10.1 Gaussian Elimination with Partial Pivoting

In Chapter 1, two methods of solving a system of \( n \) linear equations in \( n \) variables were discussed. When either of these methods (Gaussian elimination and Gauss-Jordan elimination) is used with a digital computer, the computer introduces a problem that has not yet been discussed—**rounding error**.

Digital computers store real numbers in **floating point form**, 

\[ \pm M \times 10^k \]

where \( k \) is an integer and the mantissa \( M \) satisfies the inequality \( 0.1 \leq M < 1 \). For instance, the floating point forms of some real numbers are as follows.
The number of decimal places that can be stored in the mantissa depends on the computer. If $n$ places are stored, then it is said that the computer stores $n$ significant digits. Additional digits are either truncated or rounded off. When a number is truncated to $n$ significant digits, all digits after the first $n$ significant digits are simply omitted. For instance, truncated to two significant digits, the number 0.1251 becomes 0.12.

When a number is rounded to $n$ significant digits, the last retained digit is increased by 1 if the discarded portion is greater than half a digit, and the last retained digit is not changed if the discarded portion is less than half a digit. For instance, rounded to two significant digits, 0.1251 becomes 0.13 and 0.1249 becomes 0.12. For the special case in which the discarded portion is precisely half a digit, round so that the last retained digit is even. So, rounded to two significant digits, 0.125 becomes 0.12 and 0.135 becomes 0.14.

Whenever the computer truncates or rounds, a rounding error that can affect subsequent calculations is introduced. The result after rounding or truncating is called the stored value.

### Example 1
Finding the Stored Values of Numbers

Determine the stored value of each of the real numbers listed below in a computer that rounds to three significant digits.

(a) 54.7  (b) 0.1134  (c) $-8.2256$
(d) 0.08335  (e) 0.08345

<table>
<thead>
<tr>
<th>Number</th>
<th>Floating Point Form</th>
<th>Stored Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) 54.7</td>
<td>$0.547 \times 10^2$</td>
<td>$0.547 \times 10^2$</td>
</tr>
<tr>
<td>(b) 0.1134</td>
<td>$0.1134 \times 10^0$</td>
<td>$0.113 \times 10^0$</td>
</tr>
<tr>
<td>(c) $-8.2256$</td>
<td>$-0.82256 \times 10^1$</td>
<td>$-0.823 \times 10^1$</td>
</tr>
<tr>
<td>(d) 0.08335</td>
<td>$0.8335 \times 10^{-1}$</td>
<td>$0.834 \times 10^{-1}$</td>
</tr>
<tr>
<td>(e) 0.08345</td>
<td>$0.8345 \times 10^{-1}$</td>
<td>$0.834 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Note in parts (d) and (e) that when the discarded portion of a decimal is precisely half a digit, the number is rounded so that the stored value ends in an even digit.

**Remark:** Most computers store numbers in binary form (base 2) rather than decimal form (base 10). Because rounding occurs in both systems, however, this discussion will be restricted to the more familiar base 10.

Rounding error tends to propagate as the number of arithmetic operations increases. This phenomenon is illustrated in the next example.
EXAMPLE 2  Propagation of Rounding Error

Evaluate the determinant of the matrix

\[
A = \begin{bmatrix}
0.12 & 0.23 \\
0.12 & 0.12
\end{bmatrix}
\]

rounding each intermediate calculation to two significant digits. Then find the exact solution and compare the two results.

**SOLUTION**

Rounding each intermediate calculation to two significant digits produces

\[
|A| = (0.12)(0.12) - (0.12)(0.23) \\
= 0.0144 - 0.0276 \\
\approx 0.014 - 0.028 \quad \text{Round to two significant digits.}
\]

\[
= -0.014.
\]

The exact solution is

\[
|A| = 0.0144 - 0.0276 = -0.0132.
\]

So, to two significant digits, the correct solution is \(-0.013\). Note that the rounded solution is not correct to two significant digits, even though each arithmetic operation was performed with two significant digits of accuracy. This is what is meant when it is said that arithmetic operations tend to propagate rounding error.

In Example 2, rounding at the intermediate steps introduced a rounding error of

\[-0.0132 - (-0.014) = 0.0008. \quad \text{Rounding error}\]

Although this error may seem slight, it represents a **percentage error** of

\[
\frac{0.0008}{0.0132} = 0.061 = 6.1\%. \quad \text{Percentage error}
\]

In most practical applications, a percentage error of this magnitude would be intolerable. Keep in mind that this particular percentage error arose with only a few arithmetic steps. When the number of arithmetic steps increases, the likelihood of a large percentage error also increases.

**Gaussian Elimination with Partial Pivoting**

For large systems of linear equations, Gaussian elimination can involve hundreds of arithmetic computations, each of which can produce rounding error. The next straightforward example illustrates the potential magnitude of this problem.
**EXAMPLE 3**  
**Gaussian Elimination and Rounding Error**

Use Gaussian elimination to solve the following system.

\[
\begin{align*}
0.143x_1 + 0.357x_2 + 2.01x_3 &= -5.173 \\
-1.31x_1 + 0.911x_2 + 1.99x_3 &= -5.458 \\
11.2x_1 - 4.30x_2 - 0.605x_3 &= 4.415
\end{align*}
\]

After each intermediate calculation, round the result to three significant digits.

**SOLUTION**  
Applying Gaussian elimination to the augmented matrix for this system produces

\[
\begin{bmatrix}
0.143 & 0.357 & 2.01 & -5.17 \\
-1.31 & 0.911 & 1.99 & -5.46 \\
11.2 & -4.30 & -0.605 & 4.42 \\
0.00 & 4.19 & 20.5 & -52.9 \\
11.2 & -4.30 & -0.605 & 4.42 \\
\end{bmatrix}
\]

Adding \(0.31\) times the first row to the second row produces a new second row.

Dividing the second row by 4.19 produces a new second row.

Adding \(-11.2\) times the first row to the third row produces a new third row.

Dividing the second row by 4.19 produces a new second row.

Adding 32.3 times the second row to the third row produces a new third row.

Multiplying the third row by \(-1\) produces a new third row.

So \(x_3 = -2.00\), and using back-substitution, you can obtain \(x_2 = -2.82\) and \(x_1 = -0.900\). Try checking this “solution” in the original system of equations to see that it is not correct. (The correct solution is \(x_1 = 1, x_2 = 2, \text{ and } x_3 = -3\).)

What went wrong with the Gaussian elimination procedure used in Example 3? Clearly, rounding error propagated to such an extent that the final “solution” became hopelessly inaccurate.
Part of the problem is that the original augmented matrix contains entries that differ in orders of magnitude. For instance, the first column of the matrix
\[
\begin{bmatrix}
0.143 & 0.357 & 2.01 & -5.17 \\
-1.31 & 0.911 & 1.99 & -5.46 \\
11.2 & -4.30 & -0.605 & 4.42 \\
\end{bmatrix}
\]
has entries that increase roughly by powers of 10 as one moves down the column. In subsequent elementary row operations, the first row was multiplied by 1.31 and the second row was multiplied by 32.3. When floating point arithmetic is used, such large row multipliers tend to propagate rounding error. This type of error propagation can be lessened by appropriate row interchanges that produce smaller multipliers. One method of restricting the size of the multipliers is called **Gaussian elimination with partial pivoting**.

**Example 4** shows what happens when this partial pivoting technique is used on the system of linear equations from Example 3.

**EXAMPLE 4**

**Gaussian Elimination with Partial Pivoting**

1. Find the entry in the left column with the largest absolute value. This entry is called the **pivot**.
2. Perform a row interchange, if necessary, so that the pivot is in the first row.
3. Divide the first row by the pivot. (This step is unnecessary if the pivot is 1.)
4. Use elementary row operations to reduce the remaining entries in the first column to 0.

The completion of these four steps is called a **pass**. After performing the first pass, ignore the first row and first column and repeat the four steps on the remaining submatrix. Continue this process until the matrix is in row-echelon form.

Example 4 shows what happens when this partial pivoting technique is used on the system of linear equations from Example 3.

**SOLUTION**

As in Example 3, the augmented matrix for this system is
\[
\begin{bmatrix}
0.143 & 0.357 & 2.01 & -5.17 \\
-1.31 & 0.911 & 1.99 & -5.46 \\
11.2 & -4.30 & -0.605 & 4.42 \\
\end{bmatrix}
\]

In the left column, 11.2 is the pivot because it is the entry that has the largest absolute value. So, interchange the first and third rows and apply elementary row operations as follows.
This completes the first pass. For the second pass, consider the submatrix formed by deleting the first row and first column. In this matrix the pivot is 0.412, which means that the second and third rows should be interchanged. Then proceed with Gaussian elimination, as shown below.

<table>
<thead>
<tr>
<th>11.2</th>
<th>-4.30</th>
<th>-0.605</th>
<th>4.42</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.31</td>
<td>0.911</td>
<td>1.99</td>
<td>-5.46</td>
</tr>
<tr>
<td>0.143</td>
<td>0.357</td>
<td>2.01</td>
<td>-5.17</td>
</tr>
</tbody>
</table>

Adding 1.31 times the first row to the second row produces a new second row.

<table>
<thead>
<tr>
<th>1.00</th>
<th>-0.384</th>
<th>-0.0540</th>
<th>0.395</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.31</td>
<td>0.911</td>
<td>1.99</td>
<td>-5.46</td>
</tr>
<tr>
<td>0.143</td>
<td>0.357</td>
<td>2.01</td>
<td>-5.17</td>
</tr>
</tbody>
</table>

Dividing the first row by 11.2 produces a new first row.

Adding 0.143 times the first row to the third row produces a new third row.

<table>
<thead>
<tr>
<th>1.00</th>
<th>-0.384</th>
<th>-0.0540</th>
<th>0.395</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.408</td>
<td>1.92</td>
<td>-4.94</td>
</tr>
<tr>
<td>0.143</td>
<td>0.357</td>
<td>2.01</td>
<td>-5.17</td>
</tr>
</tbody>
</table>

Interchange the first and third rows.

<table>
<thead>
<tr>
<th>1.00</th>
<th>-0.384</th>
<th>-0.0540</th>
<th>0.395</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.412</td>
<td>2.02</td>
<td>-5.23</td>
</tr>
<tr>
<td>0.00</td>
<td>0.408</td>
<td>1.92</td>
<td>-4.94</td>
</tr>
</tbody>
</table>

Dividing the second row by 0.412 produces a new second row.

Adding -0.408 times the second row to the third row produces a new third row.

<table>
<thead>
<tr>
<th>1.00</th>
<th>-0.384</th>
<th>-0.0540</th>
<th>0.395</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.00</td>
<td>4.90</td>
<td>-12.7</td>
</tr>
<tr>
<td>0.00</td>
<td>0.408</td>
<td>1.92</td>
<td>-4.94</td>
</tr>
</tbody>
</table>

Adding -0.0800 times the second row to the third row produces a new third row.

<table>
<thead>
<tr>
<th>1.00</th>
<th>-0.384</th>
<th>-0.0540</th>
<th>0.395</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.00</td>
<td>4.90</td>
<td>-12.7</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>-0.0800</td>
<td>0.240</td>
</tr>
</tbody>
</table>

This completes the second pass, and you can complete the entire procedure by dividing the third row by -0.0800, as follows.

