8

Complex Vector Spaces

8.1 Complex Numbers
8.2 Conjugates and Division of Complex Numbers
8.3 Polar Form and DeMoivre’s Theorem
8.4 Complex Vector Spaces and Inner Products
8.5 Unitary and Hermitian Matrices

CHAPTER OBJECTIVES

- Graphically represent complex numbers in the complex plane.
- Perform operations with complex numbers.
- Represent complex numbers as vectors.
- Use the Quadratic Formula to find all zeros of a quadratic polynomial.
- Perform operations with complex matrices.
- Find the determinant of a complex matrix.
- Find the conjugate, modulus, and argument of a complex number.
- Multiply and divide complex numbers.
- Find the inverse of a complex matrix.
- Determine the polar form of a complex number.
- Convert a complex number from standard form to polar form and from polar form to standard form.
- Multiply and divide complex numbers in polar form.
- Find roots and powers of complex numbers in polar form.
- Use DeMoivre’s Theorem to find roots of complex numbers in polar form.
- Recognize complex vector spaces, $C^n$.
- Perform vector operations in $C^n$.
- Represent a vector in $C^n$ by a basis.
- Find the Euclidian inner product and the Euclidian norm of a vector in $C^n$.
- Find the Euclidian distance between two vectors in $C^n$.
- Find the conjugate transpose $A^*$ of a complex matrix $A$.
- Determine if a matrix $A$ is unitary or Hermitian.
- Find the eigenvalues and eigenvectors of a Hermitian matrix.
- Diagonalize a Hermitian matrix.
- Determine if a Hermitian matrix is normal.
Chapter 8 Complex Vector Spaces

8.1 Complex Numbers

So far in the text, the scalar quantities used have been real numbers. In this chapter, you will expand the set of scalars to include complex numbers.

In algebra it is often necessary to solve quadratic equations such as \( x^2 - 3x + 2 = 0 \). The general quadratic equation is \( ax^2 + bx + c = 0 \), and its solutions are given by the Quadratic Formula,

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\]

where the quantity under the radical, \( b^2 - 4ac \), is called the discriminant. If \( b^2 - 4ac \geq 0 \), then the solutions are ordinary real numbers. But what can you conclude about the solutions of a quadratic equation whose discriminant is negative? For example, the equation \( x^2 + 4 = 0 \) has a discriminant of \( b^2 - 4ac = -16 \). From your experience with ordinary algebra, it is clear that there is no real number whose square is \(-16\). By writing

\[
\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16}\sqrt{-1} = 4\sqrt{-1},
\]

you can see that the essence of the problem is that there is no real number whose square is \(-1\). To solve the problem, mathematicians invented the imaginary unit \( i \), which has the property \( i^2 = -1 \). In terms of this imaginary unit, you can write

\[
\sqrt{-16} = 4\sqrt{-1} = 4i.
\]

The imaginary unit \( i \) is defined as follows.

**Definition of the Imaginary Unit \( i \)**

The number \( i \) is called the _imaginary unit_ and is defined as

\[
i = \sqrt{-1}
\]

where \( i^2 = -1 \).

**Remark:** When working with products involving square roots of negative numbers, be sure to convert to a multiple of \( i \) before multiplying. For instance, consider the following operations.

\[
\sqrt{-1}\sqrt{-1} = i \cdot i = i^2 = -1 \quad \text{Correct}
\]

\[
\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1 \quad \text{Incorrect}
\]

With this single addition of the imaginary unit \( i \) to the real number system, the system of complex numbers can be developed.
Some examples of complex numbers written in standard form are $2 = 2 + 0i$, $4 + 3i$, and $-6i = 0 - 6i$. The set of real numbers is a subset of the set of complex numbers. To see this, note that every real number can be written as a complex number using $b = 0$. That is, for every real number, $a = a + 0i$.

A complex number is uniquely determined by its real and imaginary parts. So, you can say that two complex numbers are equal if and only if their real and imaginary parts are equal. That is, if $a + bi$ and $c + di$ are two complex numbers written in standard form, then

$$a + bi = c + di$$

if and only if $a = c$ and $b = d$.

### The Complex Plane

Because a complex number is uniquely determined by its real and imaginary parts, it is natural to associate the number $a + bi$ with the ordered pair $(a, b)$. With this association, you can graphically represent complex numbers as points in a coordinate plane called the **complex plane**. This plane is an adaptation of the rectangular coordinate plane. Specifically, the horizontal axis is the **real axis** and the vertical axis is the **imaginary axis**. For instance, Figure 8.1 shows the graph of two complex numbers, $3 + 2i$ and $-2 - i$. The number $3 + 2i$ is associated with the point $(3, 2)$ and the number $-2 - i$ is associated with the point $(-2, -1)$.

Another way to represent the complex number $a + bi$ is as a vector whose **horizontal** component is $a$ and whose **vertical** component is $b$. (See Figure 8.2.) (Note that the use of the letter $i$ to represent the imaginary unit is unrelated to the use of $i$ to represent a unit vector.)
Addition and Scalar Multiplication of Complex Numbers

Because a complex number consists of a real part added to a multiple of $i$, the operations of addition and multiplication are defined in a manner consistent with the rules for operating with real numbers. For instance, to add (or subtract) two complex numbers, add (or subtract) the real and imaginary parts separately.

The **sum** and **difference** of $a + bi$ and $c + di$ are defined as follows.

- Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

**EXAMPLE 1** Adding and Subtracting Complex Numbers

(a) $(2 - 4i) + (3 + 4i) = (2 + 3) + (-4 + 4)i = 5$
(b) $(1 - 3i) - (3 + i) = (1 - 3) + (-3 - 1)i = -2 - 4i$

**Remark:** Note in part (a) of Example 1 that the sum of two complex numbers can be a real number.

Using the vector representation of complex numbers, you can add or subtract two complex numbers geometrically using the parallelogram rule for vector addition, as shown in Figure 8.3.
Many of the properties of addition of real numbers are valid for complex numbers as well. For instance, addition of complex numbers is both associative and commutative. Moreover, to find the sum of three or more complex numbers, extend the definition of addition in the natural way. For example,

\[(2 + i) + (3 - 2i) + (-2 + 4i) = (2 + 3 - 2) + (1 - 2 + 4)i = 3 + 3i.\]

To multiply a complex number by a real scalar, use the definition below.

**Definition of Scalar Multiplication**

If \(c\) is a real number and \(a + bi\) is a complex number, then the scalar multiple of \(a + bi\) is defined as

\[c(a + bi) = ca + cbi.\]

Geometrically, multiplication of a complex number by a real scalar corresponds to the multiplication of a vector by a scalar, as shown in Figure 8.4.

**EXAMPLE 2**

Operations with Complex Numbers

(a) \(3(2 + 7i) + 4(8 - i) = 6 + 21i + 32 - 4i = 38 + 17i\)

(b) \(-4(1 + i) + 2(3 - i) - 3(1 - 4i) = -4 - 4i + 6 - 2i - 3 + 12i = -1 + 6i\)

With addition and scalar multiplication, the set of complex numbers forms a vector space of dimension 2 (where the scalars are the real numbers). You are asked to verify this in Exercise 57.
**Chapter 8 Complex Vector Spaces**

**Multiplication of Complex Numbers**

The operations of addition, subtraction, and multiplication by a real number have exact counterparts with the corresponding vector operations. By contrast, there is no direct counterpart for the multiplication of two complex numbers.

Rather than try to memorize this definition of the product of two complex numbers, you should simply apply the distributive property, as follows.

\[(a + bi)(c + di) = (ac - bd) + (ad + bc)i.\]

This is demonstrated in the next example.

**EXAMPLE 3**

**Multiplying Complex Numbers**

(a) \((-2)(1 - 3i) = -2 + 6i\)

(b) \((2 - i)(4 + 3i) = (8 + 6i - 4i - 3i^2) = 8 + 6i - 4i - 3(-1) = 8 + 3 + 6i - 4i = 11 + 2i\)

**Technology Note**

Many computer software programs and graphing utilities are capable of calculating with complex numbers. For example, on some graphing utilities, you can express a complex number \(a + bi\) as an ordered pair \((a, b)\). Try verifying the result of Example 3(b) by multiplying \((2, -1)\) and \((4, 3)\). You should obtain the ordered pair \((11, 2)\).

**EXAMPLE 4**

**Complex Zeros of a Polynomial**

Use the Quadratic Formula to find the zeros of the polynomial

\[p(x) = x^2 - 6x + 13\]

and verify that \(p(x) = 0\) for each zero.
SOLUTION

Using the Quadratic Formula, you have

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 52}}{2} \]

\[ = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i. \]

Substituting these values of \( x \) into the polynomial \( p(x) \), you have

\[ p(3 + 2i) = (3 + 2i)^2 - 6(3 + 2i) + 13 \]
\[ = (3 + 2i)(3 + 2i) - 6(3 + 2i) + 13 \]
\[ = 9 + 6i + 6i - 4 - 18 - 12i + 13 \]
\[ = 9 + 12i - 4 - 18 - 12i + 13 = 0 \]

and

\[ p(3 - 2i) = (3 - 2i)^2 - 6(3 - 2i) + 13 \]
\[ = (3 - 2i)(3 - 2i) - 6(3 - 2i) + 13 \]
\[ = 9 - 6i - 6i - 4 - 18 + 12i + 13 \]
\[ = 9 - 12i - 4 - 18 + 12i + 13 = 0. \]

In Example 4, the two complex numbers \( 3 + 2i \) and \( 3 - 2i \) are complex conjugates of each other (together they form a conjugate pair). A well-known result from algebra states that the complex zeros of a polynomial with real coefficients must occur in conjugate pairs. (See Review Exercise 86.) More will be said about complex conjugates in Section 8.2.

**Complex Matrices**

Now that you are able to add, subtract, and multiply complex numbers, you can apply these operations to matrices whose entries are complex numbers. Such a matrix is called complex.

**Definition of a Complex Matrix**

A matrix whose entries are complex numbers is called a complex matrix.

All of the ordinary operations with matrices also work with complex matrices, as demonstrated in the next two examples.
Chapter 8  Complex Vector Spaces

EXAMPLE 5  Operations with Complex Matrices

Let $A$ and $B$ be the complex matrices below

$$A = \begin{bmatrix} 2 - 3i & 1 + i \\ 4 & 2 - 3i \\ 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2i & 0 \\ i & 1 + 2i \end{bmatrix}$$

and determine each of the following.

(a) $3A$  (b) $(2 - i)B$  (c) $A + B$  (d) $BA$

**SOLUTION**

(a) $3A = 3\begin{bmatrix} 2 - 3i & 1 + i \\ 4 & 2 - 3i \\ 4 \end{bmatrix} = \begin{bmatrix} 6 - 9i & 3 + 3i \\ 12 & 6 - 9i \end{bmatrix}$

(b) $(2 - i)B = (2 - i)^2 \begin{bmatrix} 2i & 0 \\ i & 1 + 2i \end{bmatrix} = \begin{bmatrix} 2 + 4i & 0 \\ 1 + 2i & 4 + 3i \end{bmatrix}$

(c) $A + B = \begin{bmatrix} 2 - 3i & 1 + i \\ 4 & 2 - 3i \\ 4 \end{bmatrix} + \begin{bmatrix} 2i & 0 \\ i & 1 + 2i \end{bmatrix} = \begin{bmatrix} 2 - 2i & 1 + i \\ 2 + 3i & 5 + 2i \end{bmatrix}$

(d) $BA = \begin{bmatrix} 2 + 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 + i \\ 2 - 3i \\ 4 \end{bmatrix} = -2 + 0 \begin{bmatrix} 1 + 2i \\ 2i - 2 + 0 \\ i - 1 + 4 + 8i \end{bmatrix} = \begin{bmatrix} -2 - 2 + 2i \\ 7 + i \end{bmatrix}$

example 6  Finding the Determinant of a Complex Matrix

Find the determinant of the matrix

$$A = \begin{bmatrix} 2 - 4i \\ 3 & 2 - 3i \end{bmatrix}.$$  

**SOLUTION**

$$\text{det}(A) = \begin{vmatrix} 2 - 4i & 2 \\ 3 & 2 - 3i \end{vmatrix} = (2 - 4i)(5 - 3i) - (2)(3)$$

$$= 10 - 20i - 6i - 12 - 6$$

$$= -8 - 26i$$

Technology Note

Many computer software programs and graphing utilities are capable of performing matrix operations on complex matrices. Try verifying the calculation of the determinant of the matrix in Example 6. You should obtain the same answer, $(-8, -26)$.  


