digits 3 in the expansion are sufficient to locate the number 1/3 on a coordinate line. The exact value is given by

\[ \frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots \]

How can we sum an infinite number of terms? The n-th partial sums of a series is the sum of the first n terms. Table 3 lists several partial sums for the decimal expansion of 1/3. In this example each successive partial sum adds one more 3 to the end of the expansion and gets closer to the exact value of the fraction.

The sequence of partial sums of the series

\[ a_1 + a_2 + a_3 + \ldots \]

is the sequence of numbers

\[ S_1 = a_1, S_2 = a_1 + a_2, S_3 = a_1 + a_2 + a_3, \ldots \]

If as n increases indefinitely, the numbers in the sequence of partial sums approach a unique number, then that number is called the sum of the series. This is written as:

\[ \text{if } S_n \to L \text{ as } n \to \infty, \text{ then } \sum_{n=1}^{\infty} a_n = L. \]

EXAMPLE 1 Find the sum of the series

\[ \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}. \]
Solution  The sum of an infinite series, if it exists, depends on the behavior of the partial sums. Several partial sums are

\[ S_1 = \frac{1}{2^0} = 1, \quad S_2 = 1 + \frac{1}{2} \quad \text{and} \quad S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1 + \frac{3}{4}. \]

The data in Table 4 suggests the sum of the series is 2. In fact, the \( n \)-th partial sum is given by \( 1 + \frac{2^n}{2^n} \), which approaches 2 as \( n \) goes to infinity, so

\[
\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2. \]

Table 4  
<table>
<thead>
<tr>
<th>( n )</th>
<th>( n )-th Partial Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>3</td>
<td>1.75</td>
</tr>
<tr>
<td>4</td>
<td>1.875</td>
</tr>
<tr>
<td>5</td>
<td>1.9375</td>
</tr>
<tr>
<td>6</td>
<td>1.96875</td>
</tr>
<tr>
<td>7</td>
<td>1.984375</td>
</tr>
<tr>
<td>8</td>
<td>1.9921875</td>
</tr>
<tr>
<td>9</td>
<td>1.99609375</td>
</tr>
<tr>
<td>10</td>
<td>1.998046875</td>
</tr>
</tbody>
</table>

What value, if any, can we assign to the series

\[ 1 - 1 + 1 - 1 + 1 - 1 + \ldots \]

where the terms alternate between plus 1 and minus 1? Several partial sums are

\[ S_1 = 1, \quad S_2 = 1 - 1 = 0, \quad S_3 = 1 - 1 + 1 = 1, \quad \text{and} \quad S_4 = 1 - 1 + 1 - 1 = 0, \]

so the partial sums alternate between 1 and 0. Regardless of how large the index \( n \) becomes the terms of the partial sums will never approach a unique value. This series does not have a sum.

A geometric series is the sum of the terms of a geometric sequence.

Geometric Series

A geometric series is a sum of the form

\[
\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots
\]

with first term \( a \) and common ratio \( r \).

The series we considered in Example 1 is a geometric series with first term 1 and common ratio 1/2. Geometric series are particularly useful because there is an easy check for when a sum exists.

The Sum of a Geometric Series

If \( |r| < 1 \), then the sum of the geometric series is

\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}.
\]
Using the formula for the sum of a finite number of terms of a geometric sequence, the $n$-th partial sum of a geometric series is

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a(1 + r + r^2 + \ldots + r^{n-1}) = \frac{a(1 - r^n)}{1 - r}.$$ 

If $-1 < r < 1$, then $r^n \to 0$ as $n \to \infty$ and when $r > 1$ or $r < -1$, the magnitude of the terms of $r^n$ grow without bound, as illustrated in Figure 6. This implies a geometric series will have the sum $a/(1 - r)$ when $|r| < 1$ and will not have a sum when $|r| \geq 1$.