<table>
<thead>
<tr>
<th>1.00</th>
<th>-0.384</th>
<th>-0.0540</th>
<th>0.395</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>1.00</td>
<td>4.90</td>
<td>-12.7</td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>-3.00</td>
</tr>
</tbody>
</table>

So \( x_3 = -3.00 \), and back-substitution produces \( x_2 = 2.00 \) and \( x_1 = 1.00 \), which agrees with the exact solution of \( x_1 = 1, x_2 = 2, \) and \( x_3 = -3 \) when rounded to three significant digits.
Remark: Note that the row multipliers used in Example 4 are 1.31, −0.143, and −0.408, as contrasted with the multipliers of 1.31, 11.2, and 32.3 encountered in Example 3.

The term partial in partial pivoting refers to the fact that in each pivot search only entries in the left column of the matrix or submatrix are considered. This search can be extended to include every entry in the coefficient matrix or submatrix; the resulting technique is called Gaussian elimination with complete pivoting. Unfortunately, neither complete pivoting nor partial pivoting solves all problems of rounding error. Some systems of linear equations, called ill-conditioned systems, are extremely sensitive to numerical errors. For such systems, pivoting is not much help. A common type of system of linear equations that tends to be ill-conditioned is one for which the determinant of the coefficient matrix is nearly zero. The next example illustrates this problem.

**Example 5** An Ill-Conditioned System of Linear Equations

Use Gaussian elimination to solve the system of linear equations.

\[
\begin{align*}
x + y &= 0 \\
x + \frac{401}{400} y &= 20
\end{align*}
\]

Round each intermediate calculation to four significant digits.

**SOLUTION** Using Gaussian elimination with rational arithmetic, you can find the exact solution to be \(y = 8000\) and \(x = −8000\). But rounding \(401/400 = 1.0025\) to four significant digits introduces a large rounding error, as follows.

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1.002 & 20 \\
1 & 1 & 0 \\
0 & 0.002 & 20 \\
1 & 1 & 0 \\
0 & 1.00 & 10,000
\end{bmatrix}
\]

So, \(y = 10,000\) and back-substitution produces

\[
\begin{align*}
x &= −y \\
&= −10,000.
\end{align*}
\]

This “solution” represents a percentage error of 25% for both the \(x\)-value and the \(y\)-value. Note that this error was caused by a rounding error of only 0.0005 (when you rounded 1.0025 to 1.002).
In Exercises 1–8, express the real number in floating point form.

1. 4281  
2. 321.61  
3. –2.62  
4. –21.001  
5. –0.00121  
6. 0.00026  
7. \(\frac{1}{5}\)  
8. \(16\frac{1}{2}\)

In Exercises 9–16, determine the stored value of the real number in a computer that rounds to (a) three significant digits and (b) four significant digits.

9. 331  
10. 21.4  
11. –92.646  
12. 216.964  
13. \(\frac{7}{16}\)  
14. \(\frac{5}{72}\)  
15. \(\frac{1}{7}\)  
16. \(\frac{1}{6}\)

In Exercises 17 and 18, evaluate the determinant of the matrix, rounding each intermediate calculation to three significant digits. Then compare this solution with the exact solution.

17. \[
\begin{bmatrix}
1.24 & 56.00 \\
66.00 & 1.02
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
2.12 & 4.22 \\
1.07 & 2.12
\end{bmatrix}
\]

In Exercises 19 and 20, use Gaussian elimination to solve the system of linear equations. After each intermediate calculation, round the result to three significant digits. Then compare this solution with the exact solution.

19. \(1.21x + 16.7y = 28.8\)  
20. \(14.4x - 17.1y = 31.5\)  

\[4.66x + 64.4y = 111.0\]  
\[81.6x - 97.4y = 179.0\]

In Exercises 21–24, use Gaussian elimination without partial pivoting to solve the system of linear equations, rounding to three significant digits after each intermediate calculation. Then use partial pivoting to solve the same system, again rounding to three significant digits after each intermediate calculation. Finally, compare both solutions with the exact solution provided.

21. \(x + 1.04y = 2.04\)  
22. \(0.51x + 92.6y = 97.7\)  

\[6x + 6.20y = 12.20\]  
\[99.00x - 449.0y = 541.0\]  

(Exact: \(x = 1, y = 1\))  
(Exact: \(x = 10, y = 1\))

23. \(x + 4.01y + 0.00445z = 0.00\)  

\[-x - 4.00y + 0.00600z = 0.21\]  
\[2x - 4.05y + 0.05000z = -0.385\]

(Exact: \(x = -0.49, y = 0.1, z = 20\))

24. \(0.007x + 61.20y + 0.093z = 61.3\)  

\[4.810x - 5.92y + 1.110z = 0.0\]  
\[81.400x + 1.12y + 1.180z = 83.7\]

(Exact: \(x = 1, y = 1, z = 1\))

In Exercises 25 and 26, use Gaussian elimination to solve the ill-conditioned system of linear equations, rounding each intermediate calculation to three significant digits. Then compare this solution with the exact solution provided.

25. \(x + y = 2\)  
\(x + 0.001y = 20\)

(Exact: \(x = 10,820\), \(y = -10,818\))

26. \(x - \frac{800}{401}y = 10\)  
\(\frac{x + 600}{401}y = 20\)

(Exact: \(x = 48,010\), \(y = 48,060\))

27. Consider the ill-conditioned systems

\(x + y = 2\) and \(x + y = 2\)

\(x + 1.0001y = 2\)  
\(x + 1.0001y = 2.0001\)

Calculate the solution to each system. Notice that although the systems are almost the same, their solutions differ greatly.

28. Repeat Exercise 27 for the systems

\(x - y = 0\) and \(x - y = 0\)

\(-1.001x + y = 0.001\) \(-1.001x + y = 0\)

29. The Hilbert matrix of size \(n \times n\) is the \(n \times n\) symmetric matrix \(H_n = [a_{ij}]\), where \(a_{ij} = 1/(i + j - 1)\). As \(n\) increases, the Hilbert matrix becomes more and more ill-conditioned. Use Gaussian elimination to solve the system of linear equations shown below, rounding to two significant digits after each intermediate calculation. Compare this solution with the exact solution \((x_1 = 3, x_2 = -24,\) and \(x_3 = 30)\).

\(x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = 1\)

\(\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = 1\)

\(\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = 1\)

30. Repeat Exercise 29 for \(H_n x = b\), where \(b = (1, 1, 1)^T\), rounding to four significant digits. Compare this solution with the exact solution \((x_1 = -4, x_2 = 60, x_3 = -180,\) and \(x_4 = 140)\).

31. The inverse of the \(n \times n\) Hilbert matrix \(H_n\) has integer entries. Use a computer software program or graphing utility to calculate the inverses of the Hilbert matrices \(H_n\) for \(n = 4, 5, 6,\) and \(7\). For what values of \(n\) do the inverses appear to be accurate?
True or False? In Exercises 32–35, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

32. The real number 5436 in floating point form is $5.436 \times 10^3$.
33. The real number 0.00234 in floating point form is $0.234 \times 10^{-2}$.
34. One type of ill-conditioned system that is extremely sensitive to numerical errors is a system in which the determinant of the coefficient matrix is nearly zero.
35. When a computer truncates or rounds a number, a rounding error that affects subsequent calculations is introduced, and the result after rounding or truncating is called the stored value.
10.2 Iterative Methods for Solving Linear Systems

As a numerical technique, Gaussian elimination is rather unusual because it is direct. That is, a solution is obtained after a single application of Gaussian elimination. Once a "solution" has been obtained, Gaussian elimination offers no method of refinement. The lack of refinement can be a problem because, as the preceding section shows, Gaussian elimination is sensitive to rounding error.

Numerical techniques more commonly involve an iterative method. For example, in calculus you probably studied Newton's iterative method for approximating the zeros of a differentiable function. In this section you will look at two iterative methods for approximating the solution of a system of $n$ linear equations in $n$ variables.

**The Jacobi Method**

The first iterative technique is called the Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: (1) that the system

$$
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots & \quad \vdots \\
    a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
$$

has a unique solution and (2) that the coefficient matrix $A$ has no zeros on its main diagonal. If any of the diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has all nonzero entries on the main diagonal.

To begin the Jacobi method, solve the first equation for $x_1$, the second equation for $x_2$, and so on, as follows.

$$
\begin{align*}
    x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\
    x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\
    \vdots & \quad \vdots \\
    x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})
\end{align*}
$$

Then make an initial approximation of the solution,

$$(x_1, x_2, x_3, \ldots, x_n),$$

and substitute these values of $x_i$ on the right-hand sides of the rewritten equations to obtain the first approximation. After this procedure has been completed, one iteration has been performed. In the same way, the second approximation is formed by substituting the first
approximation’s \( x \)-values on the right-hand sides of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often converges to the actual solution. This procedure is illustrated in Example 1.

**EXAMPLE 1** Applying the Jacobi Method

Use the Jacobi method to approximate the solution of the following system of linear equations.

\[
\begin{align*}
5x_1 - 2x_2 + 3x_3 &= -1 \\
-3x_1 + 9x_2 + x_3 &= 2 \\
2x_1 - x_2 - 7x_3 &= 3
\end{align*}
\]

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

**SOLUTION** To begin, write the system in the form

\[
\begin{align*}
x_1 &= -\frac{1}{5} + \frac{3}{3}x_2 - \frac{3}{3}x_3 \\
x_2 &= \frac{2}{9} + \frac{3}{3}x_1 - \frac{1}{9}x_3 \\
x_3 &= -\frac{3}{7} + \frac{2}{3}x_1 - \frac{1}{7}x_2.
\end{align*}
\]

Because you do not know the actual solution, choose

\[
x_1 = 0, \quad x_2 = 0, \quad x_3 = 0 \quad \text{Initial approximation}
\]

as a convenient initial approximation. So, the first approximation is

\[
\begin{align*}
x_1 &= -\frac{1}{5} + \frac{3}{3}(0) - \frac{3}{3}(0) = -0.200 \\
x_2 &= \frac{2}{9} + \frac{3}{3}(0) - \frac{1}{9}(0) = 0.222 \\
x_3 &= -\frac{3}{7} + \frac{2}{3}(0) - \frac{1}{7}(0) = -0.429.
\end{align*}
\]

Continuing this procedure, you obtain the sequence of approximations shown in Table 10.1.

**TABLE 10.1**

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>\begin{align*} 0.000 \end{align*} &amp; \begin{align*} -0.200 \end{align*} &amp; \begin{align*} 0.146 \end{align*} &amp; \begin{align*} 0.191 \end{align*} &amp; \begin{align*} 0.181 \end{align*} &amp; \begin{align*} 0.186 \end{align*} &amp; \begin{align*} 0.186 \end{align*} &amp; \begin{align*} 0.186 \end{align*}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_2 )</td>
<td>\begin{align*} 0.000 \end{align*} &amp; \begin{align*} 0.222 \end{align*} &amp; \begin{align*} 0.203 \end{align*} &amp; \begin{align*} 0.328 \end{align*} &amp; \begin{align*} 0.332 \end{align*} &amp; \begin{align*} 0.329 \end{align*} &amp; \begin{align*} 0.331 \end{align*} &amp; \begin{align*} 0.331 \end{align*}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_3 )</td>
<td>\begin{align*} 0.000 \end{align*} &amp; \begin{align*} -0.429 \end{align*} &amp; \begin{align*} -0.517 \end{align*} &amp; \begin{align*} -0.416 \end{align*} &amp; \begin{align*} -0.421 \end{align*} &amp; \begin{align*} -0.424 \end{align*} &amp; \begin{align*} -0.423 \end{align*} &amp; \begin{align*} -0.423 \end{align*}</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Because the last two columns in Table 10.1 are identical, you can conclude that to three significant digits, the solution is

\[ x_1 = 0.186, \quad x_2 = 0.331, \quad x_3 = -0.423. \]

For the system of linear equations in Example 1, the Jacobi method is said to converge. That is, repeated iterations succeed in producing an approximation that is correct to three significant digits. As is generally true for iterative methods, greater accuracy would require more iterations.