SECTION 8.1 Exercises

In Exercises 1–6, determine the value of the expression.
1. \( \sqrt{-2} \sqrt{-3} \)
2. \( \sqrt{8} \sqrt{-8} \)
3. \( \sqrt{-4} \sqrt{-4} \)
4. \( i^3 \)
5. \( i^4 \)
6. \( (-i)^7 \)

In Exercises 7–12, plot the complex number.
7. \( z = 6 - 2i \)
8. \( z = 3i \)
9. \( z = -5 + 5i \)
10. \( z = 7 \)
11. \( z = 1 + 5i \)
12. \( z = 1 - 5i \)

In Exercises 13 and 14, use vectors to illustrate the operations graphically. Be sure to graph the original vector.
13. \( -u \) and \( 2u \), where \( u = 3 - i \)
14. \( 3u \) and \( -\frac{3}{2}u \), where \( u = 2 + i \)

In Exercises 15–18, determine \( x \) such that the complex numbers in each pair are equal.
15. \( x + 3i, 6 + 3i \)
16. \( (2x - 8) + (x - 1)i, 2 + 4i \)
17. \( (x^2 + 6) + (2x)i, 15 + 6i \)
18. \( (-x + 4) + (x + 1)i, x + 3i \)

In Exercises 19–26, find the sum or difference of the complex numbers. Use vectors to illustrate your answer graphically.
19. \( (2 + 6i) + (3 - 3i) \)
20. \( (1 + \sqrt{2}i) + (2 - \sqrt{2}i) \)
21. \( (5 + i) - (5 - i) \)
22. \( i - (3 + i) \)
23. \( 6 - (-2i) \)
24. \( (12 - 7i) - (3 + 4i) \)
25. \( (2 + i) + (2 + i) \)
26. \( (2 + i) + (2 - i) \)

In Exercises 27–36, find the product.
27. \( (5 - 3i)(1 + 3i) \)
28. \( (3 + i)(\frac{3}{2} + i) \)
29. \( (\sqrt{7} - i)(\sqrt{7} + i) \)
30. \( (4 + \sqrt{2}i)(4 - \sqrt{2}i) \)
31. \( (a + bi)^2 \)
32. \( (a + bi)(a - bi) \)
33. \( (1 + i)^3 \)
34. \( (2 - i)(2 + 2i)(4 + i) \)
35. \( (a + bi)^3 \)
36. \( (1 + i)^2(1 - i)^2 \)

In Exercises 37–42, determine the zeros of the polynomial function.
37. \( p(x) = 2x^2 + 2x + 5 \)
38. \( p(x) = x^2 + x + 1 \)
39. \( p(x) = x^2 - 5x + 6 \)
40. \( p(x) = x^2 - 4x + 5 \)
41. \( p(x) = x^4 - 16 \)
42. \( p(x) = x^4 + 10x^2 + 9 \)

In Exercises 43–46, use the given zero to find all zeros of the polynomial function.
43. \( p(x) = x^3 - 3x^2 + 4x - 2 \) Zero: \( x = 1 \)
44. \( p(x) = x^3 - 2x^2 - 11x + 52 \) Zero: \( x = -4 \)
45. \( p(x) = 2x^3 + 3x^2 + 50x + 75 \) Zero: \( x = 5i \)
46. \( p(x) = x^3 + x^2 + 9x + 9 \) Zero: \( x = 3i \)

In Exercises 47–56, perform the indicated matrix operation using the complex matrices \( A \) and \( B \).
47. \( A + B \)
48. \( B - A \)
49. \( 2A \)
50. \( \frac{1}{2}B \)
51. \( 2iA \)
52. \( \frac{1}{2}iB \)
53. \( \det(A + B) \)
54. \( \det(B) \)
55. \( 5AB \)
56. \( BA \)

57. Prove that the set of complex numbers, with the operations of addition and scalar multiplication (with real scalars), is a vector space of dimension 2.

58. (a) Evaluate \( i^n \) for \( n = 1, 2, 3, 4, \) and \( 5 \).
   (b) Calculate \( i^{2010} \).
   (c) Find a general formula for \( i^n \) for any positive integer \( n \).

59. Let \( A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \)
   (a) Calculate \( A^n \) for \( n = 1, 2, 3, 4, \) and \( 5 \).
   (b) Calculate \( A^{2010} \).
   (c) Find a general formula for \( A^n \) for any positive integer \( n \).
60. Prove that if the product of two complex numbers is zero, then at least one of the numbers must be zero.

**True or False?** In Exercises 61 and 62, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

61. \(\sqrt{-2} \times \sqrt{-2} = \sqrt{4} = 2\)

62. \((\sqrt{-10})^2 = \sqrt{100} = 10\)
Section 8.2 Conjugates and Division of Complex Numbers

In Section 8.1, it was mentioned that the complex zeros of a polynomial with real coefficients occur in conjugate pairs. For instance, in Example 4 you saw that the zeros of 
\[ p(x) = x^2 - 6x + 13 \] are \[ 3 + 2i \] and \[ 3 - 2i \].

In this section, you will examine some additional properties of complex conjugates. You will begin with the definition of the conjugate of a complex number.

**Definition of the Conjugate of a Complex Number**

The **conjugate** of the complex number \( z = a + bi \) is denoted by \( \bar{z} \) and is given by

\[
\bar{z} = a - bi.
\]

From this definition, you can see that the conjugate of a complex number is found by changing the sign of the imaginary part of the number, as demonstrated in the next example.

**Example 1** Finding the Conjugate of a Complex Number

<table>
<thead>
<tr>
<th>Complex Number</th>
<th>Conjugate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( z = -2 + 3i )</td>
<td>( \bar{z} = -2 - 3i )</td>
</tr>
<tr>
<td>(b) ( z = 4 - 5i )</td>
<td>( \bar{z} = 4 + 5i )</td>
</tr>
<tr>
<td>(c) ( z = -2i )</td>
<td>( \bar{z} = 2i )</td>
</tr>
<tr>
<td>(d) ( z = 5 )</td>
<td>( \bar{z} = 5 )</td>
</tr>
</tbody>
</table>

**Remark:** In part (d) of Example 1, note that 5 is its own complex conjugate. In general, it can be shown that a number is its own complex conjugate if and only if the number is real. (See Exercise 39.)

Geometrically, two points in the complex plane are conjugates if and only if they are reflections about the real (horizontal) axis, as shown in Figure 8.5 on the next page.
Chapter 8 Complex Vector Spaces

Conjugate of a Complex Number

Figure 8.5

Complex conjugates have many useful properties. Some of these are shown in Theorem 8.1.

**Theorem 8.1** Properties of Complex Conjugates

For a complex number \( z = a + bi \), the following properties are true.

1. \( \overline{z} \overline{z} = a^2 + b^2 \)
2. \( \overline{z} \overline{z} \geq 0 \)
3. \( \overline{z} \overline{z} = 0 \) if and only if \( z = 0 \).
4. \( \overline{\overline{z}} = z \)

**Proof**

To prove the first property, let \( z = a + bi \). Then \( \overline{z} = a - bi \) and

\[
\overline{z} \overline{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2.
\]

The second and third properties follow directly from the first. Finally, the fourth property follows the definition of the complex conjugate. That is,

\[
\overline{\overline{z}} = \overline{a + bi} = a - bi = a + bi = z.
\]

**Example 2** Finding the Product of Complex Conjugates

Find the product of \( z = 1 - 2i \) and its complex conjugate.

**Solution**

Because \( \overline{z} = 1 + 2i \), you have

\[
z \overline{z} = (1 - 2i)(1 + 2i) = 1^2 + 2^2 = 1 + 4 = 5.
\]
The Modulus of a Complex Number

Because a complex number can be represented by a vector in the complex plane, it makes sense to talk about the length of a complex number. This length is called the modulus of the complex number.

Definition of the Modulus of a Complex Number

The modulus of the complex number \( z = a + bi \) is denoted by \( |z| \) and is given by

\[
|z| = \sqrt{a^2 + b^2}.
\]

Remark: The modulus of a complex number is also called the absolute value of the number. In fact, when \( z = a + 0i \) is a real number, you have

\[
|z| = \sqrt{a^2 + 0^2} = |a|.
\]

Example 3 Finding the Modulus of a Complex Number

For \( z = 2 + 3i \) and \( w = 6 - i \), determine the value of each modulus.

(a) \( |z| \)  
(b) \( |w| \)  
(c) \( |zw| \)

SOLUTION

(a) \( |z| = \sqrt{2^2 + 3^2} = \sqrt{13} \)

(b) \( |w| = \sqrt{6^2 + (-1)^2} = \sqrt{37} \)

(c) Because \( zw = (2 + 3i)(6 - i) = 15 + 16i \), you have

\[
|zw| = \sqrt{15^2 + 16^2} = \sqrt{481}.
\]

Note that in Example 3, \( |zw| = |z||w| \). In Exercise 40, you are asked to prove that this multiplicative property of the modulus always holds. Theorem 8.2 states that the modulus of a complex number is related to its conjugate.

Theorem 8.2 The Modulus of a Complex Number

For a complex number \( z \),

\[
|z|^2 = z \overline{z}.
\]

Proof Let \( z = a + bi \), then \( \overline{z} = a - bi \) and you have

\[
z \overline{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.
\]
**Division of Complex Numbers**

One of the most important uses of the conjugate of a complex number is in performing division in the complex number system. To define division of complex numbers, consider \( z = a + bi \) and \( w = c + di \) and assume that \( c \) and \( d \) are not both 0. If the quotient \( \frac{z}{w} \) is to make sense, it has to be true that 

\[
z = w(x + yi) = (c + di)(x + yi) = (cx - dy) + (dx + cy)i.
\]

But, because \( z = a + bi \), you can form the linear system below.

\[
\begin{align*}
    cx - dy &= a \\
dx + cy &= b
\end{align*}
\]

Solving this system of linear equations for \( x \) and \( y \) yields

\[
x = \frac{ac + bd}{w\overline{w}} \quad \text{and} \quad y = \frac{bc - ad}{w\overline{w}}.
\]

Now, because \( \overline{w} = (a + bi)(c - di) = (ac + bd) + (bc - ad)i \), the definition below is obtained.

**Definition of Division of Complex Numbers**

The quotient of the complex numbers \( z = a + bi \) and \( w = c + di \) is defined as

\[
\frac{z}{w} = \frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i = \frac{1}{|w|^2}(\overline{z\overline{w}})
\]

provided \( c^2 + d^2 \neq 0 \).

**Remark:** If \( c^2 + d^2 = 0 \), then \( c = d = 0 \), and \( w = 0 \). In other words, as is the case with real numbers, division of complex numbers by zero is not defined.

In practice, the quotient of two complex numbers can be found by multiplying the numerator and the denominator by the conjugate of the denominator, as follows.

\[
\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \left( \frac{c - di}{c - di} \right) = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}
\]

\[
= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i.
\]
**EXAMPLE 4**  Division of Complex Numbers

(a) \[
\frac{1}{1 + i} = \frac{1 - i}{1^2 - i^2} = \frac{1 - i}{2} = \frac{1}{2} - \frac{1}{2}i
\]

(b) \[
\frac{2 - i}{3 + 4i} = \frac{2 - i(3 - 4i)}{3 + 4i(3 - 4i)} = \frac{2 - 11i}{9 + 16} = \frac{2}{25} - \frac{11}{25}i
\]

Now that you can divide complex numbers, you can find the (multiplicative) inverse of a complex matrix, as demonstrated in Example 5.

**EXAMPLE 5**  Finding the Inverse of a Complex Matrix

Find the inverse of the matrix

\[
A = \begin{bmatrix}
2 - i & -5 + 2i \\
3 - i & -6 + 2i
\end{bmatrix}
\]

and verify your solution by showing that \(AA^{-1} = I_2\).

**SOLUTION**  Using the formula for the inverse of a \(2 \times 2\) matrix from Section 2.3, you have

\[
A^{-1} = \frac{1}{|A|} \begin{bmatrix}
-6 + 2i & 5 - 2i \\
3 + i & 2 - i
\end{bmatrix}
\]

Furthermore, because

\[
|A| = (2 - i)(-6 + 2i) - (-5 + 2i)(3 - i)
\]

\[
= (-12 + 6i + 4i + 2) - (-15 + 6i + 5i + 2)
\]

\[
= 3 - i
\]

you can write

\[
A^{-1} = \frac{1}{3 - i} \begin{bmatrix}
-6 + 2i & 5 - 2i \\
3 + i & 2 - i
\end{bmatrix}
\]

\[
= \frac{1}{3 - i} \left( \frac{1}{3 + i} \right) \begin{bmatrix}
-6 + 2i & 5 - 2i \\
3 + i & 2 - i
\end{bmatrix}
\]

\[
= \frac{1}{10} \begin{bmatrix}
-20 & 17 - i \\
-10 & 7 - i
\end{bmatrix}
\]

To verify your solution, multiply \(A\) and \(A^{-1}\) as follows.

\[
AA^{-1} = \begin{bmatrix}
2 - i & -5 + 2i \\
3 - i & -6 + 2i
\end{bmatrix} \frac{1}{10} \begin{bmatrix}
-20 & 17 - i \\
-10 & 7 - i
\end{bmatrix}
\]

\[
= \frac{1}{10} \begin{bmatrix}
10 & 0 \\
0 & 10
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
In Exercises 1–6, find the complex conjugate $z$ and graphically represent both $z$ and $\bar{z}$.

1. $z = 6 - 3i$  
2. $z = 2 + 5i$  
3. $z = -8i$  
4. $z = 2i$  
5. $z = 4$  
6. $z = -3$

In Exercises 7–12, find the indicated modulus, where $z = 2 + i$, $w = -3 + 2i$, and $v = -5i$.

7. $|z|$  
8. $|z^2|$  
9. $|zw|$  
10. $|wz|$  
11. $|v|$  
12. $|v^2|$  

In Exercises 13–12, perform the indicated operations.

13. Verify that $|zw| = |w||z| = |zw|$, where $z = 1 + i$ and $w = -1 + 2i$.  
14. Verify that $|zw^2| = |z||w^2| = |zw|^2$, where $z = 1 + 2i$ and $w = -2 - 3i$.  

15. $\frac{2 + i}{i}$  
16. $\frac{1}{6 + 3i}$  
17. $\frac{3 - \sqrt{2}i}{3 + \sqrt{2}i}$  
18. $\frac{5 + i}{4 + i}$  
19. $\frac{(2 + i)(3 - i)}{4 - 2i}$  
20. $\frac{3 - i}{(2 - i)(5 + 2i)}$
In Exercises 21–24, perform the operation and write the result in standard form.

21. \[\frac{2}{1 + i} - \frac{3}{1 - i}\]
22. \[\frac{2i}{2 + i} + \frac{5}{2 - i}\]
23. \[\frac{i}{3 - i} + \frac{2i}{3 + i}\]
24. \[\frac{1 + i}{i} - \frac{3}{4 - i}\]

In Exercises 25–28, use the given zero to find all zeros of the polynomial function.