![Figure 6](image-url)

**EXAMPLE 2** Find the sum of the geometric series.

a. $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$
b. $\sum_{n=1}^{\infty} \left(-1\right)^n \frac{4^n}{5^n}$
c. $\frac{4}{15} + \frac{4}{135} + \frac{4}{1350} + \ldots$

**Solution**

a. If we set $n = 1, 2, 3, 4, \text{ and } 5$, to determine the first several terms of the series, we have

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{2}{3}\right)^{n-1}.$$ 

Since the first term of the series is $2/3$ and the common ratio is also $2/3$, which is less than 1, the sum of the geometric series is

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \frac{2}{3} \left(\frac{1}{1 - \frac{2}{3}}\right) = \frac{2}{3} \left(\frac{3}{1}\right) = \frac{2}{3} (3) = 2.$$ 

b. We first adjust the exponents on the 4 and the 5 to make them the same, so

$$\frac{4^n}{5^{n+1}} = \frac{4^n}{5^n} \text{ and } \sum_{n=1}^{\infty} \left(-1\right)^n \frac{4^n}{5^{n+1}} = \frac{1}{5} \sum_{n=1}^{\infty} \left(-\frac{4}{5}\right)^n.$$
The resulting series has first term \(-\frac{4}{5}\) and common ratio \(-\frac{4}{5}\). Since \(-1 < -\frac{4}{5} < 1\), the sum of the series is

\[
\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{5^{n+1}} = \frac{1}{5} \sum_{n=1}^{\infty} \left( -\frac{4}{5} \right)^n = \frac{1}{5} \left( \frac{-4/5}{1 - (-4/5)} \right) = -\frac{4}{45}.
\]

c. When can remove common factors from both the numerators and denominators of the fractions in the sum to obtain

\[
\frac{4}{15} + \frac{4}{45} + \frac{4}{135} + \ldots = \frac{4}{5} \left( \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \right) = \frac{4}{5} \left( \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots \right) = \frac{4}{15} \left( 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots \right) = \frac{4}{15} \cdot \frac{3}{2} = \frac{2}{5}.
\]

The Koch Island

The **Koch Island** is an example of a curve whose construction requires an infinite number of steps. The process begins with an equilateral triangle and approximations are obtained by removing the middle third of each side and replacing it with two line segments of the same length to form the sides of three additional equilateral triangles. The process is then repeated. The first four approximations to the Koch Island are shown in Figure 7.

![Figure 7](image_url)

**EXAMPLE 3** Find the area and perimeter of the Koch Island constructed from an equilateral triangle with side 1.

**Solution** Let \(A\) denote the area of an equilateral triangle with side \(a\). The height is

\[
h = \sqrt{a^2 - \left( \frac{a}{2} \right)^2} = \sqrt{\frac{3a^2}{4}} = \frac{\sqrt{3}a}{2}, \text{ so } A = \frac{1}{2} \cdot a \cdot \frac{\sqrt{3}a}{2} = \frac{\sqrt{3}a^2}{4},
\]

as shown in Figure 8. If we now consider an equilateral triangle of side \(\frac{a}{3}\), the area of the smaller triangle is

\[
\frac{1}{2} \cdot \frac{a}{3} \cdot \frac{\sqrt{3} \cdot \frac{a}{3}}{2} = \frac{1}{9} \cdot \frac{\sqrt{3}a^2}{4} = \frac{1}{9} \cdot A.
\]

The first 5 steps in the construction of the Koch Island are given in Table 5.
The area of the island is the sum of the areas of all the triangles in the construction, so the area is

\[
\frac{\sqrt{3}}{4} + 3 \left( \frac{1}{9} \right) \frac{\sqrt{3}}{4} + 3 \cdot 4 \left( \frac{1}{9} \right)^2 \frac{\sqrt{3}}{4} + 3 \cdot 4^2 \left( \frac{1}{9} \right)^3 \frac{\sqrt{3}}{4} + 3 \cdot 4^3 \left( \frac{1}{9} \right)^4 \frac{\sqrt{3}}{4} + \ldots
\]