**The Gauss-Seidel Method**

You will now look at a modification of the Jacobi method called the Gauss-Seidel method, named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). This modification is no more difficult to use than the Jacobi method, and it often requires fewer iterations to produce the same degree of accuracy.

With the Jacobi method, the values of obtained in the \( n \)th approximation remain unchanged until the entire \( (n + 1) \)th approximation has been calculated. On the other hand, with the Gauss-Seidel method, you use the new values of each \( x_i \) as soon as they are known. That is, once you have determined \( x_1 \) from the first equation, its value is then used in the second equation to obtain the new \( x_2 \). Similarly, the new \( x_1 \) and \( x_2 \) are used in the third equation to obtain the new \( x_3 \), and so on. This procedure is demonstrated in Example 2.

**EXAMPLE 2 Applying the Gauss-Seidel Method**

Use the Gauss-Seidel iteration method to approximate the solution to the system of equations in Example 1.

**SOLUTION**

The first computation is identical to that in Example 1. That is, using \( (x_1, x_2, x_3) = (0, 0, 0) \) as the initial approximation, you obtain the new value of \( x_1 \).

\[ x_1 = -\frac{1}{3} + \frac{2}{5}(0) - \frac{3}{7}(0) = -0.200 \]

Now that you have a new value of \( x_1 \), use it to compute a new value of \( x_2 \). That is,

\[ x_2 = \frac{2}{5} + \frac{3}{7}(-0.200) - \frac{1}{9}(0) \approx 0.156. \]

Similarly, use \( x_1 = -0.200 \) and \( x_2 = 0.156 \) to compute a new value of \( x_3 \). That is,

\[ x_3 = -\frac{2}{7} + \frac{3}{7}(-0.200) - \frac{1}{9}(0.156) \approx -0.508. \]

So, the first approximation is \( x_1 = -0.200, \ x_2 = 0.156, \) and \( x_3 = -0.508 \). Continued iterations produce the sequence of approximations shown in Table 10.2.
Chapter 10  Numerical Methods

Note that after only six iterations of the Gauss-Seidel method, you achieved the same accuracy as was obtained with seven iterations of the Jacobi method in Example 1.

Neither of the iterative methods presented in this section always converges. That is, it is possible to apply the Jacobi method or the Gauss-Seidel method to a system of linear equations and obtain a divergent sequence of approximations. In such cases, it is said that the method diverges.

**EXAMPLE 3  An Example of Divergence**

Apply the Jacobi method to the system
\[
\begin{align*}
    x_1 - 5x_2 &= -4 \\
    7x_1 - x_2 &= 6,
\end{align*}
\]
using the initial approximation \((x_1, x_2) = (0, 0)\), and show that the method diverges.

**SOLUTION**

As usual, begin by rewriting the system in the form
\[
\begin{align*}
    x_1 &= -4 + 5x_2 \\
    x_2 &= -6 + 7x_1,
\end{align*}
\]
Then the initial approximation \((0, 0)\) produces
\[
\begin{align*}
    x_1 &= -4 + 5(0) = -4 \\
    x_2 &= -6 + 7(0) = -6
\end{align*}
\]
as the first approximation. Repeated iterations produce the sequence of approximations shown in Table 10.3.

**TABLE 10.3**

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>0</td>
<td>-4</td>
<td>-34</td>
<td>-174</td>
<td>-1224</td>
<td>-6124</td>
<td>-42,874</td>
<td>-214,374</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0</td>
<td>-6</td>
<td>-34</td>
<td>-244</td>
<td>-1224</td>
<td>-8574</td>
<td>-42,874</td>
<td>-300,124</td>
</tr>
</tbody>
</table>
For this particular system of linear equations you can determine that the actual solution is \( x_1 = 1 \) and \( x_2 = 1 \). So you can see from Table 10.3 that the approximations provided by the Jacobi method become progressively worse instead of better, and you can conclude that the method diverges.

The problem of divergence in Example 3 is not resolved by using the Gauss-Seidel method rather than the Jacobi method. In fact, for this particular system the Gauss-Seidel method diverges even more rapidly, as shown in Table 10.4.

<table>
<thead>
<tr>
<th>TABLE 10.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
</tr>
<tr>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
</tr>
</tbody>
</table>

With an initial approximation of \((x_1, x_2) = (0, 0)\), neither the Jacobi method nor the Gauss-Seidel method converges to the solution of the system of linear equations from Example 3. You will now look at a special type of coefficient matrix \( A \), called a **strictly diagonally dominant matrix**, for which it is guaranteed that both methods will converge.

**Definition of Strictly Diagonally Dominant Matrix**

An \( n \times n \) matrix \( A \) is **strictly diagonally dominant** if the absolute value of each entry on the main diagonal is greater than the sum of the absolute values of the other entries in the same row. That is,

\[
|a_{11}| > |a_{12}| + |a_{13}| + \cdots + |a_{1n}|
\]

\[
|a_{22}| > |a_{21}| + |a_{23}| + \cdots + |a_{2n}|
\]

\[
\vdots
\]

\[
|a_{nn}| > |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}|
\]

**EXAMPLE 4**

**Strictly Diagonally Dominant Matrices**

Which of the systems of linear equations shown below has a strictly diagonally dominant coefficient matrix?

(a) \( 3x_1 - x_2 = -4 \)
\[
2x_1 + 5x_2 = 2
\]

(b) \( 4x_1 + 2x_2 - x_3 = -1 \)
\[
x_1 + 2x_3 = -4
\]
\[
3x_1 - 5x_2 + x_3 = 3
\]
SOLUTION

(a) The coefficient matrix

\[
A = \begin{bmatrix}
3 & -1 \\
2 & 5
\end{bmatrix}
\]

is strictly diagonally dominant because \(|3| > |-1|\) and \(|5| > |2|\).

(b) The coefficient matrix

\[
A = \begin{bmatrix}
4 & 2 & -1 \\
1 & 0 & 2 \\
3 & -5 & 1
\end{bmatrix}
\]

is not strictly diagonally dominant because the entries in the second and third rows do not conform to the definition. For instance, in the second row, \(a_{21} = 1, a_{22} = 0,\) and \(a_{23} = 2,\) and it is not true that \(|a_{22}| > |a_{21}| + |a_{23}|.\) Interchanging the second and third rows in the original system of linear equations, however, produces the coefficient matrix

\[
A' = \begin{bmatrix}
4 & 2 & -1 \\
3 & -5 & 1 \\
1 & 0 & 2
\end{bmatrix}
\]

which is strictly diagonally dominant.

The next theorem, which is listed without proof, states that strict diagonal dominance is sufficient for the convergence of either the Jacobi method or the Gauss-Seidel method.

**Theorem 10.1**

**Convergence of the Jacobi and Gauss-Seidel Methods**

If \(A\) is strictly diagonally dominant, then the system of linear equations given by \(Ax = b\) has a unique solution to which the Jacobi method and the Gauss-Seidel method will converge for any initial approximation.

In Example 3, you looked at a system of linear equations for which the Jacobi and Gauss-Seidel methods diverged. In the next example, you can see that by interchanging the rows of the system in Example 3, you can obtain a coefficient matrix that is strictly diagonally dominant. After this interchange, convergence is assured.

**Example 5**

**Interchanging Rows to Obtain Convergence**

Interchange the rows of the system

\[
\begin{align*}
x_1 - 5x_2 &= -4 \\
7x_1 - x_2 &= 6
\end{align*}
\]

to obtain one with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to four significant digits.
SOLUTION Begin by interchanging the two rows of the system to obtain

\[
\begin{align*}
7x_1 - x_2 &= 6 \\
x_1 - 5x_2 &= -4.
\end{align*}
\]

Note that the coefficient matrix of this system is strictly diagonally dominant. Then solve for \(x_1\) and \(x_2\), as shown below.

\[
\begin{align*}
x_1 &= \frac{6}{7} + \frac{1}{7}x_2 \\
x_2 &= \frac{4}{5} + \frac{1}{5}x_1
\end{align*}
\]

Using the initial approximation \((x_1, x_2) = (0, 0)\), you can obtain the sequence of approximations shown in Table 10.5.

TABLE 10.5

<table>
<thead>
<tr>
<th>(n)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>0.0000</td>
<td>0.8571</td>
<td>0.9959</td>
<td>0.9999</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>(x_2)</td>
<td>0.0000</td>
<td>0.9714</td>
<td>0.9992</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

So, you can conclude that the solution is \(x_1 = 1\) and \(x_2 = 1\).

Do not conclude from Theorem 10.1 that strict diagonal dominance is a necessary condition for convergence of the Jacobi or Gauss-Seidel methods. For instance, the coefficient matrix of the system

\[
\begin{align*}
-4x_1 + 5x_2 &= 1 \\
x_1 + 2x_2 &= 3
\end{align*}
\]

is not a strictly diagonally dominant matrix, and yet both methods converge to the solution \(x_1 = 1\) and \(x_2 = 1\) when you use an initial approximation of \((x_1, x_2) = (0, 0)\). (See Exercises 21 and 22.)
In Exercises 1–4, apply the Jacobi method to the system of linear equations, using the initial approximation \((x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\). Continue performing iterations until two successive approximations are identical when rounded to three significant digits.

1. \(3x_1 - x_2 = 2\)  \hspace{2cm} 2. \(-4x_1 + 2x_2 = -6\)
   \hspace{2cm} \(x_1 + 4x_2 = 5\)  \hspace{2cm} 3. \(3x_1 - 5x_2 = 1\)

3. \(2x_1 - x_2 = 2\)  \hspace{2cm} 4. \(4x_1 + x_2 + x_3 = 7\)
   \hspace{2cm} \(x_1 - 3x_2 + x_3 = -2\)  \hspace{2cm} \(x_1 - 7x_2 + 2x_3 = -2\)
   \hspace{2cm} \(-x_1 + x_2 - 3x_3 = -6\)  \hspace{2cm} \(3x_1 + 4x_3 = 11\)

5. Apply the Gauss-Seidel method to Exercise 1.
6. Apply the Gauss-Seidel method to Exercise 2.
7. Apply the Gauss-Seidel method to Exercise 3.
8. Apply the Gauss-Seidel method to Exercise 4.

In Exercises 9–12, show that the Gauss-Seidel method diverges for the system using the initial approximation \((x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\).

9. \(x_1 - 2x_2 = -1\)  \hspace{2cm} 10. \(-x_1 + 4x_2 = 1\)
   \hspace{2cm} \(2x_1 + x_2 = 3\)  \hspace{2cm} 3. \(3x_1 - 2x_2 = 2\)

11. \(2x_1 - 3x_2 = -7\)  \hspace{2cm} 12. \(x_1 + 3x_2 - x_3 = 5\)
    \hspace{2cm} \(x_1 + 3x_2 - 10x_3 = 9\)  \hspace{2cm} \(3x_1 - x_2 = 5\)
    \hspace{2cm} \(3x_1 + x_3 = 13\)  \hspace{2cm} \(x_2 + 2x_3 = 1\)

In Exercises 13–16, determine whether the matrix is strictly diagonally dominant.