25. \(p(x) = 3x^3 - 4x^2 + 8x + 8\)  
   Zero: \(1 - \sqrt{3}i\)
26. \(p(x) = 4x^3 + 23x^2 + 34x - 10\)  
   Zero: \(-3 + i\)
27. \(p(x) = x^4 + 3x^3 - 5x^2 - 21x + 22\)  
   Zero: \(-3 + \sqrt{2}i\)
28. \(p(x) = x^3 + 4x^2 + 14x + 20\)  
   Zero: \(-1 - 3i\)

In Exercises 29 and 30, find each power of the complex number \(z\).
(a) \(z^2\)  
(b) \(z^3\)  
(c) \(z^{-1}\)  
(d) \(z^{-2}\)
29. \(z = 2 - i\)
30. \(z = 1 + i\)

In Exercises 31–36, determine whether the complex matrix \(A\) has an inverse. If \(A\) is invertible, find its inverse and verify that \(AA^{-1} = I\).

31. \[A = \begin{bmatrix} 6 & 3i \\ 2 - i & i \end{bmatrix}\]
32. \[A = \begin{bmatrix} 2i & -2 - i \\ 3 & 3i \end{bmatrix}\]
33. \[A = \begin{bmatrix} 1 - i & 2 \\ 1 & 1 + i \end{bmatrix}\]
34. \[A = \begin{bmatrix} 1 - i & 2 \\ 0 & 1 + i \end{bmatrix}\]
35. \[A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - i & 0 \\ 0 & 0 & 1 + i \end{bmatrix}\]
36. \[A = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}\]

In Exercises 37 and 38, determine all values of the complex number \(z\) for which \(A\) is singular. (Hint: Set \(\text{det}(A) = 0\) and solve for \(z\).)

37. \[A = \begin{bmatrix} 5 & z \\ 3i & 2 - i \end{bmatrix}\]
38. \[A = \begin{bmatrix} 2 & 2i & 1 + i \\ 1 - i & -1 + i & z \\ 1 & 0 & 0 \end{bmatrix}\]

39. Prove that \(z = \bar{z}\) if and only if \(z\) is real.
40. Prove that for any two complex numbers \(z\) and \(w\), each of the statements below is true.
   (a) \(|zw| = |z||w|\)
   (b) If \(w \neq 0\), then \(|z/w| = |z|/|w|\).
41. Describe the set of points in the complex plane that satisfies each of the statements below.
   (a) \(|z| = 3\)
   (b) \(|z - 1 + i| = 5\)
   (c) \(|z - i| \leq 2\)
   (d) \(2 \leq |z| \leq 5\)

True or False? In Exercises 42 and 43, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

42. \(i + i^2 = 0\)
43. There is no complex number that is equal to its complex conjugate.

44. Describe the set of points in the complex plane that satisfies each of the statements below.
   (a) \(|z| = 4\)
   (b) \(|z - i| = 2\)
   (c) \(|z + 1| \leq 1\)
   (d) \(|z| > 3\)

45. (a) Evaluate \((1/i)^n\) for \(n = 1, 2, 3, 4,\) and \(5\).
   (b) Calculate \((1/i)^{2000}\) and \((1/i)^{2010}\).
   (c) Find a general formula for \((1/i)^n\) for any positive integer \(n\).

46. (a) Verify that \(\left(\frac{1 + i}{\sqrt{2}}\right)^2 = i\).
   (b) Find the two square roots of \(i\).
   (c) Find all zeros of the polynomial \(x^4 + 1\).
Polar Form and DeMoivre’s Theorem

At this point you can add, subtract, multiply, and divide complex numbers. However, there is still one basic procedure that is missing from the algebra of complex numbers. To see this, consider the problem of finding the square root of a complex number such as When you use the four basic operations (addition, subtraction, multiplication, and division), there seems to be no reason to guess that That is,

To work effectively with powers and roots of complex numbers, it is helpful to use a polar representation for complex numbers, as shown in Figure 8.6. Specifically, if is a nonzero complex number, then let be the angle from the positive -axis to the radial line passing through the point and let be the modulus of So, and you have from which the polar form of a complex number is obtained.

**Remark:** The polar form of is expressed as where is any angle. Because there are infinitely many choices for the argument, the polar form of a complex number is not unique. Normally, the values of that lie between and are used, although on occasion it is convenient to use other values. The value of that satisfies the inequality is called the **principal argument** and is denoted by Arg(). Two nonzero complex numbers in polar form are equal if and only if they have the same modulus and the same principal argument.

**Example 1** Finding the Polar Form of a Complex Number

Find the polar form of each of the complex numbers. (Use the principal argument.)

(a) \( z = 1 - i \) \tap \( z = 2 + 3i \) \tap \( z = i \)
SOLUTION

(a) Because $a = 1$ and $b = -1$, then $r^2 = 1^2 + (-1)^2 = 2$, which implies that $r = \sqrt{2}$. From $a = r \cos \theta$ and $b = r \sin \theta$, you have

$$\cos \theta = \frac{a}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin \theta = \frac{b}{r} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

So, $\theta = -\pi/4$ and

$$z = \sqrt{2} \left[ \cos \left(-\frac{\pi}{4}\right) + i \sin \left(-\frac{\pi}{4}\right) \right].$$

(b) Because $a = 2$ and $b = 3$, then $r^2 = 2^2 + 3^2 = 13$, which implies that $r = \sqrt{13}$. So,

$$\cos \theta = \frac{a}{r} = \frac{2}{\sqrt{13}} \quad \text{and} \quad \sin \theta = \frac{b}{r} = \frac{3}{\sqrt{13}},$$

and it follows that $\theta \approx 0.98$. So, the polar form is

$$z \approx \sqrt{13} \left[ \cos(0.98) + i \sin(0.98) \right].$$

(c) Because $a = 0$ and $b = 1$, it follows that $r = 1$ and $\theta = \pi/2$, so

$$z = \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right).$$

The polar forms derived in parts (a), (b), and (c) are depicted graphically in Figure 8.7.

---

**EXAMPLE 2**

**Converting from Polar to Standard Form**

Express the complex number in standard form.

$$z = 8 \left[ \cos \left(-\frac{\pi}{3}\right) + i \sin \left(-\frac{\pi}{3}\right) \right]$$
Because \( \cos(-\pi/3) = 1/2 \) and \( \sin(-\pi/3) = -\sqrt{3}/2 \), you can obtain the standard form
\[
z = 8 \left[ \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right] = 8 \left[ \frac{1}{2} - i \frac{\sqrt{3}}{2} \right] = 4 - 4\sqrt{3}i.
\]

The polar form adapts nicely to multiplication and division of complex numbers. Suppose you have two complex numbers in polar form
\[
z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2).
\]
Then the product of \( z_1 \) and \( z_2 \) is expressed as
\[
z_1 z_2 = r_1 r_2 (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)
= r_1 r_2 [ (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) ].
\]
Using the trigonometric identities
\[
\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2
\]
and
\[
\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2.
\]
you have
\[
z_1 z_2 = r_1 r_2 [ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) ].
\]
This establishes the first part of the next theorem. The proof of the second part is left to you. (See Exercise 65.)

**THEOREM 8.4**

**Product and Quotient of Two Complex Numbers**

Given two complex numbers in polar form
\[
z_1 = r_1 (\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)
\]
the product and quotient of the numbers are as follows.
\[
z_1 z_2 = r_1 r_2 [ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) ] \quad \text{Product}
\]
\[
\frac{z_1}{z_2} = \frac{r_1}{r_2} [ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) ], \quad z_2 \neq 0 \quad \text{Quotient}
\]

This theorem says that to multiply two complex numbers in polar form, multiply moduli and add arguments. To divide two complex numbers, divide moduli and subtract arguments. (See Figure 8.8.)
Section 8.3 Polar Form and DeMoivre’s Theorem

Figure 8.8

To multiply $z_1$ and $z_2$: Multiply moduli and add arguments.

To divide $z_1$ and $z_2$: Divide moduli and subtract arguments.

**EXAMPLE 3** **Multiplying and Dividing in Polar Form**

Find $z_1z_2$ and $z_1/z_2$ for the complex numbers

$$z_1 = 5 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \quad \text{and} \quad z_2 = \frac{1}{3} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right).$$

**SOLUTION** Because you have the polar forms of $z_1$ and $z_2$, you can apply Theorem 8.4, as follows.

Multiply

$$z_1z_2 = \left( 5 \cdot \frac{1}{3} \right) \left[ \cos \left( \frac{\pi}{4} + \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{4} + \frac{\pi}{6} \right) \right] = \frac{5}{3} \left( \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$$

Divide

$$\frac{z_1}{z_2} = \left( \frac{5}{1/3} \right) \left[ \cos \left( \frac{\pi}{4} - \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{4} - \frac{\pi}{6} \right) \right] = 15 \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

**REMARK:** Try performing the multiplication and division in Example 3 using the standard forms

$$z_1 = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i \quad \text{and} \quad z_2 = \frac{\sqrt{3}}{6} + \frac{1}{6}i.$$
DeMoivre’s Theorem

The final topic in this section involves procedures for finding powers and roots of complex numbers. Repeated use of multiplication in the polar form yields:

\[ z = r(\cos \theta + i \sin \theta) \]
\[ z^2 = r(\cos \theta + i \sin \theta)r(\cos \theta + i \sin \theta) = r^2(\cos 2\theta + i \sin 2\theta) \]
\[ z^3 = r(\cos \theta + i \sin \theta)r^2(\cos 2\theta + i \sin 2\theta) = r^3(\cos 3\theta + i \sin 3\theta). \]

Similarly,

\[ z^4 = r^4(\cos 4\theta + i \sin 4\theta) \]
\[ z^5 = r^5(\cos 5\theta + i \sin 5\theta). \]

This pattern leads to the next important theorem, named after the French mathematician Abraham DeMoivre (1667–1754). You are asked to prove this theorem in Review Exercise 85.

**THEOREM 8.5 DeMoivre’s Theorem**

If \( z = r(\cos \theta + i \sin \theta) \) and \( n \) is any positive integer, then

\[ z^n = r^n(\cos n\theta + i \sin n\theta). \]

**EXAMPLE 4 Raising a Complex Number to an Integer Power**

Find \((-1 + \sqrt{3} i)^{12}\) and write the result in standard form.

**SOLUTION**

First convert to polar form. For \(-1 + \sqrt{3} i\),

\[ r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \quad \text{and} \quad \tan \theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \]

which implies that \( \theta = 2\pi/3 \). So,

\[ -1 + \sqrt{3} i = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right). \]

By DeMoivre’s Theorem,

\[ (-1 + \sqrt{3} i)^{12} = \left[2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\right]^{12} \]
\[ = 2^{12}\left[\cos \frac{12(2\pi)}{3} + i \sin \frac{12(2\pi)}{3}\right] \]
\[ = 4096(\cos 8\pi + i \sin 8\pi) \]
\[ = 4096[1 + i(0)] = 4096. \]
Recall that a consequence of the Fundamental Theorem of Algebra is that a polynomial of degree $n$ has $n$ zeros in the complex number system. So, a polynomial such as $p(x) = x^6 - 1$ has six zeros, and in this case you can find the six zeros by factoring and using the Quadratic Formula.

Consequently, the zeros are

$$x = \pm 1, \quad x = -\frac{1 \pm \sqrt{3}i}{2}, \quad \text{and} \quad x = \frac{1 \pm \sqrt{3}i}{2}.$$

Each of these numbers is called a sixth root of 1. In general, the $n$th root of a complex number is defined as follows.

**Definition of the $n$th Root of a Complex Number**

The complex number $w = a + bi$ is an $n$th root of the complex number $z$ if

$$z = w^n = (a + bi)^n.$$

DeMoivre’s Theorem is useful in determining roots of complex numbers. To see how this is done, let $w$ be an $n$th root of $z$, where

$$w = s(\cos \beta + i \sin \beta) \quad \text{and} \quad z = r(\cos \theta + i \sin \theta).$$

Then, by DeMoivre’s Theorem you have $w^n = s^n(\cos n\beta + i \sin n\beta)$, and because $w^n = z$, it follows that

$$s^n(\cos n\beta + i \sin n\beta) = r(\cos \theta + i \sin \theta).$$

Now, because the right and left sides of this equation represent equal complex numbers, you can equate moduli to obtain $s^n = r$, which implies that $s = \sqrt[n]{r}$, and equate principal arguments to conclude that $\theta$ and $n\beta$ must differ by a multiple of $2\pi$. Note that $r$ is a positive real number and so $s = \sqrt[n]{r}$ is also a positive real number. Consequently, for some integer $k$, $n\beta = \theta + 2\pi k$, which implies that

$$\beta = \frac{\theta + 2\pi k}{n}.$$

Finally, substituting this value of $\beta$ into the polar form of $w$ produces the result stated in the next theorem.

**THEOREM 8.6**

**The $n$th Roots of a Complex Number**

For any positive integer $n$, the complex number $z = r(\cos \theta + i \sin \theta)$ has exactly $n$ distinct roots. These $n$ roots are given by

$$\sqrt[n]{r} \left[ \cos \left( \frac{\theta + 2\pi k}{n} \right) + i \sin \left( \frac{\theta + 2\pi k}{n} \right) \right]$$

where $k = 0, 1, 2, \ldots, n - 1$. 


Remark: Note that when \( k \) exceeds \( n - 1 \), the roots begin to repeat. For instance, if \( k = n \), the angle is
\[
\frac{\theta + 2\pi n}{n} = \frac{\theta}{n} + 2\pi
\]
which yields the same values for the sine and cosine as \( k = 0 \).

The formula for the \( n \)th roots of a complex number has a nice geometric interpretation, as shown in Figure 8.9. Note that because the \( n \)th roots all have the same modulus (length) \( \sqrt[n]{r} \), they will lie on a circle of radius \( \sqrt[n]{r} \) with center at the origin. Furthermore, the \( n \) roots are equally spaced around the circle, because successive \( n \)th roots have arguments that differ by \( 2\pi/n \).