We used the sum of a geometric series with common ratio \(\frac{1}{9}\).

\[
= \frac{\sqrt{3}}{4} \left[ 1 + 3 \left( \frac{1}{9} + 4 \left( \frac{1}{9} \right)^2 + 4^2 \left( \frac{1}{9} \right)^3 + 4^3 \left( \frac{1}{9} \right)^4 + \ldots \right) \right]
\]

\[
= \frac{\sqrt{3}}{4} \left[ 1 - \frac{3}{9} \left( 1 + 4 \left( \frac{1}{9} \right) + 4^2 \left( \frac{1}{9} \right)^2 + 4^3 \left( \frac{1}{9} \right)^3 + \ldots \right) \right]
\]

\[
= \frac{\sqrt{3}}{4} \left[ 1 + \frac{3}{9} \left( \frac{1}{1 - \frac{4}{9}} \right) \right] = \frac{\sqrt{3}}{4} \left[ 1 + \frac{3}{9} \cdot \frac{9}{5} \right] = \frac{\sqrt{3}}{4} \left[ \frac{8}{5} \right] = \frac{2\sqrt{3}}{5}.
\]

At each step of the construction line segments are replaced with 4 line segments of \(1/3\) the length. See Table 6 and Figure 7. At stage \(n\), the perimeter is

\[
3 \left( \frac{4}{3} \right)^{n-1}, \quad \text{so} \quad 3 \left( \frac{4}{3} \right)^{n-1} \to \infty, \quad \text{as} \quad n \to \infty.
\]

We have shown the area of the Koch Island is finite, but the coastline has infinite length.

The Koch Island is an example of a fractal. In Figure 9 is another example, called the Mandelbrot set.
Exercise Set

In Exercises 1–4, state whether the geometric series does or does not have a sum. If the series does have a sum, then find it.

1. \( \sum_{n=1}^{\infty} \frac{7}{2^n} \)

2. \( \sum_{n=1}^{\infty} \frac{6^{n+1}}{2^{2n}} \)

3. \( 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \)

4. \( -\frac{3}{8} + \frac{3}{16} - \frac{3}{32} - \frac{3}{64} + \ldots \)

In Exercises 5–8, use a geometric series to write the repeating decimal in the form \( \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \neq 0 \).

5. \( 0.2525 \)

6. \( 0.555 \)

7. \( 0.453453 \)

8. \( 0.125125 \)

9. Determine the total length removed from the unit interval \([0, 1] \) when the following procedure is followed, as shown in the figure.

- Remove the middle third of the unit interval.
- Remove the middle thirds of the two intervals remaining after the first step.
- Remove the middle thirds of the four intervals that remain after the second step.
- Continue the process indefinitely.

10. Determine how much area is removed from a square of side 1 when the following procedure is followed, as shown in the figure.

- Divide the square in four equal quarters and remove the top left quarter.
- Divide the lower right quarter in four equal quarters and remove the top left quarter.
- Continue dividing the remaining lower right quarter into four equal quarters indefinitely, removing the upper left quarter at each step of the process.

Answers to Sequences Exercises

1. \( \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{6}}{n + 1} \rightarrow 1, \) as \( n \rightarrow \infty. \)

3. \( \frac{\frac{1}{115} \cdot \frac{2}{219} \cdot \frac{9}{415} \cdot \frac{16}{515} \cdot \frac{25}{515}}{100n + 15} \rightarrow \infty, \) as \( n \rightarrow \infty. \)

5. \( \frac{\frac{1}{2} \cdot \frac{32}{4} \cdot \frac{243}{8} \cdot \frac{1024}{16} \cdot \frac{3125}{64}}{\frac{n}{2n}} \rightarrow 0, \) as \( n \rightarrow \infty. \)
21. Arithmetic Sequence
Common difference: 6
General term: $3 + (n - 1) \cdot 6$