13. \(\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\)  \hspace{2cm} 14. \(\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}\)
15. \(\begin{bmatrix} 12 & 6 & 0 \\ 2 & -3 & 2 \\ 0 & 6 & 13 \end{bmatrix}\)  \hspace{2cm} 16. \(\begin{bmatrix} 7 & 5 & -1 \\ 1 & -4 & 1 \\ 0 & 2 & -3 \end{bmatrix}\)

17. Interchange the rows of the system of linear equations in Exercise 9 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.
18. Interchange the rows of the system of linear equations in Exercise 10 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

19. Interchange the rows of the system of linear equations in Exercise 11 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

20. Interchange the rows of the system of linear equations in Exercise 12 to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

In Exercises 21 and 22, the coefficient matrix of the system of linear equations is not strictly diagonally dominant. Show that the Jacobi and Gauss-Seidel methods converge using an initial approximation of \((x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)\).

21. \(-4x_1 + 5x_2 = 1\)  \hspace{2cm} 22. \(4x_1 + 2x_2 - 2x_3 = 0\)
    \hspace{2cm} \(x_1 + 2x_2 = 3\)  \hspace{2cm} \(x_1 - 3x_2 - x_3 = 7\)
    \hspace{2cm} \(3x_1 - x_2 + 4x_3 = 5\)

In Exercises 23 and 24, write a computer program that applies the Gauss-Seidel method to solve the system of linear equations.

23. \(4x_1 + x_2 - x_3 = 3\)  \hspace{2cm} \(x_1 + 6x_2 - 2x_3 + x_4 - x_5 = -6\)
    \hspace{2cm} \(x_2 + 5x_4 - x_5 + 6x_6 - x_7 = -5\)  \hspace{2cm} \(2x_2 + 5x_4 - x_5 - x_7 - x_9 = 0\)
    \hspace{2cm} \(-x_3 - x_4 + 6x_5 - x_6 - x_8 = 12\)  \hspace{2cm} \(-x_3 - x_5 + 5x_6 = -12\)
    \hspace{2cm} \(-x_4 + 4x_7 - x_8 = -2\)  \hspace{2cm} \(-x_4 - x_5 + x_7 + 5x_8 = 2\)

24. \(4x_1 - x_2 - x_3 = 18\)  \hspace{2cm} \(-x_1 + 4x_2 - x_3 - x_4 = 18\)
    \hspace{2cm} \(-x_2 + 4x_3 - x_4 - x_5 = 4\)  \hspace{2cm} \(-x_3 + 4x_4 - x_5 - x_6 - x_7 = 4\)
    \hspace{2cm} \(-x_4 + 4x_5 - x_6 - x_7 = 26\)  \hspace{2cm} \(-x_5 + 4x_6 - x_7 - x_8 = 16\)
    \hspace{2cm} \(-x_6 + 4x_7 - x_8 = 10\)  \hspace{2cm} \(-x_7 + 4x_8 = 32\)
True or False? In Exercises 25–28, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

25. The Jacobi method of solving systems of equations makes two assumptions: the system has a unique solution, and the coefficient matrix has no zeros on its main diagonal.

26. The Jacobi method is said to converge if it produces a sequence of repeated iterations accurate to within a specific number of decimal places.

27. A matrix $A$ is strictly diagonally dominant if the absolute value of each entry on its main diagonal is greater than the sum of the other entries in the same column.

28. If a matrix $A$ is strictly diagonally dominant, then the system of linear equations represented by $Ax = b$ has no unique solution.
10.3 Power Method for Approximating Eigenvalues

In Chapter 7 you saw that the eigenvalues of an $n \times n$ matrix $A$ are obtained by solving its characteristic equation

$$\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \cdots + c_0 = 0.$$ 

For large values of $n$, polynomial equations such as this one are difficult and time consuming to solve. Moreover, numerical techniques for approximating roots of polynomial equations of high degree are sensitive to rounding errors. In this section you will look at an alternative method for approximating eigenvalues. As presented here, the method can be used only to find the eigenvalue of $A$ that is largest in absolute value—this eigenvalue is called the **dominant eigenvalue** of $A$. Although this restriction may seem severe, dominant eigenvalues are of primary interest in many physical applications.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix $A$. $\lambda_1$ is called the **dominant eigenvalue** of $A$ if

$$|\lambda_1| > |\lambda_i|, \quad i = 2, \ldots, n.$$ 

The eigenvectors corresponding to $\lambda_1$ are called **dominant eigenvectors** of $A$.

Not every matrix has a dominant eigenvalue. For instance, the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 1$ and $\lambda_2 = -1$) has no dominant eigenvalue. Similarly, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(with eigenvalues of $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 1$) has no dominant eigenvalue.

**EXAMPLE 1** Finding a Dominant Eigenvalue

Find the dominant eigenvalue and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$
SOLUTION

From Example 4 in Section 7.1 you know that the characteristic polynomial of $A$ is $\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$. So the eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = -2$, of which the dominant one is $\lambda_2 = -2$. From the same example you know that the dominant eigenvectors of $A$ (those corresponding to $\lambda_2 = -2$) are of the form

$$x = \begin{bmatrix} t \\ 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

The Power Method

Like the Jacobi and Gauss-Seidel methods, the power method for approximating eigenvalues is iterative. First assume that the matrix $A$ has a dominant eigenvalue with corresponding dominant eigenvectors. Then choose an initial approximation $x_0$ of one of the dominant eigenvectors of $A$. This initial approximation must be a nonzero vector in $\mathbb{R}^n$. Finally, form the sequence

$$x_1 = Ax_0$$
$$x_2 = Ax_1 = A(Ax_0) = A^2x_0$$
$$x_3 = Ax_2 = A(A^2x_0) = A^3x_0$$
$$\vdots$$
$$x_k = Ax_{k-1} = A(A^{k-1}x_0) = A^kx_0.$$

For large powers of $k$, and by properly scaling this sequence, you will see that you obtain a good approximation of the dominant eigenvector of $A$. This procedure is illustrated in Example 2.

EXAMPLE 2

Approximating a Dominant Eigenvector by the Power Method

Complete six iterations of the power method to approximate a dominant eigenvector of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

SOLUTION

Begin with an initial nonzero approximation of

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Then obtain the approximations shown below.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>“Scaled” Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{x}_1 = A \mathbf{x}_0 = \begin{bmatrix} 2 &amp; -12 \ 1 &amp; -5 \end{bmatrix} \begin{bmatrix} 1 \ 1 \end{bmatrix} = \begin{bmatrix} -10 \ -4 \end{bmatrix} )</td>
<td>( \rightarrow 4 \begin{bmatrix} 2.50 \ 1.00 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \mathbf{x}_2 = A \mathbf{x}_1 = \begin{bmatrix} 2 &amp; -12 \ 1 &amp; -5 \end{bmatrix} \begin{bmatrix} -10 \ -4 \end{bmatrix} = \begin{bmatrix} 28 \ 10 \end{bmatrix} )</td>
<td>( \rightarrow 10 \begin{bmatrix} 2.80 \ 1.00 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \mathbf{x}_3 = A \mathbf{x}_2 = \begin{bmatrix} 2 &amp; -12 \ 1 &amp; -5 \end{bmatrix} \begin{bmatrix} 28 \ 10 \end{bmatrix} = \begin{bmatrix} -64 \ -22 \end{bmatrix} )</td>
<td>( \rightarrow -22 \begin{bmatrix} 2.91 \ 1.00 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \mathbf{x}_4 = A \mathbf{x}_3 = \begin{bmatrix} 2 &amp; -12 \ 1 &amp; -5 \end{bmatrix} \begin{bmatrix} -64 \ -22 \end{bmatrix} = \begin{bmatrix} 136 \ 46 \end{bmatrix} )</td>
<td>( \rightarrow 46 \begin{bmatrix} 2.96 \ 1.00 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \mathbf{x}_5 = A \mathbf{x}_4 = \begin{bmatrix} 2 &amp; -12 \ 1 &amp; -5 \end{bmatrix} \begin{bmatrix} 136 \ 46 \end{bmatrix} = \begin{bmatrix} -280 \ -94 \end{bmatrix} )</td>
<td>( \rightarrow -94 \begin{bmatrix} 2.98 \ 1.00 \end{bmatrix} )</td>
</tr>
<tr>
<td>( \mathbf{x}_6 = A \mathbf{x}_5 = \begin{bmatrix} 2 &amp; -12 \ 1 &amp; -5 \end{bmatrix} \begin{bmatrix} -280 \ -94 \end{bmatrix} = \begin{bmatrix} 568 \ 190 \end{bmatrix} )</td>
<td>( \rightarrow 190 \begin{bmatrix} 2.99 \ 1.00 \end{bmatrix} )</td>
</tr>
</tbody>
</table>

Note that the approximations in Example 2 appear to be approaching scalar multiples of \( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \), which you know from Example 1 is a dominant eigenvector of the matrix

\[
A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.
\]

In Example 2, the power method was used to approximate a dominant eigenvector of the matrix \( A \). In that example you already knew that the dominant eigenvalue of \( A \) was \( \lambda = -2 \). For the sake of demonstration, however, assume that you do not know the dominant eigenvalue of \( A \). The next theorem provides a formula for determining the eigenvalue corresponding to a given eigenvector. This theorem is credited to the English physicist John William Rayleigh (1842–1919).

**THEOREM 10.2**

*Determining an Eigenvalue from an Eigenvector*

If \( \mathbf{x} \) is an eigenvector of a matrix \( A \), then its corresponding eigenvalue is given by

\[
\lambda = \frac{A \mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}.
\]

This quotient is called the **Rayleigh quotient**.
**Chapter 10  Numerical Methods**

**PROOF** Because \( x \) is an eigenvector of \( A \), you know that \( Ax = \lambda x \) and you can write

\[
\frac{Ax \cdot x}{x \cdot x} = \frac{\lambda x \cdot x}{x \cdot x} = \frac{\lambda(x \cdot x)}{x \cdot x} = \lambda.
\]

In cases for which the power method generates a good approximation of a dominant eigenvector, the Rayleigh quotient provides a correspondingly good approximation of the dominant eigenvalue. The use of the Rayleigh quotient is demonstrated in Example 3.

**EXAMPLE 3  Approximating a Dominant Eigenvalue**

Use the result of Example 2 to approximate the dominant eigenvalue of the matrix

\[
A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.
\]

**SOLUTION** After the sixth iteration of the power method in Example 2, you obtained

\[
x_6 = \begin{bmatrix} 568 \\ 190 \end{bmatrix} \approx 190 \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}.
\]

With \( x = (2.99, 1) \) as the approximation of a dominant eigenvector of \( A \), use the Rayleigh quotient to obtain an approximation of the dominant eigenvalue of \( A \). First compute the product \( Ax \).

\[
Ax = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = \begin{bmatrix} -6.02 \\ -2.01 \end{bmatrix}
\]

Then, because

\[
Ax \cdot x = (-6.02)(2.99) + (-2.01)(1) \approx -20.0
\]

and

\[
x \cdot x = (2.99)(2.99) + (1)(1) \approx 9.94,
\]

you can compute the Rayleigh quotient to be

\[
\lambda = \frac{Ax \cdot x}{x \cdot x} \approx \frac{-20.0}{9.94} \approx -2.01,
\]

which is a good approximation of the dominant eigenvalue \( \lambda = -2 \).