You have already found the sixth roots of 1 by factoring and the Quadratic Formula. Try solving the same problem using Theorem 8.6 to see if you get the roots shown in Figure 8.10. When Theorem 8.6 is applied to the real number 1, the \( n \)th roots have a special name—the \( n \)th roots of unity.

Example 5: Finding the \( n \)th Roots of a Complex Number

Determine the fourth roots of \( i \).

Solution

In polar form, you can write \( i \) as
\[
i = 1 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)
\]
so that \( r = 1 \) and \( \theta = \pi/2 \). Then, by applying Theorem 8.6, you have
\[
i^{1/4} = \sqrt[4]{1} \left[ \cos \left( \frac{\pi/2}{4} + \frac{2k\pi}{4} \right) + i \sin \left( \frac{\pi/2}{4} + \frac{2k\pi}{4} \right) \right]
\]
\[
= \cos \left( \frac{\pi}{8} + \frac{k\pi}{2} \right) + i \sin \left( \frac{\pi}{8} + \frac{k\pi}{2} \right).
\]
Section 8.3 Polar Form and DeMoivre’s Theorem

Setting \( k = 0, 1, 2, \) and \( 3, \) you obtain the four roots

\[
\begin{align*}
    z_1 &= \cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \\
    z_2 &= \cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8} \\
    z_3 &= \cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8} \\
    z_4 &= \cos \frac{13\pi}{8} + i \sin \frac{13\pi}{8}
\end{align*}
\]

as shown in Figure 8.11.

**Remark:** In Figure 8.11, note that when each of the four angles \( \pi/8, 5\pi/8, 9\pi/8, \) and \( 13\pi/8 \) is multiplied by 4, the result is of the form \( (\pi/2) + 2k\pi. \)

---

**SECTION 8.3 Exercises**

In Exercises 1–4, express the complex number in polar form.

1. Imaginary axis

2. Imaginary axis

3. Imaginary axis

4. Imaginary axis
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In Exercises 5–16, represent the complex number graphically, and give the polar form of the number. (Use the principal argument.)

5. \(-2 - 2i\)  
6. \(2 + 2i\)  
7. \(-2(1 + \sqrt{3}i)\)  
8. \(\frac{5}{3}(\sqrt{3} - i)\)  
9. \(6i\)  
10. \(-2i\)  
11. 7  
12. 4  
13. \(3 + \sqrt{3}i\)  
14. \(2\sqrt{2} - i\)  
15. \(-1 - 2i\)  
16. \(5 + 2i\)

In Exercises 17–26, represent the complex number graphically, and give the standard form of the number.

17. \(2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\)  
18. \(5\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\)  
19. \(\frac{3}{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)\)  
20. \(\frac{3}{4}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)\)  
21. \(3.75\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\)  
22. \(8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\)  
23. \(4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)\)  
24. \(6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\)  
25. \(7\cos0 + i\sin0\)  
26. \(6\cos\pi + i\sin\pi\)

In Exercises 27–34, perform the indicated operation and leave the result in polar form.

27. \(3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left[4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]\)  
28. \(\frac{3}{4}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left[6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]\)  
29. \(0.5(\cos\pi + i\sin\pi)\left[0.5(\cos[-\pi] + i\sin[-\pi])\right]\)  
30. \(3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left[\frac{1}{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]\)  
31. \(\frac{2[\cos(2\pi/3) + i\sin(2\pi/3)]}{4[\cos(5\pi/6) + i\sin(5\pi/6)]}\)  
32. \(\frac{\cos(5\pi/3) + i\sin(5\pi/3)}{\cos\pi + i\sin\pi}\)  
33. \(\frac{12[\cos(\pi/3) + i\sin(\pi/3)]}{3[\cos(\pi/6) + i\sin(\pi/6)]}\)  
34. \(\frac{9[\cos(3\pi/4) + i\sin(3\pi/4)]}{5[\cos(-\pi/4) + i\sin(-\pi/4)]}\)

In Exercises 35–44, use DeMoivre’s Theorem to find the indicated powers of the complex number. Express the result in standard form.

35. \((1 + i)^4\)  
36. \((2 + 2i)^6\)  
37. \((-1 + i)^{10}\)  
38. \((\sqrt{3} + i)^7\)  
39. \((1 - \sqrt{3}i)^3\)  
40. \([5\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)]^3\)  
41. \([3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)]^4\)  
42. \([\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}]^{10}\)  
43. \([2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)]^8\)  
44. \([5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)]^4\)

In Exercises 45–56, (a) use DeMoivre’s Theorem to find the indicated roots, (b) represent each of the roots graphically, and (c) express each of the roots in standard form.

45. Square roots: \(16\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\)  
46. Square roots: \(9\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\)  
47. Fourth roots: \(16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)\)  
48. Fifth roots: \(32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\)

49. Square roots: \(-25i\)  
50. Fourth roots: \(625i\)  
51. Cube roots: \(-\frac{1+i}{2}\)\((1 + \sqrt{3}i)\)  
52. Cube roots: \(-4\sqrt{2}(1 - i)\)  
53. Cube roots: \(8\)  
54. Fourth roots: \(i\)  
55. Fourth roots: \(1\)  
56. Cube roots: \(1000\)

In Exercises 57–64, find all the solutions to the equation and represent your solutions graphically.

57. \(x^4 - i = 0\)  
58. \(x^4 + 16i = 0\)  
59. \(x^3 + 1 = 0\)  
60. \(x^3 - 27 = 0\)  
61. \(x^4 + 243 = 0\)  
62. \(x^4 - 81 = 0\)  
63. \(x^3 + 64i = 0\)  
64. \(x^4 + i = 0\)

65. When provided with two complex numbers \(z_1 = r_1(\cos \theta_1 + i \sin \theta_1)\) and \(z_2 = r_2(\cos \theta_2 + i \sin \theta_2)\), with \(z_2 \neq 0\), prove that

\[
\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].
\]
66. Show that the complex conjugate of \( z = r(\cos \theta + i \sin \theta) \) is 
\[ \overline{z} = r[\cos(-\theta) + i \sin(-\theta)] . \]

67. Use the polar forms of \( z \) and \( \overline{z} \) in Exercise 66 to find each of the following.
(a) \( z\overline{z} \)
(b) \( z/\overline{z}, \overline{z} \neq 0 \)

68. Show that the negative of \( z = r(\cos \theta + i \sin \theta) \) is 
\[ -z = r[\cos(\theta + \pi) + i \sin(\theta + \pi)] . \]

69. Writing
(a) Let \( z = r(\cos \theta + i \sin \theta) = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \).
Sketch \( z \), \( iz \), and \( z/i \) in the complex plane.
(b) What is the geometric effect of multiplying a complex number \( z \) by \( i \)? What is the geometric effect of dividing \( z \) by \( i \)?

70. Calculus  Recall that the Maclaurin series for \( e^x \), \( \sin x \), and \( \cos x \) are
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \]
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \]
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots . \]

(a) Substitute \( x = i\theta \) in the series for \( e^x \) and show that 
\[ e^{i\theta} = \cos \theta + i \sin \theta . \]
(b) Show that any complex number \( z = a + bi \) can be expressed in polar form as \( z = re^{i\theta} . \)
(c) Prove that if \( z = re^{i\theta} \), then \( \overline{z} = re^{-i\theta} . \)
(d) Prove the amazing formula \( e^{i\pi} = -1 . \)

True or False? In Exercises 71 and 72, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

71. Although the square of the complex number \( bi \) is given by \( (bi)^2 = -b^2 \), the absolute value of the complex number \( z = a + bi \) is defined as \( |a + bi| = \sqrt{a^2 + b^2} . \)

72. Geometrically, the \( n \)th roots of any complex number \( z \) are all equally spaced around the unit circle centered at the origin.
Complex Vector Spaces and Inner Products

All the vector spaces you have studied so far in the text have been real vector spaces because the scalars have been real numbers. A complex vector space is one in which the scalars are complex numbers. So, if \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \) are vectors in a complex vector space, then a linear combination is of the form

\[
c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m
\]

where the scalars \( c_1, c_2, \ldots, c_m \) are complex numbers. The complex version of \( \mathbb{R}^n \) is the complex vector space \( \mathbb{C}^n \) consisting of ordered \( n \)-tuples of complex numbers. So, a vector in \( \mathbb{C}^n \) has the form

\[
\mathbf{v} = (a_1 + b_1i, a_2 + b_2i, \ldots, a_n + b_ni).
\]

It is also convenient to represent vectors in \( \mathbb{C}^n \) by column matrices of the form

\[
\mathbf{v} = \begin{bmatrix} a_1 + b_1i \\ a_2 + b_2i \\ \vdots \\ a_n + b_ni \end{bmatrix}.
\]

As with \( \mathbb{R}^n \), the operations of addition and scalar multiplication in \( \mathbb{C}^n \) are performed component by component.

**EXAMPLE 1** Vector Operations in \( \mathbb{C}^n \)

Let

\[
\mathbf{v} = (1 + 2i, 3 - i) \quad \text{and} \quad \mathbf{u} = (-2 + i, 4)
\]

be vectors in the complex vector space \( \mathbb{C}^2 \). Determine each vector.

(a) \( \mathbf{v} + \mathbf{u} \) \quad (b) \( (2 + i)\mathbf{v} \) \quad (c) \( 3\mathbf{v} - (5 - i)\mathbf{u} \)

**SOLUTION**

(a) In column matrix form, the sum \( \mathbf{v} + \mathbf{u} \) is

\[
\mathbf{v} + \mathbf{u} = \begin{bmatrix} 1 + 2i \\ 3 - i \end{bmatrix} + \begin{bmatrix} -2 + i \\ 4 \end{bmatrix} = \begin{bmatrix} -1 + 3i \\ 7 - i \end{bmatrix}.
\]

(b) Because \((2 + i)(1 + 2i) = 5i\) and \((2 + i)(3 - i) = 7 + i\), you have

\[
(2 + i)\mathbf{v} = (2 + i)(1 + 2i, 3 - i) = (5i, 7 + i).
\]

(c) \(3\mathbf{v} - (5 - i)\mathbf{u} = 3(1 + 2i, 3 - i) - (5 - i)(-2 + i, 4)\)

\[
= (3 + 6i, 9 - 3i) - (-9 + 7i, 20 - 4i)
\]

\[
= (12 - i, -11 + i)
\]
Chapter 8  Complex Vector Spaces

Many of the properties of $\mathbb{R}^n$ are shared by $\mathbb{C}^n$. For instance, the scalar multiplicative identity is the scalar 1 and the additive identity in $\mathbb{C}^n$ is $0 = (0, 0, 0, \ldots, 0)$. The **standard basis** for $\mathbb{C}^n$ is simply

\[
e_1 = (1, 0, 0, \ldots, 0) \\
e_2 = (0, 1, 0, \ldots, 0) \\
\ldots \\
e_n = (0, 0, 0, \ldots, 1)
\]

which is the standard basis for $\mathbb{R}^n$. Because this basis contains $n$ vectors, it follows that the dimension of $\mathbb{C}^n$ is $n$. Other bases exist; in fact, any linearly independent set of $n$ vectors in $\mathbb{C}^n$ can be used, as demonstrated in Example 2.

**Example 2  Verifying a Basis**

Show that

\[
S = \{(i, 0, 0), (i, i, 0), (0, 0, i)\}
\]

is a basis for $\mathbb{C}^3$.

**Solution**

Because $\mathbb{C}^3$ has a dimension of 3, the set $\{v_1, v_2, v_3\}$ will be a basis if it is linearly independent. To check for linear independence, set a linear combination of the vectors in $S$ equal to $0$, as follows.

\[
c_1v_1 + c_2v_2 + c_3v_3 = (0, 0, 0) \\
(c_1i, 0, 0) + (c_2i, c_2i, 0) + (0, 0, c_3i) = (0, 0, 0) \\
((c_1 + c_2)i, c_2i, c_3i) = (0, 0, 0)
\]

This implies that

\[
(c_1 + c_2)i = 0 \\
c_2i = 0 \\
c_3i = 0.
\]

So, $c_1 = c_2 = c_3 = 0$, and you can conclude that $\{v_1, v_2, v_3\}$ is linearly independent.

**Example 3  Representing a Vector in $\mathbb{C}^n$ by a Basis**

Use the basis $S$ in Example 2 to represent the vector

\[
v = (2, i, 2 - i).
\]
**Section 8.4 Complex Vector Spaces and Inner Products**

**SOLUTION**  
By writing  
\[ v = c_1v_1 + c_2v_2 + c_3v_3 \]  
\[ = ((c_1 + c_2)i, c_2i, c_3i) \]  
\[ = (2, i, 2 - i), \]  
you can obtain  
\[ (c_1 + c_2)i = 2 \]  
\[ c_2i = i \]  
\[ c_3i = 2 - i \]  
which implies that \( c_2 = 1 \),  
\[ c_1 = \frac{2 - i}{i} = -1 - 2i, \quad \text{and} \quad c_3 = \frac{2 - i}{i} = -1 - 2i. \]  
So,  
\[ v = (-1 - 2i)v_1 + v_2 + (-1 - 2i)v_3. \]  
Try verifying that this linear combination yields \( (2, i, 2 - i) \).

Other than \( C^n \), there are several additional examples of complex vector spaces. For instance, the set of \( m \times n \) complex matrices with matrix addition and scalar multiplication forms a complex vector space. Example 4 describes a complex vector space in which the vectors are functions.