23. Geometric Sequence
Common ratio: $\frac{2}{3}$
General term: $\frac{1}{5} \left( \frac{2}{3} \right)^{n-1}$

25. a. $12 + 3 \cdot (n - 1)$
b. No, since $27431 = 12 + 3 \cdot (n - 1)$ implies $3n = 27422$ and 3 does not divide evenly into 27422.
c. Since $882 = \frac{2}{3} \left( 12 + (12 + 3(n-1)) \right)$ implies

$$0 = 3n^2 + 21n - 1764 = 3(n+28)(n-21),$$
21 terms are required.
d. No, since the roots of

$$3n^2 + 21n - 25134 = 0$$
are $n \approx 88.1$ and $n \approx -95.1$. So the least $n$ is 89.

27. a. $3(4)^{n-1}$
b. No, since $196604 = 3(4)^{n-1}$ implies $n - 1 = \ln \frac{196604}{3}$ implies $n = \ln \frac{196604}{3} + 1$ implies $n \approx 12.1$,
c. Yes, since

$$3 \left( \frac{1 - 4^{n+1}}{1 - 4} \right) = 4194303$$
implies $n = 10$, so 10 terms are required.

29. a. $1000 \cdot 2^{n-1}$
b. $1000 \cdot 2^4 = 16000$
c. No, since

$$1000 \cdot 2^{n-1} \to \infty \text{ as } n \to \infty.$$
Answers to Annuities Exercises

1. 
\[350 \frac{\left(1 + \frac{0.05}{12}\right)^{12 \cdot 30}}{\frac{0.05}{12}} \approx \$291,290\]
implies \(A \approx 18087\), so you can afford to pay approximately \(\$18,087.00\).

9. The amount the buyers can afford is

\[A = 850 \frac{\left(1 - \left(1 - \frac{0.045}{12}\right)^{-60}\right)}{\frac{0.045}{12}} \approx 45493,\]

so they will need a down payment of approximately \(\$139,407.00\).

11. The number of withdrawals is

\[N = -\frac{\ln\left(\frac{4500 - \frac{0.045 \cdot 500000}{12}}{4500}\right)}{\ln\left(1 + \frac{0.04}{12}\right)} \approx 139\]

or approximately 11 years and 5 months.

Answers to Geometric Series Exercises

1. 
\[\sum_{n=1}^{\infty} \frac{7^n + 2}{8^n} = \frac{7}{8} \sum_{n=1}^{\infty} \frac{7^{n-1}}{8^{n-1}} = \frac{7}{8} \sum_{n=1}^{\infty} \left(\frac{7}{8}\right)^{n-1} = \frac{7^3}{8} \frac{1}{1 - \frac{7}{8}} = \frac{7^3}{3} = 343\]

3. 
\[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}\]

5. 
\[0.2925 = \frac{25}{100} + \frac{25}{10000} + \cdots = \frac{25}{10^2} \left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) = \frac{25}{10^2} \frac{1}{1 - \frac{1}{10^2}} = \frac{25}{99}\]

7. 
\[0.453453 = \frac{453}{1000} + \frac{453}{10^6} + \cdots = \frac{453}{10^3} \left(1 + \frac{1}{10^3} + \frac{1}{10^6} + \cdots\right) = \frac{453}{10^3} \frac{1}{1 - \frac{1}{10^3}} = \frac{453}{999}\]
9. At each successive step the number of intervals removed doubles and the lengths decrease by a third, so the total amount removed has length

\[
\frac{1}{3} + 2 \left( \frac{1}{9} \right) + 4 \left( \frac{1}{27} \right) + 8 \left( \frac{1}{81} \right) + \cdots = \frac{1}{3} \left( 1 + \frac{2}{3} + \frac{4}{9} + \cdots \right) = \frac{1}{3} \frac{1}{1 - \frac{2}{3}} = 1.
\]