From Example 2 you can see that the power method tends to produce approximations with large entries. In practice it is best to “scale down” each approximation before proceeding to the next iteration. One way to accomplish this **scaling** is to determine the component of \( Ax \) that has the largest absolute value and multiply the vector \( Ax \) by the reciprocal of this component. The resulting vector will then have components whose absolute values are less than or equal to 1. (Other scaling techniques are possible. For examples, see Exercises 27 and 28.)
EXAMPLE 4  The Power Method with Scaling

Calculate six iterations of the power method with scaling to approximate a dominant eigenvector of the matrix

\[
A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}.
\]

Use \( x_0 = (1, 1, 1) \) as the initial approximation.

**SOLUTION**

One iteration of the power method produces

\[
Ax_0 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix},
\]

and by scaling you obtain the approximation

\[
x_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}.
\]

A second iteration yields

\[
Ax_1 = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 2.20 \end{bmatrix}
\]

and

\[
x_2 = \frac{1}{2.20} \begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.45 \end{bmatrix}.
\]

Continuing this process, you obtain the sequence of approximations shown in Table 10.6.

**TABLE 10.6**

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1.00 )</td>
<td>( 0.60 )</td>
<td>( 0.45 )</td>
<td>( 0.48 )</td>
<td>( 0.50 )</td>
<td>( 0.50 )</td>
<td>( 0.50 )</td>
</tr>
<tr>
<td>( 1.00 )</td>
<td>( 0.20 )</td>
<td>( 0.45 )</td>
<td>( 0.55 )</td>
<td>( 0.51 )</td>
<td>( 0.50 )</td>
<td>( 0.50 )</td>
</tr>
<tr>
<td>( 1.00 )</td>
<td>( 1.00 )</td>
<td>( 1.00 )</td>
<td>( 1.00 )</td>
<td>( 1.00 )</td>
<td>( 1.00 )</td>
<td>( 1.00 )</td>
</tr>
</tbody>
</table>
From Table 10.6 you can approximate a dominant eigenvector of \( A \) to be
\[
\mathbf{x} = \begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}.
\]
Using the Rayleigh quotient, you can approximate the dominant eigenvalue of \( A \) to be \( \lambda = 3 \). (For this example, you can check that the approximations of \( \mathbf{x} \) and \( \lambda \) are exact.)

**Remark:** Note that the scaling factors used to obtain the vectors in Table 10.6,
\[
\begin{array}{cccccc}
5.00 & 2.20 & 2.80 & 3.13 & 3.03 & 3.00
\end{array}
\]
are approaching the dominant eigenvalue \( \lambda = 3 \).

In Example 4, the power method with scaling converges to a dominant eigenvector. The next theorem states that a sufficient condition for convergence of the power method is that the matrix \( A \) be diagonalizable (and have a dominant eigenvalue).

**Theorem 10.3**

**Convergence of the Power Method**

If \( A \) is an \( n \times n \) diagonalizable matrix with a dominant eigenvalue, then there exists a nonzero vector \( \mathbf{x}_0 \) such that the sequence of vectors given by
\[
A\mathbf{x}_0, \quad A^2\mathbf{x}_0, \quad A^3\mathbf{x}_0, \quad A^4\mathbf{x}_0, \quad \ldots, \quad A^n\mathbf{x}_0, \quad \ldots
\]
approaches a multiple of the dominant eigenvector of \( A \).

**Proof**

Because \( A \) is diagonalizable, you know from Theorem 7.5 that it has \( n \) linearly independent eigenvectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) with corresponding eigenvalues of \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Assume that these eigenvalues are ordered so that \( \lambda_1 \) is the dominant eigenvalue (with a corresponding eigenvector of \( \mathbf{x}_1 \)). Because the \( n \) eigenvectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) are linearly independent, they must form a basis for \( \mathbb{R}^n \). For the initial approximation \( \mathbf{x}_0 \), choose a nonzero vector such that the linear combination
\[
\mathbf{x}_0 = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n
\]
has nonzero leading coefficients. (If \( c_1 = 0 \), the power method may not converge, and a different \( \mathbf{x}_0 \) must be used as the initial approximation. See Exercises 21 and 22.) Now, multiplying both sides of this equation by \( A \) produces
\[
A\mathbf{x}_0 = A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n)
\]
\[
= c_1(A\mathbf{x}_1) + c_2(A\mathbf{x}_2) + \cdots + c_n(A\mathbf{x}_n)
\]
\[
= c_1(\lambda_1\mathbf{x}_1) + c_2(\lambda_2\mathbf{x}_2) + \cdots + c_n(\lambda_n\mathbf{x}_n).
\]
Repeated multiplication of both sides of this equation by $A$ produces

$$A^kx_0 = c_1(\lambda_1^kx_1) + c_2(\lambda_2^kx_2) + \cdots + c_n(\lambda_n^kx_n),$$

which implies that

$$A^kx_0 = \lambda_1^k\left[c_1x_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^kx_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^kx_n\right].$$

Now, from the original assumption that $\lambda_1$ is larger in absolute value than the other eigenvalues, it follows that each of the fractions

$$\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}$$

is less than 1 in absolute value. So each of the factors

$$\left(\frac{\lambda_2}{\lambda_1}\right)^k, \left(\frac{\lambda_3}{\lambda_1}\right)^k, \ldots, \left(\frac{\lambda_n}{\lambda_1}\right)^k$$

must approach 0 as $k$ approaches infinity. This implies that the approximation

$$A^kx_0 = \lambda_1^k c_1 x_1, \quad c_1 \neq 0$$

improves as $k$ increases. Because $x_1$ is a dominant eigenvector, it follows that any scalar multiple of $x_1$ is also a dominant eigenvector, which shows that $A^kx_0$ approaches a multiple of the dominant eigenvector of $A$.

The proof of Theorem 10.3 provides some insight into the rate of convergence of the power method. That is, if the eigenvalues of $A$ are ordered so that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|,$$

then the power method will converge quickly if $|\lambda_2|/|\lambda_1|$ is small, and slowly if $|\lambda_2|/|\lambda_1|$ is close to 1. This principle is illustrated in Example 5.

**EXAMPLE 5** The Rate of Convergence of the Power Method

(a) The matrix

$$A = \begin{bmatrix} 4 & 5 \\ 6 & 5 \end{bmatrix}$$

has eigenvalues of $\lambda_1 = 10$ and $\lambda_2 = -1$. So the ratio $|\lambda_2|/|\lambda_1|$ is 0.1. For this matrix, only four iterations are required to obtain successive approximations that agree when rounded to three significant digits. (See Table 10.7.)
(b) The matrix

\[ A = \begin{bmatrix} -4 & 10 \\ 7 & 5 \end{bmatrix} \]

has eigenvalues of \( \lambda_1 = 10 \) and \( \lambda_2 = -9 \). For this matrix, the ratio \( |\lambda_2|/|\lambda_1| \) is 0.9, and the power method does not produce successive approximations that agree to three significant digits until 68 iterations have been performed, as shown in Table 10.8.

**TABLE 10.8**

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>0.500</td>
<td>0.941</td>
<td>0.715</td>
<td>0.714</td>
<td>0.714</td>
</tr>
<tr>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

In this section you have seen the use of the power method to approximate the dominant eigenvalue of a matrix. This method can be modified to approximate other eigenvalues through use of a procedure called deflation. Moreover, the power method is only one of several techniques that can be used to approximate the eigenvalues of a matrix. Another popular method is called the \textit{QR} algorithm.

This is the method used in most computer programs and calculators for finding eigenvalues and eigenvectors. The \textit{QR} algorithm uses the \textit{QR}-factorization of the matrix, as presented in Chapter 5. Discussions of the deflation method and the \textit{QR} algorithm can be found in most texts on numerical methods.
In Exercises 1–6, use the techniques presented in Chapter 7 to find the eigenvalues of the matrix $A$. If $A$ has a dominant eigenvalue, find a corresponding dominant eigenvector.

1. $A = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}$
2. $A = \begin{bmatrix} -3 & 0 \\ 1 & 3 \end{bmatrix}$
3. $A = \begin{bmatrix} 1 & -5 \\ -3 & -1 \end{bmatrix}$
4. $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$
5. $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$
6. $A = \begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}$

In Exercises 7–10, use the Rayleigh quotient to compute the eigenvalue $\lambda$ of $A$ corresponding to the eigenvector $x$.

7. $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$, $x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$
8. $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$
9. $A = \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -6 & 6 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$
10. $A = \begin{bmatrix} 3 & 2 \\ -3 & -4 \\ -1 & 2 \end{bmatrix}$, $x = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

In Exercises 11–14, use the power method with scaling to approximate a dominant eigenvector of the matrix $A$. Start with $x_0 = (1, 1, 1)$ and calculate five iterations. Then use $x_4$ to approximate the dominant eigenvalue of $A$.

11. $A = \begin{bmatrix} 2 & 1 \\ 0 & -7 \end{bmatrix}$
12. $A = \begin{bmatrix} -1 & 0 \\ 1 & 6 \end{bmatrix}$
13. $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$
14. $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

In Exercises 15–18, use the power method with scaling to approximate a dominant eigenvector of the matrix $A$. Start with $x_0 = (1, 1, 1)$ and calculate four iterations. Then use $x_4$ to approximate the dominant eigenvalue of $A$.

15. $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 8 \end{bmatrix}$
16. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
17. $A = \begin{bmatrix} -1 & -6 & 0 \\ 2 & 7 & 0 \\ 1 & 2 & -1 \end{bmatrix}$
18. $A = \begin{bmatrix} 0 & 6 & 0 \\ 0 & -4 & 0 \\ 2 & 1 & 1 \end{bmatrix}$

In Exercises 19 and 20, the matrix $A$ does not have a dominant eigenvalue. Apply the power method with scaling, starting with $x_0 = (1, 1, 1)$, and observe the results of the first four iterations.

19. $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$
20. $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

21. Writing
(a) Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$.
(b) Calculate two iterations of the power method with scaling, starting with $x_0 = (1, 1)$.
(c) Explain why the method does not seem to converge to a dominant eigenvector.

22. Writing Repeat Exercise 21 using $x_0 = (1, 1, 1)$, for the matrix $A = \begin{bmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$.

23. The matrix $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$ has a dominant eigenvalue of $\lambda = -2$. Observe that $Ax = \lambda x$ implies that $A^{-1}x = \frac{1}{\lambda}x$.

Apply five iterations of the power method (with scaling) on $A^{-1}$ to compute the eigenvalue of $A$ with the smallest magnitude.

24. Repeat Exercise 23 for the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

25. (a) Compute the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.
(b) Apply four iterations of the power method with scaling to each matrix in part (a), starting with $x_0 = (-1, 2)$.
(c) Compute the ratios $\lambda_2/\lambda_1$ for $A$ and $B$. For which matrix do you expect faster convergence?
26. Use the proof of Theorem 10.3 to show that
\[ A(A^k x_0) = \lambda_1(A^k x_0) \]
for large values of \( k \). That is, show that the scale factors obtained
by the power method approach the dominant eigenvalue.