---

**Example 4** The Space of Complex-Valued Functions

Consider the set \( S \) of complex-valued functions of the form  
\[ f(x) = f_1(x) + if_2(x) \]  
where \( f_1 \) and \( f_2 \) are real-valued functions of a real variable. The set of complex numbers form the scalars for \( S \), and vector addition is defined by  
\[ f(x) + g(x) = [f_1(x) + i f_2(x)] + [g_1(x) + i g_2(x)] \]  
\[ = [f_1(x) + g_1(x)] + i [f_2(x) + g_2(x)]. \]  
It can be shown that \( S \), scalar multiplication, and vector addition form a complex vector space. For instance, to show that \( S \) is closed under scalar multiplication, let \( c = a + bi \) be a complex number. Then  
\[ cf(x) = (a + bi)[f_1(x) + if_2(x)] \]  
\[ = [af_1(x) - bf_2(x)] + i[bf_1(x) + af_2(x)] \]  
is in \( S \).
The definition of the Euclidean inner product in $C^n$ is similar to the standard dot product in $R^n$, except that here the second factor in each term is a complex conjugate.

**Definition of the Euclidean Inner Product in $C^n$**

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $C^n$. The **Euclidean inner product** of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1\bar{v}_1 + u_2\bar{v}_2 + \cdots + u_n\bar{v}_n.$$  

**Remark:** Note that if $\mathbf{u}$ and $\mathbf{v}$ happen to be “real,” then this definition agrees with the standard inner (or dot) product in $R^n$.

**Example 5**

**Finding the Euclidean Inner Product in $C^3$**

Determine the Euclidean inner product of the vectors

$$\mathbf{u} = (2 + i, 0, 4 - 5i) \quad \text{and} \quad \mathbf{v} = (1 + i, 2 + i, 0).$$

**Solution**

$$\mathbf{u} \cdot \mathbf{v} = u_1\bar{v}_1 + u_2\bar{v}_2 + u_3\bar{v}_3$$  
$$= (2 + i)(1 - i) + 0(2 - i) + (4 - 5i)(0)$$  
$$= 3 - i$$

Several properties of the Euclidean inner product $C^n$ are stated in the following theorem.

**Theorem 8.7**

**Properties of the Euclidean Inner Product**

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $C^n$ and let $k$ be a complex number. Then the following properties are true.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
4. $\mathbf{u} \cdot (k\mathbf{v}) = \overline{k}(\mathbf{u} \cdot \mathbf{v})$
5. $\mathbf{u} \cdot \mathbf{u} \geq 0$
6. $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

**Proof**

The proof of the first property is shown below, and the proofs of the remaining properties have been left to you. Let

$$\mathbf{u} = (u_1, u_2, \ldots, u_n) \quad \text{and} \quad \mathbf{v} = (v_1, v_2, \ldots, v_n).$$
Then
\[ \langle \mathbf{v}, \mathbf{u} \rangle = v_1 \overline{u_1} + v_2 \overline{u_2} + \cdots + v_n \overline{u_n} \]
\[ = v_1 \overline{u_1} + v_2 \overline{u_2} + \cdots + v_n \overline{u_n} \]
\[ = \overline{\mathbf{v}} \cdot \mathbf{u} \]
\[ = \mathbf{u} \cdot \mathbf{v}. \]

You will now use the Euclidean inner product in \( \mathbb{C}^n \) to define the Euclidean norm (or length) of a vector in \( \mathbb{C}^n \) and the Euclidean distance between two vectors in \( \mathbb{C}^n \).

**Definitions of the Euclidean Norm and Distance in \( \mathbb{C}^n \)**

The **Euclidean norm** (or length) of \( \mathbf{u} \) in \( \mathbb{C}^n \) is denoted by \( \| \mathbf{u} \| \) and is
\[ \| \mathbf{u} \| = (\langle \mathbf{u}, \mathbf{u} \rangle)^{1/2}. \]

The **Euclidean distance** between \( \mathbf{u} \) and \( \mathbf{v} \) is
\[ d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \|. \]

The Euclidean norm and distance may be expressed in terms of components as
\[ \| \mathbf{u} \| = (|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2)^{1/2} \]
\[ d(\mathbf{u}, \mathbf{v}) = (|u_1 - v_1|^2 + |u_2 - v_2|^2 + \cdots + |u_n - v_n|^2)^{1/2}. \]

**EXAMPLE 6** Finding the Euclidean Norm and Distance in \( \mathbb{C}^n \)

Determine the norms of the vectors
\[ \mathbf{u} = (2 + i, 0, 4 - 5i) \quad \text{and} \quad \mathbf{v} = (1 + i, 2 + i, 0) \]
and find the distance between \( \mathbf{u} \) and \( \mathbf{v} \).

**SOLUTION** The norms of \( \mathbf{u} \) and \( \mathbf{v} \) are expressed as follows.
\[ \| \mathbf{u} \| = (|u_1|^2 + |u_2|^2 + |u_3|^2)^{1/2} \]
\[ = [(2^2 + 1^2) + (0^2 + 0^2) + (4^2 + 5^2)]^{1/2} \]
\[ = (5 + 0 + 41)^{1/2} = \sqrt{46} \]
\[ \| \mathbf{v} \| = (|v_1|^2 + |v_2|^2 + |v_3|^2)^{1/2} \]
\[ = [(1^2 + 1^2) + (2^2 + 1^2) + (0^2 + 0^2)]^{1/2} \]
\[ = (2 + 5 + 0)^{1/2} = \sqrt{7} \]
A Complex Inner Product

The distance between \( \mathbf{u} \) and \( \mathbf{v} \) is expressed as
\[
d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|
\]
\[
= \|(1, -2 - i, 4 - 5i)\|
\]
\[
= \left[\left(1^2 + 0^2\right) + \left((-2)^2 + (-1)^2\right) + \left(4^2 + (-5)^2\right)\right]^{1/2}
\]
\[
= (1 + 5 + 41)^{1/2} = \sqrt{47}.
\]

**Complex Inner Product Spaces**

The Euclidean inner product is the most commonly used inner product in \( \mathbb{C}^n \). On occasion, however, it is useful to consider other inner products. To generalize the notion of an inner product, use the properties listed in Theorem 8.7.

**Definition of a Complex Inner Product**

Let \( \mathbf{u} \) and \( \mathbf{v} \) be vectors in a complex vector space. A function that associates \( \mathbf{u} \) and \( \mathbf{v} \) with the complex number \( \langle \mathbf{u}, \mathbf{v} \rangle \) is called a **complex inner product** if it satisfies the following properties.

1. \( \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle \)
2. \( \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \)
3. \( \langle k \mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle \)
4. \( \langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \) and \( \langle \mathbf{u}, \mathbf{u} \rangle = 0 \) if and only if \( \mathbf{u} = \mathbf{0} \).

A complex vector space with a complex inner product is called a **complex inner product space** or **unitary space**.

**Example 7**

A Complex Inner Product Space

Let \( \mathbf{u} = (u_1, u_2) \) and \( \mathbf{v} = (v_1, v_2) \) be vectors in the complex space \( \mathbb{C}^2 \). Show that the function defined by
\[
\langle \mathbf{u}, \mathbf{v} \rangle = u_1 \overline{v}_1 + 2u_2 \overline{v}_2
\]
is a complex inner product.

**Solution**

Verify the four properties of a complex inner product as follows.

1. \( \langle \mathbf{v}, \mathbf{u} \rangle = \overline{v}_1 u_1 + 2 \overline{v}_2 u_2 = u_1 \overline{v}_1 + 2u_2 \overline{v}_2 = \langle \mathbf{u}, \mathbf{v} \rangle \)
2. \( \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = (u_1 + v_1) \overline{w}_1 + 2(u_2 + v_2) \overline{w}_2
\]
\[
= (u_1 \overline{w}_1 + 2u_2 \overline{w}_2) + (v_1 \overline{w}_1 + 2v_2 \overline{w}_2)
\]
\[
= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle
\]
3. \( \langle k \mathbf{u}, \mathbf{v} \rangle = (ku_1) \overline{v}_1 + 2(ku_2) \overline{v}_2
\]
\[
= ku_1 \overline{v}_1 + 2ku_2 \overline{v}_2
\]
\[
= k \langle \mathbf{u}, \mathbf{v} \rangle
\]
4. \( \langle \mathbf{u}, \mathbf{u} \rangle = u_1 \overline{u}_1 + 2u_2 \overline{u}_2 = |u_1|^2 + 2|u_2|^2 \geq 0 \)

Moreover, \( \langle \mathbf{u}, \mathbf{u} \rangle = 0 \) if and only if \( u_1 = u_2 = 0 \). Because all properties hold, \( \langle \mathbf{u}, \mathbf{v} \rangle \) is a complex inner product.
In Exercises 17–24, determine the Euclidean norm of the following basis vectors.

1. \(3u\)
2. \(4i\omega\)
3. \((1 + 2i)w\)
4. \(iv + 3w\)
5. \(u - (2 - i)v\)
6. \((6 + 3i)v - (2 + 2i)w\)
7. \(u + iv + 2iw\)
8. \(2iw - (3 - i)w + u\)

In Exercises 9–12, determine whether \(S\) is a basis for \(\mathbb{C}^n\).

9. \(S = \{(1, i), (i, -1)\}\)
10. \(S = \{(1, i), (i, 1)\}\)
11. \(S = \{(i, 0, 0), (0, i, i), (0, 0, 1)\}\)
12. \(S = \{(1 - i, 0, 1), (2, i, 1 + i), (1 - i, 1, 1)\}\)

In Exercises 13–16, express \(v\) as a linear combination of each of the following basis vectors.

(a) \(\{i, 0, 0\}, \{i, i, 0\}, \{i, i, i\}\)
(b) \(\{1, 0, 0\}, \{1, 1, 0\}, \{0, 0, 1 + i\}\)
13. \(\mathbf{v} = (1, 2, 0)\)
14. \(\mathbf{v} = (1 - i, 1 + i, -3)\)
15. \(\mathbf{v} = (-i, 2 + i, -1)\)
16. \(\mathbf{v} = (i, i, i)\)

In Exercises 17–24, determine the Euclidean norm of \(v\).

17. \(\mathbf{v} = (i, -i)\)
18. \(\mathbf{v} = (1, 0)\)
19. \(\mathbf{v} = 3(6 + i, 2 - i)\)
20. \(\mathbf{v} = (2 + 3i, 2 - 3i)\)
21. \(\mathbf{v} = (1, 2 + i, -i)\)
22. \(\mathbf{v} = (0, 0, 0)\)
23. \(\mathbf{v} = (1 - 2i, i, 3i, 1 + i)\)
24. \(\mathbf{v} = (2, -1 + i, 2 - i, 4i)\)

In Exercises 25–30, determine the Euclidean distance between \(u\) and \(v\).

25. \(\mathbf{u} = (1, 0), \mathbf{v} = (i, i)\)
26. \(\mathbf{u} = (2 + i, 4, -i), \mathbf{v} = (2 + i, 4, -i)\)
27. \(\mathbf{u} = (i, 2i, 3i), \mathbf{v} = (0, 1, 0)\)
28. \(\mathbf{u} = (\sqrt{2}, 2i, -i), \mathbf{v} = (i, i, i)\)
29. \(\mathbf{u} = (1, 0), \mathbf{v} = (0, 1)\)
30. \(\mathbf{u} = (1, 2, 1, -2i), \mathbf{v} = (i, 2i, i, 2)\)

In Exercises 31–34, determine whether the set of vectors is linearly independent or linearly dependent.

31. \(\{(1, i), (i, -1)\}\)
32. \(\{(1 + i, 1 - i, 1), (i, 0, 1), (-2, -1 + i, 0)\}\)
33. \(\{(i, 1, 1 + i), (0, 1, -i), (0, 0, 1)\}\)
34. \(\{(1 + i, 1 - i, 0), (1 - i, 0, 0), (0, 1, 1)\}\)

In Exercises 35–38, determine whether the function is a complex inner product, where \(u = (u_1, u_2)\) and \(v = (v_1, v_2)\).

35. \(\langle u, v \rangle = u_1 + u_2 v_2\)
36. \(\langle u, v \rangle = (u_1 + v_1) + 2(u_2 + v_2)\)
37. \(\langle u, v \rangle = 4u_1 \overline{v_1} + 6u_2 \overline{v_2}\)
38. \(\langle u, v \rangle = u_1 v_1 - u_2 v_2\)

In Exercises 39–42, use the inner product \(\langle u, v \rangle = u_1 \overline{v_1} + 2u_2 \overline{v_2}\) to find \(\langle u, v \rangle\).

39. \(\mathbf{u} = (2i, -i)\) and \(\mathbf{v} = (i, 4i)\)
40. \(\mathbf{u} = (3 + i, i)\) and \(\mathbf{v} = (2 - i, 2i)\)
41. \(\mathbf{u} = (2 - i, 2 + i)\) and \(\mathbf{v} = (3 - i, 3 + 2i)\)
42. \(\mathbf{u} = (4 + 2i, 3)\) and \(\mathbf{v} = (2 - 3i, -2)\)

43. Let \(\mathbf{v}_1 = (i, 0, 0)\) and \(\mathbf{v}_2 = (i, i, 0)\). If \(\mathbf{v}_3 = (z_1, z_2, z_3)\) and the set \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is not a basis for \(\mathbb{C}^3\), what does this imply about \(z_1, z_2,\) and \(z_3\) ?

44. Let \(\mathbf{v}_1 = (i, i, i)\) and \(\mathbf{v}_2 = (1, 0, 1)\). Determine a vector \(\mathbf{v}_3\) such that \(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}\) is a basis for \(\mathbb{C}^3\).

In Exercises 45–49, prove the property, where \(u, v,\) and \(w\) are vectors in \(\mathbb{C}^n\) and \(k\) is a complex number.

45. \(\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle\)
46. \(\langle k u, v \rangle = k \langle u, v \rangle\)
47. \(\langle u, (kv) \rangle = \overline{k} \langle u, v \rangle\)
48. \(\langle u, u \rangle \geq 0\)
49. \(\langle u, u \rangle = 0\) if and only if \(u = 0\).

50. Writing Let \(\langle u, v \rangle\) be a complex inner product and let \(k\) be a complex number. How are \(\langle u, v \rangle\) and \(\langle u, kv \rangle\) related?