In Exercises 27 and 28, apply four iterations of the power method
(with scaling) to approximate the dominant eigenvalue of the
matrix. After each iteration, scale the approximation by dividing by
the length so that the resulting approximation will be a unit vector.

27. \( A = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix} \) 28. \( A = \begin{bmatrix} 7 & -4 & 2 \\ 16 & -9 & 6 \\ 8 & -4 & 5 \end{bmatrix} \)

**True or False?** In Exercises 29–32, determine whether each
statement is true or false. If a statement is true, give a reason or cite
an appropriate statement from the text. If a statement is false,
provide an example that shows the statement is not true in all cases
or cite an appropriate statement from the text.

29. The eigenvalue with the greatest absolute value is called the
dominant eigenvalue of the matrix \( A \), and the eigenvectors
 corresponding to that eigenvalue are the dominant eigenvectors.

30. If \( x \) is an eigenvector of a matrix \( A \), then its corresponding
eigenvalue is represented by \( \lambda = \frac{A x \cdot x}{x \cdot x} \), which is called the
Rayleigh quotient.

31. One method of scaling down each approximation before
proceeding to the next iteration is to determine the component
of \( A x \) that has the largest absolute value and multiply the vector
\( A x \) by the reciprocal of this component.

32. The method of deflation is used to approximate eigenvalues
other than the dominant eigenvalue of a matrix found by the
power method.
10.4 Applications of Numerical Methods

Applications of Gaussian Elimination with Pivoting

In Section 2.5 you used least squares regression analysis to find linear mathematical models that best fit a set of \( n \) points in the plane. This procedure can be extended to cover polynomial models of any degree, as follows.

The least squares regression polynomial of degree \( m \) for the points \( \{ (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \} \) is given by

\[
y = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0,
\]

where the coefficients are determined by the following system of \( m + 1 \) linear equations.

\[
na_0 + (\Sigma x_i)a_1 + (\Sigma x_i^2)a_2 + \cdots + (\Sigma x_i^m)a_m = \Sigma y_i
\]
\[
(\Sigma x_i)a_0 + (\Sigma x_i^2)a_1 + (\Sigma x_i^3)a_2 + \cdots + (\Sigma x_i^{m+1})a_m = \Sigma x_i y_i
\]
\[
(\Sigma x_i^2)a_0 + (\Sigma x_i^3)a_1 + (\Sigma x_i^4)a_2 + \cdots + (\Sigma x_i^{m+2})a_m = \Sigma x_i^2 y_i
\]
\[
\vdots
\]
\[
(\Sigma x_i^m)a_0 + (\Sigma x_i^{m+1})a_1 + (\Sigma x_i^{m+2})a_2 + \cdots + (\Sigma x_i^{2m})a_m = \Sigma x_i^m y_i
\]

Note that if \( m = 1 \), this system of equations reduces to

\[
n a_0 + (\Sigma x_i)a_1 = \Sigma y_i
\]
\[
(\Sigma x_i)a_0 + (\Sigma x_i^2)a_1 = \Sigma x_i y_i
\]

which has a solution of

\[
a_1 = \frac{n(\Sigma x_i y_i) - (\Sigma x_i)(\Sigma y_i)}{n(\Sigma x_i^2) - (\Sigma x_i)^2} \quad \text{and} \quad a_0 = \frac{\Sigma y_i}{n} - a_1 \frac{\Sigma x_i}{n}.
\]

Exercise 16 asks you to show that this formula is equivalent to the matrix formula for linear regression that was presented in Section 2.5.

Example 1 illustrates the use of regression analysis to find a second-degree polynomial model.

**Example 1** Least Squares Regression Analysis

The world populations in billions for selected years from 1970 to 2005 are shown in Table 10.9. (Source: U.S. Census Bureau)
TABLE 10.9

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>3.71</td>
<td>4.08</td>
<td>4.45</td>
<td>4.84</td>
<td>5.27</td>
<td>5.68</td>
<td>6.07</td>
<td>6.45</td>
</tr>
</tbody>
</table>

Find the second-degree least squares regression polynomial for these data and use the resulting model to predict the world populations in 2010 and 2015.

**SOLUTION**

Begin by letting $x = 0$ represent 1970, $x = 1$ represent 1975, and so on. So, the collection of points is represented by $\{(0, 3.71), (1, 4.08), (2, 4.45), (3, 4.84), (4, 5.27), (5, 5.68), (6, 6.07), (7, 6.45)\}$, which yields

\[
\begin{align*}
    n &= 8, \\
    \sum_{i=1}^{8} x_i &= 28, \\
    \sum_{i=1}^{8} x_i^2 &= 140, \\
    \sum_{i=1}^{8} x_i^3 &= 784, \\
    \sum_{i=1}^{8} x_i^4 &= 4676, \\
    \sum_{i=1}^{8} y_i &= 40.55, \\
    \sum_{i=1}^{8} x_i y_i &= 158.55, \\
    \sum_{i=1}^{8} x_i^2 y_i &= 826.33.
\end{align*}
\]

So, the system of linear equations giving the coefficients of the quadratic model $y = ax^2 + bx + c$ is

\[
\begin{align*}
    8a_0 + 28a_1 + 140a_2 &= 40.55 \\
    28a_0 + 140a_1 + 784a_2 &= 158.55 \\
    140a_0 + 784a_1 + 4676a_2 &= 826.33.
\end{align*}
\]

Gaussian elimination with pivoting on the matrix

\[
\begin{bmatrix}
    8 & 28 & 140 & 40.55 \\
    28 & 140 & 784 & 158.55 \\
    140 & 784 & 4676 & 826.33
\end{bmatrix}
\]

produces

\[
\begin{bmatrix}
    1 & 5.6 & 33.4 & 5.9024 \\
    0 & 1 & 7.5714 & 0.3970 \\
    0 & 0 & 1 & 0.0020
\end{bmatrix}
\]

So, by back-substitution you find the solution to be

\[
a_2 = 0.0020, \quad a_1 = 0.3819, \quad a_0 = 3.6970,
\]

and the regression quadratic is

\[
y = 0.002x^2 + 0.3819x + 3.6970.
\]
Figure 10.1 compares this model with the given points. To predict the world population in 2010, let $x = 8$, and obtain

$$y = 0.002(8^2) + 0.3819(8) + 3.6970 = 6.88 \text{ billion}.$$  

Similarly, the prediction for 2015 ($x = 9$) is

$$y = 0.002(9^2) + 0.3819(9) + 3.6970 = 7.30 \text{ billion}.$$  

---

**EXAMPLE 2**  

**Least Squares Regression Analysis**

Find the third-degree least squares regression polynomial

$$y = a_3x^3 + a_2x^2 + a_1x + a_0$$

for the points

$$\{(0, 0), (1, 2), (2, 3), (3, 2), (4, 1), (5, 2), (6, 4)\}.$$  

**SOLUTION**

For this set of points the linear system

$$na_0 + (\Sigma x_i)a_1 + (\Sigma x_i^2)a_2 + (\Sigma x_i^3)a_3 = \Sigma y_i$$  

$$(\Sigma x_i)a_0 + (\Sigma x_i^2)a_1 + (\Sigma x_i^3)a_2 + (\Sigma x_i^4)a_3 = \Sigma x_iy_i$$  

$$(\Sigma x_i^2)a_0 + (\Sigma x_i^3)a_1 + (\Sigma x_i^4)a_2 + (\Sigma x_i^5)a_3 = \Sigma x_i^2y_i$$  

$$(\Sigma x_i^3)a_0 + (\Sigma x_i^4)a_1 + (\Sigma x_i^5)a_2 + (\Sigma x_i^6)a_3 = \Sigma x_i^3y_i$$

becomes

$$7a_0 + 21a_1 + 91a_2 + 441a_3 = 14$$  

$$21a_0 + 91a_1 + 441a_2 + 2275a_3 = 52$$  

$$91a_0 + 441a_1 + 2275a_2 + 12,201a_3 = 242$$  

$$441a_0 + 2275a_1 + 12,201a_2 + 67,171a_3 = 1258.$$  

Using Gaussian elimination with pivoting on the matrix

$$\begin{bmatrix}
7 & 21 & 91 & 441 & 14 \\
21 & 91 & 441 & 2275 & 52 \\
91 & 441 & 2275 & 12,201 & 242 \\
441 & 2275 & 12,201 & 67,171 & 1258
\end{bmatrix}$$

produces
Chapter 10 Numerical Methods

which implies

So the cubic model is

\[ y = 0.1667x^3 - 1.5003x^2 + 3.6912x - 0.0718. \]

Figure 10.2 compares this model with the given points.

Applications of the Gauss-Seidel Method

**EXAMPLE 3** An Application to Probability

Figure 10.3 is a diagram of a maze used in a laboratory experiment. The experiment begins by placing a mouse at one of the ten interior intersections of the maze. Once the mouse emerges in the outer corridor, it cannot return to the maze. When the mouse is at an interior intersection, its choice of paths is assumed to be random. What is the probability that the mouse will emerge in the “food corridor” when it begins at the \( i \)th intersection?

Let the probability of winning (getting food) when starting at the \( i \)th intersection be represented by \( p_i \). Then form a linear equation involving \( p_i \) and the probabilities associated with the intersections bordering the \( i \)th intersection. For instance, at the first intersection the mouse has a probability of \( \frac{1}{4} \) of choosing the upper right path and losing, a probability of \( \frac{1}{4} \) of choosing the upper left path and losing, a probability of \( \frac{1}{4} \) of choosing the lower left path (at which point it has a probability of \( p_3 \) of winning), and a probability of \( \frac{1}{4} \) of choosing the lower right path (at which point it has a probability of \( p_2 \) of winning). So

\[ p_1 = \frac{1}{4} p(0) + \frac{1}{4} p(0) + \frac{1}{4} p_3 + \frac{1}{4} p_2. \]

The \( \begin{bmatrix} 1.0000 & 5.1587 & 27.6667 & 152.3152 & 2.8526 \\ 0.0000 & 1.0000 & 8.5313 & 58.3482 & 0.6183 \\ 0.0000 & 0.0000 & 1.0000 & 9.7714 & 0.1286 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.1667 \end{bmatrix} \]

which implies

\[ a_3 = 0.1667, \quad a_2 = -1.5003, \quad a_1 = 3.6912, \quad a_0 = -0.0718. \]
Using similar reasoning, the other nine probabilities can be represented by the following equations.

\[
\begin{align*}
p_2 &= \frac{1}{3}(0) + \frac{1}{3}p_1 + \frac{1}{3}p_3 + \frac{1}{3}p_4 + \frac{1}{3}p_5 \\
p_3 &= \frac{1}{5}(0) + \frac{1}{5}p_1 + \frac{1}{5}p_2 + \frac{1}{5}p_6 \\
p_4 &= \frac{1}{5}(0) + \frac{1}{5}p_2 + \frac{1}{5}p_5 + \frac{1}{5}p_7 + \frac{1}{5}p_8 \\
p_5 &= \frac{1}{6}p_2 + \frac{1}{6}p_3 + \frac{1}{6}p_4 + \frac{1}{6}p_6 + \frac{1}{6}p_8 + \frac{1}{6}p_9 \\
p_6 &= \frac{1}{5}(0) + \frac{1}{5}p_3 + \frac{1}{5}p_5 + \frac{1}{5}p_6 + \frac{1}{5}p_{10} \\
p_7 &= \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}p_4 + \frac{1}{4}p_8 \\
p_8 &= \frac{1}{5}(1) + \frac{1}{5}p_4 + \frac{1}{5}p_5 + \frac{1}{5}p_7 + \frac{1}{5}p_9 \\
p_9 &= \frac{1}{5}(1) + \frac{1}{5}p_5 + \frac{1}{5}p_6 + \frac{1}{5}p_8 + \frac{1}{5}p_{10} \\
p_{10} &= \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}p_6 + \frac{1}{4}p_9
\end{align*}
\]