In Exercises 51 and 52, use the inner product
\[
\langle u, v \rangle = u_{11} \overline{v}_{11} + u_{12} \overline{v}_{12} + u_{21} \overline{v}_{21} + u_{22} \overline{v}_{22}
\]
where
\[
\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}
\]
and
\[
\mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}
\]
to find \(\langle u, v \rangle\).

51. \[
\mathbf{u} = \begin{bmatrix} 0 & i \\ 1 & -2i \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 & 1 - 2i \\ 0 & i \end{bmatrix}
\]
52. \[
\mathbf{u} = \begin{bmatrix} 1 & 2i \\ 1 + i & 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} i & -2i \\ 3i & -1 \end{bmatrix}
\]
In Exercises 53 and 54, determine the linear transformation \( T: \mathbb{C}^m \to \mathbb{C}^n \) that has the given characteristics.

53. \( T(1, 0) = (2 + i, 1), T(0, 1) = (0, -i) \)

54. \( T(i, 0) = (2 + i, 1), T(0, i) = (0, -i) \)

In Exercises 55–58, the linear transformation is shown by \( T(v) = Av \). Find the image of \( v \) and the preimage of \( w \).

55. \( A = \begin{bmatrix} 1 & 0 \\ i & i \end{bmatrix}, \quad v = \begin{bmatrix} 1 + i \\ 1 - i \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

56. \( A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad v = \begin{bmatrix} i \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

57. \( A = \begin{bmatrix} 1 & 0 \\ i & 0 \\ i & i \end{bmatrix}, \quad v = \begin{bmatrix} 2 - i \\ 3 + 2i \\ 2i \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ 2i \\ 3i \end{bmatrix} \)

58. \( A = \begin{bmatrix} 0 & 1 \\ i & -1 \\ 0 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad w = \begin{bmatrix} 1 - i \\ 1 + i \\ 0 \end{bmatrix} \)

59. Find the kernel of the linear transformation from Exercise 55.

60. Find the kernel of the linear transformation from Exercise 56.

In Exercises 61 and 62, find the image of \( v = (i, i) \) for the indicated composition, where \( T_1 \) and \( T_2 \) are the matrices below.

\[
T_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} -i & i \\ i & -i \end{bmatrix}
\]

61. \( T_2 \circ T_1 \)

62. \( T_1 \circ T_2 \)

63. Determine which of the sets below are subspaces of the vector space of \( 2 \times 2 \) complex matrices.

(a) The set of \( 2 \times 2 \) symmetric matrices.

(b) The set of \( 2 \times 2 \) matrices \( A \) satisfying \( (A^T)^T = A \).

(c) The set of \( 2 \times 2 \) matrices in which all entries are real.

(d) The set of \( 2 \times 2 \) diagonal matrices.

64. Determine which of the sets below are subspaces of the vector space of complex-valued functions (see Example 4).

(a) The set of all functions \( f \) satisfying \( f(i) = 0 \).

(b) The set of all functions \( f \) satisfying \( f(0) = 1 \).

(c) The set of all functions \( f \) satisfying \( f(i) = f(-i) \).

**True or False?** In Exercises 65 and 66, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

65. Using the Euclidean inner product of \( u \) and \( v \) in \( \mathbb{C}^n \), \( u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n \).

66. The Euclidean norm of \( u \) in \( \mathbb{C}^n \) denoted by \( ||u|| \) is \( (u \cdot u)^2 \).
8.5 Unitary and Hermitian Matrices

Problems involving diagonalization of complex matrices and the associated eigenvalue problems require the concepts of **unitary** and **Hermitian** matrices. These matrices roughly correspond to orthogonal and symmetric real matrices. In order to define unitary and Hermitian matrices, the concept of the **conjugate transpose** of a complex matrix must first be introduced.

The **conjugate transpose** of a complex matrix $A$, denoted by $A^*$, is given by

$$A^* = \overline{A}^T$$

where the entries of $\overline{A}$ are the complex conjugates of the corresponding entries of $A$.

Note that if $A$ is a matrix with real entries, then $A^* = A^T$. To find the conjugate transpose of a matrix, first calculate the complex conjugate of each entry and then take the transpose of the matrix, as shown in the following example.

**EXAMPLE 1** Finding the Conjugate Transpose of a Complex Matrix

Determine $A^*$ for the matrix

$$A = \begin{bmatrix} 3 + 7i & 0 \\ 2i & 4 - i \end{bmatrix}.$$ 

**SOLUTION**

$$\overline{A} = \begin{bmatrix} 3 + 7i & 0 \\ 2i & 4 - i \end{bmatrix} = \begin{bmatrix} 3 - 7i & 0 \\ -2i & 4 + i \end{bmatrix}$$

$$A^* = \overline{A}^T = \begin{bmatrix} 3 - 7i & -2i \\ 0 & 4 + i \end{bmatrix}$$

Several properties of the conjugate transpose of a matrix are listed in the following theorem. The proofs of these properties are straightforward and are left for you to supply in Exercises 55–58.

**THEOREM 8.8** Properties of the Conjugate Transpose

If $A$ and $B$ are complex matrices and $k$ is a complex number, then the following properties are true.

1. $(A^*)^* = A$
2. $(A + B)^* = A^* + B^*$
3. $(kA)^* = kA^*$
4. $(AB)^* = B^*A^*$
Unitary Matrices

Recall that a real matrix $A$ is orthogonal if and only if $A^{-1} = A^T$. In the complex system, matrices having the property that $A^{-1} = A^*$ are more useful, and such matrices are called unitary.

**Definition of Unitary Matrix**

A complex matrix $A$ is **unitary** if $A^{-1} = A^*$.

**Example 2**

**A Unitary Matrix**

Show that the matrix $A$ is unitary.

$$A = \frac{1}{2} \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + 1 \end{bmatrix}$$

**Solution**

Because

$$AA^* = \frac{1}{2} \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + 1 \end{bmatrix} \begin{bmatrix} 1 - i & 1 + i \\ 1 + i & 1 - i \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

you can conclude that $A^* = A^{-1}$. So, $A$ is a unitary matrix.

In Section 7.3, you saw that a real matrix is orthogonal if and only if its row (or column) vectors form an orthonormal set. For complex matrices, this property characterizes matrices that are unitary. Note that a set of vectors

$$\{v_1, v_2, \ldots, v_m\}$$

in $C^n$ (a complex Euclidean space) is called orthonormal if the statements below are true.

1. $\|v_i\| = 1$, $i = 1, 2, \ldots, m$

2. $v_i \cdot v_j = 0$, $i \neq j$

The proof of the next theorem is similar to the proof of Theorem 7.8 presented in Section 7.3.

**Theorem 8.9**

**Unitary Matrices**

An $n \times n$ complex matrix $A$ is unitary if and only if its row (or column) vectors form an orthonormal set in $C^n$. 
EXAMPLE 3  The Row Vectors of a Unitary Matrix

Show that the complex matrix \( A \) is unitary by showing that its set of row vectors forms an orthonormal set in \( C^3 \).

\[
A = \begin{bmatrix}
\frac{1}{2} & \frac{1+i}{2} & -\frac{1}{2} \\
-\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{5i}{2\sqrt{15}} & \frac{3+i}{2\sqrt{15}} & \frac{4+3i}{2\sqrt{15}}
\end{bmatrix}
\]

SOLUTION  Let \( r_1, r_2, \) and \( r_3 \) be defined as follows.

\[
r_1 = \left( \frac{1}{2}, -\frac{1+i}{2}, -\frac{1}{2} \right)
\]

\[
r_2 = \left( -\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)
\]

\[
r_3 = \left( \frac{5i}{2\sqrt{15}}, \frac{3+i}{2\sqrt{15}}, \frac{4+3i}{2\sqrt{15}} \right)
\]

The length of \( r_1 \) is

\[
\|r_1\| = (r_1 \cdot r_1)^{1/2} = \left[ \left( \frac{1}{2} \right)^2 + \left( \frac{1+i}{2} \right)^2 + \left( -\frac{1}{2} \right)^2 \right]^{1/2} = 1.
\]

The vectors \( r_2 \) and \( r_3 \) can also be shown to be unit vectors. The inner product of \( r_1 \) and \( r_2 \) is

\[
r_1 \cdot r_2 = \left( \frac{1}{2} \right) \left( -\frac{i}{\sqrt{3}} \right) + \left( \frac{1+i}{2} \right) \left( \frac{i}{\sqrt{3}} \right) + \left( -\frac{1}{2} \right) \left( \frac{1}{\sqrt{3}} \right) = 0.
\]

Similarly, \( r_1 \cdot r_3 = 0 \) and \( r_2 \cdot r_3 = 0 \). So, you can conclude that \( \{r_1, r_2, r_3\} \) is an orthonormal set. (Try showing that the column vectors of \( A \) also form an orthonormal set in \( C^3 \).)
Hermitian Matrices

A real matrix is called symmetric if it is equal to its own transpose. In the complex system, the more useful type of matrix is one that is equal to its own conjugate transpose. Such a matrix is called Hermitian after the French mathematician Charles Hermite (1822–1901).

**Definition of a Hermitian Matrix**

A square matrix $A$ is **Hermitian** if

$$A = A^*.$$

As with symmetric matrices, you can easily recognize Hermitian matrices by inspection. To see this, consider the $2 \times 2$ matrix $A$.

$$A = \begin{bmatrix} a_1 + a_2i & b_1 + b_2i \\ c_1 + c_2i & d_1 + d_2i \end{bmatrix}$$

The conjugate transpose of $A$ has the form

$$A^* = \overline{A}^T = \begin{bmatrix} a_1 + a_2i & c_1 + c_2i \\ b_1 + b_2i & d_1 + d_2i \end{bmatrix}$$

$$= \begin{bmatrix} a_1 - a_2i & c_1 - c_2i \\ b_1 - b_2i & d_1 - d_2i \end{bmatrix}.$$

If $A$ is Hermitian, then $A = A^*$. So, you can conclude that $A$ must be of the form

$$A = \begin{bmatrix} a_1 & b_1 + b_2i \\ b_1 - b_2i & d_1 \end{bmatrix}.$$

Similar results can be obtained for Hermitian matrices of order $n \times n$. In other words, a square matrix $A$ is Hermitian if and only if the following two conditions are met.

1. The entries on the main diagonal of $A$ are real.
2. The entry $a_{ij}$ in the $i$th row and the $j$th column is the complex conjugate of the entry $a_{ji}$ in the $j$th row and the $i$th column.

**Example 4**

Which matrices are Hermitian?

(a) \[
\begin{bmatrix}
1 & 3 - i \\
3 + i & i
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
0 & 3 - 2i \\
3 - 2i & 4
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
3 & 2 - i & -3i \\
2 + i & 0 & 1 - i \\
3i & 1 + i & 0
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 0 & -1 \\
3 & -1 & 4
\end{bmatrix}
\]
Section 8.5 Unitary and Hermitian Matrices

SOLUTION

(a) This matrix is not Hermitian because it has an imaginary entry on its main diagonal.
(b) This matrix is symmetric but not Hermitian because the entry in the first row and second column is not the complex conjugate of the entry in the second row and first column.
(c) This matrix is Hermitian.
(d) This matrix is Hermitian because all real symmetric matrices are Hermitian.

One of the most important characteristics of Hermitian matrices is that their eigenvalues are real. This is formally stated in the next theorem.

THEOREM 8.10
The Eigenvalues of a Hermitian Matrix

If $A$ is a Hermitian matrix, then its eigenvalues are real numbers.

PROOF

Let $\lambda$ be an eigenvalue of $A$ and let

$$v = \begin{bmatrix} a_1 + b_1i \\ a_2 + b_2i \\ \vdots \\ a_n + b_ni \end{bmatrix}$$

be its corresponding eigenvector. If both sides of the equation $Av = \lambda v$ are multiplied by the row vector $v^*$, then

$$v^*Av = v^*(\lambda v) = \lambda(v^*v) = \lambda(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots + a_n^2 + b_n^2).$$

Furthermore, because

$$(v^*Av)^* = v^*A^*(v^*)^* = v^*Av,$$

it follows that $v^*Av$ is a Hermitian $1 \times 1$ matrix. This implies that $v^*Av$ is a real number, so $\lambda$ is real.

REMARK: Note that this theorem implies that the eigenvalues of a real symmetric matrix are real, as stated in Theorem 7.7.

To find the eigenvalues of complex matrices, follow the same procedure as for real matrices.
EXAMPLE 5  Finding the Eigenvalues of a Hermitian Matrix

Find the eigenvalues of the matrix \( A \).

\[
A = \begin{bmatrix}
3 & 2 - i & -3i \\
2 + i & 0 & 1 - i \\
3i & 1 + i & 0
\end{bmatrix}
\]

**SOLUTION**  The characteristic polynomial of \( A \) is

\[
| \lambda I - A | = \begin{bmatrix}
\lambda - 3 & -2 + i & 3i \\
-2 - i & \lambda & -1 + i \\
-3i & -1 - i & \lambda
\end{bmatrix}
\]

\[
= (\lambda - 3)(\lambda^2 - 2) - (-2 + i)[(-2 - i)\lambda - (3i + 3)] \\
+ 3i[(1 + 3i) + 3\lambda]
\]

\[
= (\lambda^3 - 3\lambda^2 - 2\lambda + 6) - (5\lambda + 9 + 3i) + (3i - 9 - 9\lambda)
\]

\[
= \lambda^3 - 3\lambda^2 - 16\lambda - 12
\]

\[
= (\lambda + 1)(\lambda - 6)(\lambda + 2).
\]

This implies that the eigenvalues of \( A \) are \(-1, 6, \) and \(-2\).