Rewriting these equations in standard form produces the following system of ten linear equations in ten variables.

\[
\begin{align*}
4p_1 - p_2 - p_3 &= 0 \\
-p_1 + 5p_2 - p_3 - p_4 - p_5 &= 0 \\
-p_1 - p_2 + 5p_3 - p_5 - p_6 &= 0 \\
-p_2 + 5p_4 - p_5 - p_7 - p_8 &= 0 \\
-p_2 - p_3 - p_4 + 6p_5 - p_6 - p_8 - p_9 &= 0 \\
-p_3 - p_5 + 5p_6 - p_9 - p_{10} &= 0 \\
-p_4 - 4p_7 - p_8 &= 1 \\
-p_4 - p_5 - p_7 + 5p_8 - p_9 &= 1 \\
-p_5 - p_6 - p_8 + 5p_9 - p_{10} &= 1 \\
-p_6 - p_9 + 4p_{10} &= 1
\end{align*}
\]

The augmented matrix for this system is

\[
\begin{bmatrix}
4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 5 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 5 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 5 & -1 & 0 & -1 & -1 & 0 & 0 \\
0 & -1 & -1 & 6 & -1 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 5 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 5 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 5 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4
\end{bmatrix}
\]
Using the Gauss-Seidel method with an initial approximation of \( p_1 = p_2 = \ldots = p_{10} = 0 \) produces (after 18 iterations) an approximation of:

\[
\begin{align*}
 p_1 &= 0.090, & p_2 &= 0.180 \\
p_3 &= 0.180, & p_4 &= 0.298 \\
p_5 &= 0.333, & p_6 &= 0.298 \\
p_7 &= 0.455, & p_8 &= 0.522 \\
p_9 &= 0.522, & p_{10} &= 0.455.
\end{align*}
\]

The structure of the probability problem described in Example 3 is related to a technique called **finite element analysis**, which is used in many engineering problems.

Note that the matrix developed in Example 3 has mostly zero entries. Such matrices are called **sparse**. For solving systems of equations with sparse coefficient matrices, the Jacobi and Gauss-Seidel methods are much more efficient than Gaussian elimination.

## Applications of the Power Method

Section 7.4 introduced the idea of an **age transition matrix** as a model for population growth. Recall that this model was developed by grouping the population into \( n \) age classes of equal duration. So, for a maximum life span of \( L \) years, the age classes are represented by the intervals listed below.

\[
\begin{align*}
\text{First age class} & \quad \left[ 0, \frac{L}{n} \right), \\
\text{Second age class} & \quad \left[ \frac{L}{n}, \frac{2L}{n} \right), \ldots, \\
\text{nth age class} & \quad \left[ \frac{(n-1)L}{n}, L \right]
\end{align*}
\]

The number of population members in each age class is then represented by the age distribution vector

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},
\]

where

\[
\begin{align*}
\text{Number in first age class} & = x_1, \\
\text{Number in second age class} & = x_2, \\
\vdots & , \\
\text{Number in nth age class} & = x_n.
\end{align*}
\]

Over a period of \( L/n \) years, the probability that a member of the \( i \)th age class will survive to become a member of the \((i + 1)\)th age class is given by \( p_i \), where \( 0 \leq p_i \leq 1, \ i = 1, 2, \ldots, n - 1 \). The average number of offspring produced by a member of the \( i \)th age class is given by \( b_i \), where \( 0 \leq b_i \), \( i = 1, 2, \ldots, n \). These numbers can be written in matrix form as follows.
Multiplying this age transition matrix by the age distribution vector for a specific period of time produces the age distribution vector for the next period of time. That is,

$$Ax_i = x_{i+1}.$$  

In Section 7.4 you saw that the growth pattern for a population is stable if the same percentage of the total population is in each age class each year. That is,

$$Ax_i = x_{i+1} = \lambda x_i.$$  

For populations with many age classes, the solution to this eigenvalue problem can be found with the power method, as illustrated in Example 4.

**EXAMPLE 4**  
**A Population Growth Model**

Assume that a population of human females has the characteristics listed below.

<table>
<thead>
<tr>
<th>Age Class (in years)</th>
<th>Average Number of Female Children During 10-Year Period</th>
<th>Probability of Surviving to Next Age Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ≤ age &lt; 10</td>
<td>0.000</td>
<td>0.985</td>
</tr>
<tr>
<td>10 ≤ age &lt; 20</td>
<td>0.174</td>
<td>0.996</td>
</tr>
<tr>
<td>20 ≤ age &lt; 30</td>
<td>0.782</td>
<td>0.994</td>
</tr>
<tr>
<td>30 ≤ age &lt; 40</td>
<td>0.263</td>
<td>0.990</td>
</tr>
<tr>
<td>40 ≤ age &lt; 50</td>
<td>0.022</td>
<td>0.975</td>
</tr>
<tr>
<td>50 ≤ age &lt; 60</td>
<td>0.000</td>
<td>0.940</td>
</tr>
<tr>
<td>60 ≤ age &lt; 70</td>
<td>0.000</td>
<td>0.866</td>
</tr>
<tr>
<td>70 ≤ age &lt; 80</td>
<td>0.000</td>
<td>0.680</td>
</tr>
<tr>
<td>80 ≤ age &lt; 90</td>
<td>0.000</td>
<td>0.361</td>
</tr>
<tr>
<td>90 ≤ age &lt; 100</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Find a stable age distribution for this population.
The age transition matrix for this population is

$$A = \begin{bmatrix}
0.000 & 0.174 & 0.782 & 0.263 & 0.022 & 0.000 & 0.000 & 0.000 & 0.000 \\
0.985 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.996 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.994 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.990 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.975 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.940 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.866 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.680 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.361 \\
\end{bmatrix}.$$ 

To apply the power method with scaling to find an eigenvector for this matrix, use an initial approximation of $x_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. An approximation for an eigenvector of $A$, with the percentage of each age in the total population, is shown below.

$$x = \begin{bmatrix}
1.000 \\
0.925 \\
0.864 \\
0.806 \\
0.749 \\
0.686 \\
0.605 \\
0.492 \\
0.314 \\
0.106 \\
\end{bmatrix} \begin{array}{c}
\leq age < 10 \\
\leq age < 20 \\
\leq age < 30 \\
\leq age < 40 \\
\leq age < 50 \\
\leq age < 60 \\
\leq age < 70 \\
\leq age < 80 \\
\leq age < 90 \\
\leq age < 100 \\
\end{array} \begin{array}{c}
15.27 \\
14.13 \\
13.20 \\
12.31 \\
11.24 \\
10.48 \\
9.24 \\
7.51 \\
4.80 \\
1.62 \\
\end{array}$$

The eigenvalue corresponding to the eigenvector $x$ in Example 4 is $\lambda \approx 1.065$. That is,

$$Ax = A \begin{bmatrix}
1.000 \\
0.925 \\
0.864 \\
0.806 \\
0.749 \\
0.686 \\
0.605 \\
0.492 \\
0.314 \\
0.106 \\
\end{bmatrix} \approx \begin{bmatrix}
1.000 \\
0.985 \\
0.921 \\
0.859 \\
0.798 \\
0.731 \\
0.645 \\
0.524 \\
0.334 \\
0.113 \\
\end{bmatrix} \approx \begin{bmatrix}
1.000 \\
0.925 \\
0.864 \\
0.806 \\
0.749 \\
0.686 \\
0.605 \\
0.492 \\
0.314 \\
0.106 \\
\end{bmatrix}.$$ 

This means that the population in Example 4 increases by 6.5% every 10 years.
Find the third-degree least squares regression polynomial for the given data. Then graphically compare the model with the given points.

In Exercises 1–4, find the second-degree least squares regression polynomial for the given data. Then graphically compare the model with the given points.

1. \((-2, 1), (-1, 0), (0, 0), (1, 1), (3, 2)\)
2. \((0, 4), (1, 2), (2, -1), (3, 0), (4, 1), (5, 4)\)
3. \((-2, 1), (-1, 2), (0, 6), (1, 3), (2, 0), (3, -1)\)
4. \((1, 1), (2, 1), (3, 0), (4, -1), (5, -4)\)

In Exercises 5–8, find the third-degree least squares regression polynomial for the given data. Then graphically compare the model with the given points.

5. \((0, 0), (1, 2), (2, 4), (3, 1), (4, 0), (5, 1)\)
6. \((1, 1), (2, 4), (3, 4), (5, 1), (6, 2)\)
7. \((-3, 4), (-1, 1), (0, 0), (1, 2), (2, 5)\)
8. \((-7, 2), (-3, 0), (1, -1), (2, 3), (4, 6)\)

9. Find the second-degree least squares regression polynomial for the points

\[
\left(-\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{3}, \frac{1}{2}\right), (0, 1), \left(\frac{\pi}{3}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 0\right).
\]

Then use the results to approximate \(\cos(\pi/4)\). Compare the approximation with the exact value.

10. Find the third-degree least squares regression polynomial for the points

\[
\left(-\frac{\pi}{4}, -1\right), \left(-\frac{\pi}{3}, -\sqrt{3}\right), (0, 0), \left(\frac{\pi}{3}, \sqrt{3}\right), \left(\frac{\pi}{4}, 1\right).
\]

Then use the result to approximate \(\tan(\pi/6)\). Compare the approximation with the exact value.

**Remark:** Should you try duplicating the results of Example 4, you would notice that the convergence of the power method for this problem is very slow. The reason is that the dominant eigenvalue of \(\lambda \approx 1.065\) is only slightly larger in absolute value than the next largest eigenvalue.

### Exercises

**Section 10.4 Applications of Numerical Methods**

#### Applications of Gaussian Elimination with Pivoting

In Exercises 1–4, find the second-degree least squares regression polynomial for the given data. Then graphically compare the model with the given points.

<table>
<thead>
<tr>
<th>Depth (in feet)</th>
<th>35</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (in minutes)</td>
<td>310</td>
<td>200</td>
<td>100</td>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Depth (in feet)</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (in minutes)</td>
<td>40</td>
<td>30</td>
<td>25</td>
<td>20</td>
</tr>
</tbody>
</table>

(a) Find the least squares regression line for these data.
(b) Find the second-degree least squares regression polynomial for these data.
(c) Sketch the graphs of the models found in parts (a) and (b).
(d) Use the models found in parts (a) and (b) to approximate the maximum number of minutes a diver should stay at a depth of 120 feet. (The value from the Navy’s tables is 15 minutes.)

**Remark:** Should you try duplicating the results of Example 4, you would notice that the convergence of the power method for this problem is very slow. The reason is that the dominant eigenvalue of \(\lambda \approx 1.065\) is only slightly larger in absolute value than the next largest eigenvalue.