To find the eigenvectors of a complex matrix, use a procedure similar to that used for a real matrix. For instance, in Example 5, the eigenvector corresponding to the eigenvalue \( \lambda = -1 \) is obtained by solving the following equation.

\[
\begin{bmatrix}
\lambda - 3 & -2 + i & 3i \\
-2 - i & \lambda & -1 + i \\
-3i & -1 - i & \lambda
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Using Gauss-Jordan elimination, or a computer software program or graphing utility, obtain the eigenvector corresponding to \( \lambda_1 = -1 \), which is shown below.

\[
v_1 = \begin{bmatrix}
-1 \\
1 + 2i \\
1
\end{bmatrix}
\]

Eigenvectors for \( \lambda_2 = 6 \) and \( \lambda_3 = -2 \) can be found in a similar manner. They are

\[
\begin{bmatrix}
1 - 21i \\
6 - 9i \\
13
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 + 3i \\
-2 - i \\
5
\end{bmatrix},
\]
Section 8.5  Unitary and Hermitian Matrices

Just as you saw in Section 7.3 that real symmetric matrices are orthogonally diagonalizable, you will now see that Hermitian matrices are \textit{unitarily diagonalizable}. A square matrix is unitarily diagonalizable if there exists a unitary matrix such that

\[
P^{-1}AP
\]

is a diagonal matrix. Because \( P \) is unitary, \( P^{-1} = P^* \), so an equivalent statement is that \( A \) is unitarily diagonalizable if there exists a unitary matrix \( P \) such that \( P^*AP \) is a diagonal matrix. The next theorem states that Hermitian matrices are unitarily diagonalizable.

\textbf{THEOREM 8.11  Hermitian Matrices and Diagonalization}

If \( A \) is an \( n \times n \) Hermitian matrix, then
\begin{enumerate}
  \item eigenvectors corresponding to distinct eigenvalues are orthogonal.
  \item \( A \) is unitarily diagonalizable.
\end{enumerate}

\textbf{PROOF}

To prove part 1, let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be two eigenvectors corresponding to the distinct (and real) eigenvalues \( \lambda_1 \) and \( \lambda_2 \). Because \( A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \) and \( A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \), you have the equations shown below for the matrix product \( (A\mathbf{v}_1)^*\mathbf{v}_2 \).

\[
(A\mathbf{v}_1)^*\mathbf{v}_2 = \mathbf{v}_1^*A^*\mathbf{v}_2 = \mathbf{v}_1^*\lambda_2 \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^*\mathbf{v}_2
\]

\[
(A\mathbf{v}_1)^*\mathbf{v}_2 = (\lambda_1 \mathbf{v}_1)^*\mathbf{v}_2 = \lambda_1 \mathbf{v}_1^*\mathbf{v}_2
\]

So,

\[
\lambda_2 \mathbf{v}_1^*\mathbf{v}_2 - \lambda_1 \mathbf{v}_1^*\mathbf{v}_2 = 0
\]

\[
(\lambda_2 - \lambda_1) \mathbf{v}_1^*\mathbf{v}_2 = 0
\]

\[
\mathbf{v}_1^*\mathbf{v}_2 = 0 \quad \text{because} \quad \lambda_1 \neq \lambda_2,
\]

and this shows that \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are orthogonal. Part 2 of Theorem 8.11 is often called the \textit{Spectral Theorem}, and its proof is left to you.

\textbf{EXAMPLE 6  The Eigenvectors of a Hermitian Matrix}

The eigenvectors of the Hermitian matrix shown in Example 5 are mutually orthogonal because the eigenvalues are distinct. You can verify this by calculating the Euclidean inner products \( \mathbf{v}_1 \cdot \mathbf{v}_2, \mathbf{v}_1 \cdot \mathbf{v}_3, \) and \( \mathbf{v}_2 \cdot \mathbf{v}_3 \). For example,
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The other two inner products \( \mathbf{v}_1 \cdot \mathbf{v}_2 \) and \( \mathbf{v}_2 \cdot \mathbf{v}_3 \) can be shown to equal zero in a similar manner.

The three eigenvectors in Example 6 are mutually orthogonal because they correspond to distinct eigenvalues of the Hermitian matrix \( A \). Two or more eigenvectors corresponding to the same eigenvalue may not be orthogonal. Once any set of linearly independent eigenvectors is obtained for an eigenvalue, however, the Gram-Schmidt orthonormalization process can be used to find an orthogonal set.

**EXAMPLE 7** Diagonalization of a Hermitian Matrix

Find a unitary matrix \( P \) such that \( P^*AP \) is a diagonal matrix where

\[
A = \begin{bmatrix}
3 & 2 - i & -3i \\
2 + i & 0 & 1 - i \\
3i & 1 + i & 0
\end{bmatrix}.
\]

**SOLUTION**

The eigenvectors of \( A \) are shown after Example 5. Form the matrix \( P \) by normalizing these three eigenvectors and using the results to create the columns of \( P \). So, because

\[
\| \mathbf{v}_1 \| = \|(-1, 1 + 2i, 1)\| = \sqrt{1 + 5 + 1} = \sqrt{7}
\]
\[
\| \mathbf{v}_2 \| = \|(-21i, 6 - 9i, 13)\| = \sqrt{442 + 117 + 169} = \sqrt{728}
\]
\[
\| \mathbf{v}_3 \| = \|(-3i, -2 - i, 5)\| = \sqrt{10 + 5 + 25} = \sqrt{40},
\]

the unitary matrix \( P \) is obtained.

\[
P = \begin{bmatrix}
\frac{1}{\sqrt{7}} & \frac{1 - 2i}{\sqrt{728}} & \frac{1 + 3i}{\sqrt{40}} \\
\frac{1}{\sqrt{7}} & \frac{6 - 9i}{\sqrt{728}} & -\frac{2 - i}{\sqrt{40}} \\
\frac{1}{\sqrt{7}} & \frac{13}{\sqrt{728}} & \frac{5}{\sqrt{40}}
\end{bmatrix}
\]

Try computing the product \( P^*AP \) for the matrices \( A \) and \( P \) in Example 7 to see that you obtain

\[
P^*AP = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & -2
\end{bmatrix}
\]

where \(-1, 6, \) and \(-2\) are the eigenvalues of \( A \).
You have seen that Hermitian matrices are unitarily diagonalizable. It turns out that there is a larger class of matrices, called normal matrices, that are also unitarily diagonalizable. A square complex matrix $A$ is normal if it commutes with its conjugate transpose: $AA^* = A^*A$. The main theorem of normal matrices states that a complex matrix $A$ is normal if and only if it is unitarily diagonalizable. You are asked to explore normal matrices further in Exercise 65.

The properties of complex matrices described in this section are comparable to the properties of real matrices discussed in Chapter 7. The summary below indicates the correspondence between unitary and Hermitian complex matrices when compared with orthogonal and symmetric real matrices.

### Comparison of Symmetric and Hermitian Matrices

<table>
<thead>
<tr>
<th>$A$ is a symmetric matrix (real)</th>
<th>$A$ is a Hermitian matrix (complex)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Eigenvalues of $A$ are real.</td>
<td>1. Eigenvalues of $A$ are real.</td>
</tr>
<tr>
<td>2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.</td>
<td>2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.</td>
</tr>
<tr>
<td>3. There exists an orthogonal matrix $P$ such that $P^TAP$ is diagonal.</td>
<td>3. There exists a unitary matrix $P$ such that $P^*AP$ is diagonal.</td>
</tr>
</tbody>
</table>

### SECTION 8.5 Exercises

In Exercises 1–8, determine the conjugate transpose of the matrix.

1. $A = \begin{bmatrix} i & -i \\ 2 & 3i \end{bmatrix}$
2. $A = \begin{bmatrix} 1 + 2i & 2 - i \\ 1 & 1 \end{bmatrix}$
3. $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$
4. $A = \begin{bmatrix} 4 + 3i & 2 + i \\ 2 - i & 6i \end{bmatrix}$
5. $A = \begin{bmatrix} 0 & 5 + i & \sqrt{2}i \\ 5 - i & 6 & 4 \\ -\sqrt{2}i & 4 & 3 \end{bmatrix}$
6. $A = \begin{bmatrix} 2 + i & 3 - i & 4 + 5i \\ 3 - i & 2 & 6 - 2i \end{bmatrix}$
7. $A = \begin{bmatrix} 7 + 5i \\ 2i \\ 4 \end{bmatrix}$
8. $A = \begin{bmatrix} 2 & i \\ 5 & 3i \\ 0 & 6 - i \end{bmatrix}$

In Exercises 9–12, use a graphing utility or computer software program to find the conjugate transpose of the matrix.

9. $A = \begin{bmatrix} 1 + i & 0 & 1 \\ -2 & i & 2 - i \\ i & 2 + i & 2i \end{bmatrix}$
10. $A = \begin{bmatrix} i & 1 & 1 - 2i \\ 0 & 1 - i & i \\ 1 + 2i & 2i & i \end{bmatrix}$
11. $A = \begin{bmatrix} 1 + i & 0 & 1 & -i \\ 2 + i & 1 & 0 & 2i \\ 1 - i & i & 2 & 4i \\ i & 2 + i & -1 & 0 \end{bmatrix}$
12. $A = \begin{bmatrix} 2 + i & 1 & -1 & 2i \\ 0 & 2 - i & 2i & 1 - i \\ i & 2 + i & -i & 1 \\ 1 + 2i & 4 & 0 & -2i \end{bmatrix}$
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In Exercises 13–16, explain why the matrix is not unitary.
13. \( A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} \)
14. \( A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \)
15. \( A = \begin{bmatrix} \frac{1+i}{\sqrt{2}} & 0 - \frac{i}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \)
16. \( A = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} & \frac{1+i}{2} \\ -\frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{-i}{\sqrt{6}} \end{bmatrix} \)

In Exercises 17–22, determine whether \( A \) is unitary by calculating \( AA^* \).
17. \( A = \begin{bmatrix} 1+i & 1+i \\ 1-i & 1-i \end{bmatrix} \)
18. \( A = \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \)
19. \( A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \)
20. \( A = \begin{bmatrix} 1 \sqrt{2} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{bmatrix} \)
21. \( A = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ 0 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{6}} \end{bmatrix} \)
22. \( A = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \)

In Exercises 23–26, (a) verify that \( A \) is unitary by showing that its rows are orthonormal, and (b) determine the inverse of \( A \).
23. \( A = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \)
24. \( A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \)
25. \( A = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} - i & 1 + \sqrt{3}i \\ \sqrt{3} + i & 1 - \sqrt{3}i \end{bmatrix} \)
26. \( A = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1-i}{\sqrt{3}} \\ \frac{-1+i}{\sqrt{6}} & 0 \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \)

In Exercises 27–34, determine whether the matrix \( A \) is Hermitian.
27. \( A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \)
28. \( A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \)
29. \( A = \begin{bmatrix} 0 & 2 + i \\ 2 - i & 0 \end{bmatrix} \)
30. \( A = \begin{bmatrix} 2 + i & i \\ 1 & 0 \end{bmatrix} \)
31. \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)
32. \( A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \)
33. \( A = \begin{bmatrix} 1 & 2 + i \\ 2 - i & 3 - i \end{bmatrix} \)
34. \( A = \begin{bmatrix} \sqrt{2} - i & 2 \sqrt{3} + i \\ 3 - i & 6 \end{bmatrix} \)

In Exercises 35–40, determine the eigenvalues of the matrix \( A \).
35. \( A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \)
36. \( A = \begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix} \)
37. \( A = \begin{bmatrix} 3 & 1 - i \\ 1 + i & 2 \end{bmatrix} \)
38. \( A = \begin{bmatrix} 0 & 2 + i \\ 2 - i & 4 \end{bmatrix} \)
39. \( A = \begin{bmatrix} \frac{2}{\sqrt{2}} & -i \sqrt{2} \\ \frac{i}{\sqrt{2}} & 2 \end{bmatrix} \)
40. \( A = \begin{bmatrix} 1 & 4 & 1 - i \\ 0 & i & 3i \\ 0 & 0 & 2 + i \end{bmatrix} \)

In Exercises 41–44, determine the eigenvectors of the matrix.
41. The matrix in Exercise 35
42. The matrix in Exercise 38
43. The matrix in Exercise 39
44. The matrix in Exercise 36

In Exercises 45–49, find a unitary matrix \( P \) that diagonalizes the matrix \( A \).
45. \( A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \)
46. \( A = \begin{bmatrix} 0 & 2 + i \\ 2 - i & 4 \end{bmatrix} \)
63. (a) Prove that every Hermitian matrix can be written as the sum $A = B + iC$, where $B$ is a real symmetric matrix and $C$ is real and skew-symmetric.

(b) Use part (a) to write the matrix
$$A = \begin{bmatrix} 2 & 1 + i \\ 1 - i & 3 \end{bmatrix}$$
as the sum $A = B + iC$, where $B$ is a real symmetric matrix and $C$ is real and skew-symmetric.

(c) Prove that every $n \times n$ complex matrix $A$ can be written as $A = B + iC$, where $B$ and $C$ are Hermitian.

(d) Use part (c) to write the complex matrix
$$A = \begin{bmatrix} i & 2 \\ 2 + i & 1 - 2i \end{bmatrix}$$
as the sum $A = B + iC$, where $B$ and $C$ are Hermitian.

64. Determine which of the sets listed below are subspaces of the vector space of $n \times n$ complex matrices.

(a) The set of $n \times n$ Hermitian matrices
(b) The set of $n \times n$ unitary matrices
(c) The set of $n \times n$ normal matrices

65. (a) Prove that every Hermitian matrix is normal.

(b) Prove that every unitary matrix is normal.

(c) Find a $2 \times 2$ matrix that is Hermitian, but not unitary.

(d) Find a $2 \times 2$ matrix that is unitary, but not Hermitian.

(e) Find a $2 \times 2$ matrix that is normal, but neither Hermitian nor unitary.

(f) Find the eigenvalues and corresponding eigenvectors of your matrix from part (e).

(g) Show that the complex matrix
$$\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$$
is not diagonalizable. Is this matrix normal?

66. Show that $A = I_n$ is unitary by computing $AA^*$. 

**True or False?** In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

67. A complex matrix $A$ is called unitary if $A^{-1} = A^*$.

68. If $A$ is a complex matrix and $k$ is a complex number, then $(kA)^* = kA^*$. 

47. \[ A = \begin{bmatrix} 2 & -i \sqrt{2} & i \\ i \sqrt{2} & 2 & 0 \\ -i \sqrt{2} & 0 & 2 \end{bmatrix} \]

48. \[ A = \begin{bmatrix} 4 & 2 + 2i \\ 2 - 2i & 6 \end{bmatrix} \]

49. \[ A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 + i \\ 0 & -1 - i & 0 \end{bmatrix} \]

50. Let $z$ be a complex number with modulus 1. Show that the matrix $A$ is unitary.

\[ A = \frac{1}{\sqrt{2}} \begin{bmatrix} z & \bar{z} \\ iz & -i\bar{z} \end{bmatrix} \]

In Exercises 51–54, use the result of Exercise 50 to determine $a$, $b$, and $c$ such that $A$ is unitary.

51. \[ A = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & a \\ b & c \end{bmatrix} \]

52. \[ A = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 & a \\ 3 - 4i & b \\ b & c \end{bmatrix} \]

53. \[ A = \frac{1}{\sqrt{2}} \begin{bmatrix} i & a \\ b & c \end{bmatrix} \]

54. \[ A = \frac{1}{\sqrt{2}} \begin{bmatrix} 6 + 3i & a \\ 6 - 3i & b \\ b & c \end{bmatrix} \]

In Exercises 55–58, prove the formula, where $A$ and $B$ are $n \times n$ complex matrices.

55. $(A^*)^* = A$  
56. $(A + B)^* = A^* + B^*$

57. $(kA)^* = \overline{kA}$

58. $(AB)^* = B^*A^*$

59. Let $A$ be a matrix such that $A^* + A = O$. Prove that $iA$ is Hermitian.

60. Show that $\det(A) = \overline{\det(A)}$, where $A$ is a $2 \times 2$ matrix.

In Exercises 61 and 62, assume that the result of Exercise 60 is true for matrices of any size.

61. Show that $\det(A^*) = \overline{\det(A)}$.

62. Prove that if $A$ is unitary, then $|\det(A)| = 1$.

63. (a) Prove that every Hermitian matrix $A$ can be written as the sum $A = B + iC$, where $B$ is a real symmetric matrix and $C$ is real and skew-symmetric.
In Exercises 1–6, perform the operation.
1. Find \( u + z : u = 2 - 4i, z = 4i \)
2. Find \( u - z : u = 4, z = 8i \)
3. Find \( uz : u = 4 - 2i, z = 4 + 2i \)
4. Find \( uz : u = 2i, z = 1 - 2i \)
5. Find \( \frac{u}{z} : u = 6 - 2i, z = 3 - 3i \)
6. Find \( \frac{u}{z} : u = 7 + i, z = i \)

In Exercises 7–14, find all zeros of the polynomial function.
7. \( p(x) = x^2 - 4x + 8 \)
8. \( p(x) = x^2 - 4x + 7 \)
9. \( p(x) = 3x^2 + 3x + 3 \)
10. \( p(x) = x^2 + 6x + 10 \)
11. \( p(x) = x^3 + 2x^2 + 2x + 1 \)
12. \( p(x) = x^3 - 2x - 4 \)
13. \( p(x) = x^4 + x^3 + 3x^2 + 5x - 10 \)
14. \( p(x) = x^4 - x^3 + x^2 - 3x - 6 \)

In Exercises 15–22, perform the operation using
\[ A = \begin{bmatrix} 4 - i & 2 \\ 3 + i \\ \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 + i & 2i \\ 2i & 2 + i \\ \end{bmatrix} . \]
15. \( A + B \)
16. \( A - B \)
17. \( 2iB \)
18. \( -iA \)
19. \( \text{det}(A - B) \)
20. \( \text{det}(A + B) \)
21. \( 3BA \)
22. \( 2AB \)

In Exercises 23–28, perform the operation using \( w = 2 - 2i, v = 3 + i, \) and \( z = -1 + 2i . \)
23. \( \overline{z} \)
24. \( \overline{v} \)
25. \( |w| \)
26. \( |v||w| \)
27. \( \overline{w} \overline{v} \)
28. \( |\overline{w}||\overline{v}| \)

In Exercises 29–32, perform the indicated operation.
29. \( \frac{2 + i}{2 - i} \)
30. \( \frac{1 + i}{-1 + 2i} \)
31. \( \frac{(1 - 2i)(1 + 2i)}{3 - 3i} \)
32. \( \frac{5 + 2i}{(-2 + 2i)(2 - 3i)} \)

In Exercises 33 and 34, find \( A^{-1} \) (if it exists).
33. \( A = \begin{bmatrix} 3 - i & -1 - 2i \\ -23/5 + 11/3i & 2 + 3i \end{bmatrix} \)
34. \( A = \begin{bmatrix} 5 & 1 - i \\ 0 & i \end{bmatrix} \)

In Exercises 35–40, determine the polar form of the complex number.
35. \( 4 + 4i \)
36. \( 2 - 2i \)
37. \( \sqrt{3} + i \)
38. \( 1 + \sqrt{3}i \)
39. \( 7 - 4i \)
40. \( 3 + 2i \)

In Exercises 41–46, find the standard form of the complex number.
41. \( 5 \left[ \cos \left( -\frac{\pi}{6} \right) + i \sin \left( -\frac{\pi}{6} \right) \right] \)
42. \( 2 \left[ \cos \left( -\frac{\pi}{3} \right) + i \sin \left( -\frac{\pi}{3} \right) \right] \)
43. \( 4 \left[ \cos \left( \frac{5\pi}{4} \right) + i \sin \left( \frac{5\pi}{4} \right) \right] \)
44. \( 6 \left[ \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right] \)
45. \( 7 \left[ \cos \left( \frac{3\pi}{2} \right) + i \sin \left( \frac{3\pi}{2} \right) \right] \)
46. \( 4 \left[ \cos \pi + i \sin \pi \right] \)

In Exercises 47–50, perform the indicated operation. Leave the result in polar form.
47. \[ 4 \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right) \left[ 3 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right) \right] \]
48. \[ \frac{1}{2} \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right) \left[ 2 \left( \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right) \right] \]
49. \[ 9 \left[ \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right] \left[ 6 \left[ \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right] \right] \]
50. \[ 4 \left[ \cos \left( \frac{\pi}{4} \right) + i \sin \left( \frac{\pi}{4} \right) \right] \left[ 7 \left[ \cos \left( \frac{\pi}{3} \right) + i \sin \left( \frac{\pi}{3} \right) \right] \right] \]

In Exercises 51–54, find the indicated power of the number and express the result in polar form.
51. \( (-1 - i)^4 \)
52. \( (2i)^3 \)
53. \[ \sqrt{2} \left( \cos \left( \frac{\pi}{5} \right) + i \sin \left( \frac{\pi}{5} \right) \right) \]
54. \[ 5 \left( \cos \left( \frac{\pi}{3} \right) + i \sin \left( \frac{\pi}{3} \right) \right)^4 \]

In Exercises 55–58, express the roots in standard form.
55. Square roots: \( 25 \left( \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right) \)
56. Cube roots: \( 27 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right) \)
57. Cube roots: \( i \)
58. Fourth roots: 

\[ 16 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \]

In Exercises 59–62, determine the conjugate transpose of the matrix.

59. \( A = \begin{bmatrix} -1 + 4i & 3 + i \\ 3 - i & 2 + i \end{bmatrix} \)

60. \( A = \begin{bmatrix} 2 + i & 2 - i \\ 1 + 2i & 2 - 2i \end{bmatrix} \)

61. \( A = \begin{bmatrix} 5 & 2 - i & 3 + 2i \\ 2 + 2i & 3 - 2i & i \\ 3i & 2 + i & -1 - 2i \end{bmatrix} \)

62. \( A = \begin{bmatrix} 2 & 1 + i & i \\ -i & 2 + 2i & 0 \\ 1 & 1 + i & -2i \end{bmatrix} \)

In Exercises 63–66, find the indicated vector using \( u = (4i, 2 + i) \), \( v = (3, -i) \), and \( w = (3 - i, 4 + i) \).

63. \( 7u - v \)

64. \( 3iw + (4 - i)v \)

65. \( iu + iv - iw \)

66. \( (3 + 2i)u - (-2i)w \)

In Exercises 67 and 68, determine the Euclidean norm of the vector.

67. \( v = (3 - 5i, 2i) \)

68. \( v = (3i, -1 - 5i, 3 + 2i) \)

In Exercises 69 and 70, find the Euclidean distance between the vectors.

69. \( v = (2 - i, i) \)

70. \( v = (2 + i, -1 + 2i, 3i) \)

In Exercises 71–74, determine whether the matrix is unitary.

71. \( \begin{bmatrix} i & -1 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} \)

72. \( \begin{bmatrix} 2 + i & 1 + i \\ \frac{4}{i} & \frac{4}{\sqrt{3}} \end{bmatrix} \)

73. \( \begin{bmatrix} 1 & 0 \\ i & -i \end{bmatrix} \)

74. \( \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \)

In Exercises 75 and 76, determine whether the matrix is Hermitian.

75. \( \begin{bmatrix} 1 & -1 + i & 2 - i \\ 1 - i & 3 & i \\ 2 + i & -i & 4 \end{bmatrix} \)

76. \( \begin{bmatrix} 9 & 2 - i & 2 \\ 2 + i & 0 & -1 - i \\ 2 & -1 + i & 3 \end{bmatrix} \)

In Exercises 77 and 78, find the eigenvalues and corresponding eigenvectors of the matrix.

77. \( \begin{bmatrix} 4 & 2 - i \\ 2 + i & 0 \end{bmatrix} \)

78. \( \begin{bmatrix} 2 & 0 & -i \\ 0 & 3 & 0 \\ i & 0 & 2 \end{bmatrix} \)

79. Prove that if \( A \) is an invertible matrix, then \( A^* \) is also invertible.

80. Determine all complex numbers \( z \) such that \( z = -\bar{z} \).

81. Prove that if the product of two complex numbers is zero, then one of the numbers must be zero.

82. (a) Find the determinant of the Hermitian matrix

\[ \begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix} \]

(b) Prove that the determinant of any Hermitian matrix is real.

83. Let \( A \) and \( B \) be Hermitian matrices. Prove that \( AB = BA \) if and only if \( AB \) is Hermitian.

84. Let \( u \) be a unit vector in \( \mathbb{C}^n \). Define \( H = I - 2uu^* \). Prove that \( H \) is an \( n \times n \) Hermitian and unitary matrix.

85. Use mathematical induction to prove DeMoivre's Theorem.

86. Prove that if \( z \) is a zero of a polynomial equation with real coefficients, then the conjugate of \( z \) is also a zero.

87. Show that if \( z_1 + z_2 \) and \( z_1z_2 \) are both nonzero real numbers, then \( z_1 \) and \( z_2 \) are both real numbers.

88. Prove that if \( z \) and \( w \) are complex numbers, then

\[ |z + w| \leq |z| + |w| \]

89. Prove that for all vectors \( u \) and \( v \) in a complex inner product space,

\[ \langle u, v \rangle = \frac{1}{2} \left[ ||u + v||^2 - ||u - v||^2 + i||u + iv||^2 - i||u - iv||^2 \right] \]

**True or False?** In Exercises 90 and 91, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

90. A square complex matrix \( A \) is called normal if it commutes with its conjugate transpose so that \( AA^* = A^*A \).

91. A square complex matrix \( A \) is called Hermitian if \( A = A^* \).
1 Population Growth and Dynamical Systems – II

In the projects for Chapter 7, you were asked to model the populations of two species using a system of differential equations of the form

\[ y_1'(t) = ay_1(t) + by_2(t) \]
\[ y_2'(t) = cy_1(t) + dy_2(t). \]

The constants \( a, b, c, \) and \( d \) depend on the particular species being studied. In Chapter 7, you looked at an example of a predator-prey relationship, in which \( a = 0.5, \) \( b = 0.6, \) \( c = -0.4, \) and \( d = 3.0. \) Now consider a slightly different model.

\[ y_1'(t) = 0.6y_1(t) + 0.8y_2(t), \quad y_1(0) = 36 \]
\[ y_2'(t) = -0.8y_1(t) + 0.6y_2(t), \quad y_2(0) = 121 \]

1. Use the diagonalization technique to find the general solutions \( y_1(t) \) and \( y_2(t) \) at any time \( t > 0. \) Although the eigenvalues and eigenvectors of the matrix

\[ A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} \]

are complex, the same principles apply, and you can obtain complex exponential solutions.

2. Convert your complex solutions to real solutions by observing that if \( \lambda = a + bi \) is a (complex) eigenvalue of \( A \) with (complex) eigenvector \( \mathbf{v}, \) then the real and imaginary parts of \( e^{\lambda t}\mathbf{v} \) form a linearly independent pair of (real) solutions. You will need to use the formula \( e^{\lambda t} = \cos \theta + i \sin \theta. \)

3. Use the initial conditions to find the explicit form of the (real) solutions to the original equations.

4. If you have access to a computer software program or graphing utility, graph the solutions obtained in part 3 over the domain \( 0 \leq t \leq 3. \) At what moment are the two populations equal?

5. Interpret the solution in terms of the long-term population trend for the two species. Does one species ultimately disappear? Why or why not? Contrast this solution to that obtained for the model in Chapter 7.

6. If you have access to a computer software program or graphing utility that can numerically solve differential equations, use it to graph the solutions to the original system of equations. Does this numerical approximation appear to be accurate?