11. The numbers of minutes a scuba diver can stay at particular depths without acquiring decompression sickness are shown in the table. (Source: United States Navy’s Standard Air Decompression Tables)

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
</tr>
</thead>
<tbody>
<tr>
<td>Health Expenditures</td>
<td>1.359</td>
<td>1.474</td>
<td>1.608</td>
<td>1.741</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>Health Expenditures</td>
<td>1.878</td>
<td>2.016</td>
<td>2.164</td>
<td>2.320</td>
</tr>
</tbody>
</table>
(a) Find the second-degree least squares regression polynomial for these data. Let \( x = 0 \) correspond to 2000.

(b) Use the result of part (a) to predict the expenditures for the years 2008 through 2010.

13. The total numbers of people (in millions) in the United States 65 years of age or older in selected years are shown in the table. (Source: U.S. Census Bureau)

<table>
<thead>
<tr>
<th>Year</th>
<th>1990</th>
<th>1995</th>
<th>2000</th>
<th>2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of People</td>
<td>29.6</td>
<td>31.7</td>
<td>32.6</td>
<td>35.0</td>
</tr>
</tbody>
</table>

(a) Find the second-degree least squares regression polynomial for the data. Let \( x = 0 \) correspond to 1990, \( x = 1 \) correspond to 1995, and so on.

(b) Use the result of part (a) to predict the total numbers of people in the United States 65 years of age or older in 2010, 2015, and 2020.

(c) Are your predictions from part (b) realistic? Explain.

14. The total amounts of money (in billions of dollars) spent on retail prescription sales in the United States from 2000 to 2005 are shown in the table. (Source: National Association of Chain Drug Stores)

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount Spent</td>
<td>145.6</td>
<td>164.1</td>
<td>182.7</td>
<td>203.1</td>
<td>221.0</td>
<td>230.3</td>
</tr>
</tbody>
</table>

(a) Find the second-degree least squares regression polynomial for the data. Let \( x = 0 \) correspond to 2000.

(b) Use the result of part (a) to predict the total retail sales in 2010 and 2015.

(c) Are your predictions from part (b) realistic? Explain.

15. Find the least squares regression line for the population data from Example 1. Then use the model to predict the world population in 2010 and 2015, and compare the results with the predictions obtained in Example 1.

16. Show that the formula for the least squares regression line presented in Section 2.5 is equivalent to the formula presented in this section. That is, if

\[
Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix},
\]

then the matrix equation \( A = (X^T X)^{-1} X^T Y \) is equivalent to

\[
a_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2} \quad \text{and} \quad a_0 = \frac{\sum y_i - a_1 \sum x_i}{n}.
\]

Applications of the Gauss-Seidel Method

17. Suppose that the experiment in Example 3 is performed with the maze shown in Figure 10.4. Find the probability that the mouse will emerge in the food corridor when it begins in the \( i \)th intersection.

![Figure 10.4](image)

18. Suppose that the experiment in Example 3 is performed with the maze shown in Figure 10.5. Find the probability that the mouse will emerge in the food corridor when it begins in the \( i \)th intersection.

![Figure 10.5](image)
19. A square metal plate has a constant temperature on each of its four boundaries, as shown in Figure 10.6. Use a $4 \times 4$ grid to approximate the temperature distribution in the interior of the plate. Assume that the temperature at each interior point is the average of the temperatures at the four closest neighboring points.

![Figure 10.6](image)

20. A rectangular metal plate has a constant temperature on each of its four boundaries, as shown in Figure 10.7. Use a $4 \times 5$ grid to approximate the temperature distribution in the interior of the plate. Assume that the temperature at each interior point is the average of the temperatures at the four closest neighboring points.

![Figure 10.7](image)

### Applications of the Power Method

In Exercises 21–24, the matrix represents the age transition matrix for a population. Use the power method with scaling to find a stable age distribution.

21. $A = \begin{bmatrix} 1 & 4 \\ \frac{1}{2} & 0 \end{bmatrix}$  
22. $A = \begin{bmatrix} 1 & 2 \\ \frac{1}{4} & 0 \end{bmatrix}$  
23. $A = \begin{bmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$  
24. $A = \begin{bmatrix} 1 & 2 & 2 \\ \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$

25. In Example 1 in Section 7.4, a laboratory population of rabbits is described. The age transition matrix for the population is

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$  

Find a stable age distribution for this population.

26. A population has the characteristics listed below.

(a) A total of 75% of the population survives its first year. Of that 75%, 25% survives its second year. The maximum life span is three years.

(b) The average numbers of offspring for each member of the population are 2 the first year, 4 the second year, and 2 the third year.

Find a stable age distribution for this population. (See Exercise 9, Section 7.4.)

27. Apply the power method to the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

discussed in Chapter 7 (Fibonacci sequence). Use the power method to approximate the dominant eigenvalue of $A$. [The dominant eigenvalue is $\lambda = \left(1 + \sqrt{5}\right)/2$.]

28. Writing In Example 2 in Section 2.5, the stochastic matrix

$$P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}$$

represents the transition probabilities for a consumer preference model. Use the power method to approximate a dominant eigenvector for this matrix. How does the approximation relate to the steady state matrix described in the discussion in Example 3 in Section 2.5?

29. In Exercise 9 in Section 2.5, a population of 10,000 is divided into nonsmokers, moderate smokers, and heavy smokers. Use the power method to approximate a dominant eigenvector for this matrix.
CHAPTER 10  Review Exercises

In Exercises 1–6, express the real number in floating point form.

1. 528.6  
2. 475.2  
3. -4.85  
4. -22.5  
5. 3\frac{1}{2}  
6. 4\frac{7}{8}

In Exercises 7–12, determine the stored value of the real number in a computer that (a) rounds to three significant digits, and (b) rounds to four significant digits.

7. 25.2  
8. -41.2  
9. -250.231  
10. 628.742  
11. \frac{5}{13}  
12. \frac{3}{16}

In Exercises 13 and 14, evaluate the determinant of the matrix, rounding each intermediate step to three significant digits. Compare the rounded solution with the exact solution.

13. \begin{bmatrix} 12.5 & 2.5 \\ 20.24 & 6.25 \end{bmatrix}  
14. \begin{bmatrix} 8.5 & 3.2 \\ 10.25 & 8.1 \end{bmatrix}

In Exercises 15 and 16, use Gaussian elimination to solve the system of linear equations. After each intermediate step, round the result to three significant digits. Compare the rounded solution with the exact solution.

15. 2.53x + 8.5y = 29.65  
16. 12.5x - 18.2y = 56.8

In Exercises 17 and 18, use Gaussian elimination without partial pivoting to solve the system of linear equations, rounding each intermediate step to three significant digits. Then use Gaussian elimination with partial pivoting to solve the same system, again rounding each intermediate step to three significant digits. Finally, compare both solutions with the exact solution provided.

17. 2.15x + 7.25y = 13.7  
18. 4.25x + 6.3y = 16.85

In Exercises 19 and 20, use Gaussian elimination to solve the ill-conditioned system of linear equations, rounding each intermediate calculation to three significant digits. Compare the solution with the exact solution provided.

19. \begin{align*} x + \frac{999}{1000} y &= \frac{4001}{1000} \\
Exact solution: x &= 5000, y = -5001 \end{align*}

20. \begin{align*} x - y &= -1 \\
-\frac{99}{100} x + \frac{20}{1001} y &= \frac{2001}{100} \\
Exact solution: x &= 20,001, y = 20,002 \end{align*}

In Exercises 21 and 22, apply the Jacobi method to the system of linear equations, using the initial approximation \((x_1, x_2, x_3, \ldots, x_n) = (0, 0, 0, \ldots, 0)\). Continue performing iterations until two successive approximations are identical when rounded to three significant digits.

21. \begin{align*} 2x_1 - x_2 &= -1 \\
x_1 + 4x_2 &= -3 \end{align*}

22. \begin{align*} x_1 + 2x_2 &= 7 \\
x_1 + x_2 &= 1 \end{align*}

23. Apply the Gauss-Seidel method to Exercise 21.


In Exercises 25–28, determine whether the matrix is strictly diagonally dominant.

25. \begin{bmatrix} 4 & 2 \\ 0 & -3 \end{bmatrix}  
26. \begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}

27. \begin{bmatrix} 4 & 0 & 2 \\ 10 & 12 & -2 \\ 1 & -2 & 0 \end{bmatrix}  
28. \begin{bmatrix} 4 & 2 & -1 \\ 0 & -2 & -1 \\ 1 & 1 & -1 \end{bmatrix}

In Exercises 29–32, interchange the rows of the system of linear equations to obtain a system with a strictly diagonally dominant matrix. Then apply the Gauss-Seidel method to approximate the solution to four significant digits.

29. \begin{align*} x_1 + 2x_2 &= -5 \\
5x_1 - x_2 &= 8 \end{align*}

30. \begin{align*} x_1 + 4x_2 &= -4 \\
x_2 &= 6 \end{align*}

31. \begin{align*} 2x_1 + 4x_2 + x_3 &= -2 \\
4x_1 + x_2 + x_3 &= 1 \end{align*}

32. \begin{align*} x_1 + 3x_2 + x_3 &= 2 \\
x_1 + x_2 + 3x_3 &= -1 \end{align*}
In Exercises 33–36, use the techniques presented in Chapter 7 to find the eigenvalues of the matrix \( A \). If \( A \) has a dominant eigenvalue, find a corresponding dominant eigenvector.

**33.** \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\]

**34.** \[
\begin{bmatrix}
4 & -5 \\
2 & -3
\end{bmatrix}
\]

**35.** \[
\begin{bmatrix}
-2 & 2 & -3 \\
2 & 1 & -6 \\
-1 & -2 & 0
\end{bmatrix}
\]

**36.** \[
\begin{bmatrix}
2 & 3 & 1 \\
0 & -1 & 2 \\
0 & 0 & 3
\end{bmatrix}
\]

In Exercises 37–40, use the Rayleigh quotient to compute the eigenvalue \( \lambda \) of \( A \) corresponding to the eigenvector \( \mathbf{x} \).

**37.** \[
\begin{bmatrix}
2 & -12 \\
1 & -5
\end{bmatrix}
\begin{bmatrix}
x \\
x
\end{bmatrix} = \begin{bmatrix}
3 \\
1
\end{bmatrix}
\]

**38.** \[
\begin{bmatrix}
6 & -3 \\
-2 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
x
\end{bmatrix} = \begin{bmatrix}
3 \\
-1
\end{bmatrix}
\]

**39.** \[
\begin{bmatrix}
2 & 0 & 1 \\
0 & 3 & 4 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
x \\
x
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

**40.** \[
\begin{bmatrix}
1 & 2 & -2 \\
-2 & 5 & -2 \\
-6 & 6 & -3
\end{bmatrix}
\begin{bmatrix}
x \\
x \\
x
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
3
\end{bmatrix}
\]

In Exercises 41 and 42, use the power method with scaling to approximate a dominant eigenvector of the matrix \( A \). Start with \( x_0 = (1, 1) \) and calculate four iterations. Then use \( x_4 \) to approximate the dominant eigenvalue of \( A \).

**41.** \[
\begin{bmatrix}
7 & 2 \\
2 & 4
\end{bmatrix}
\]

**42.** \[
\begin{bmatrix}
-3 & 10 \\
5 & 2
\end{bmatrix}
\]

In Exercises 43 and 44, find the second-degree least squares regression polynomial for the data. Then use a graphing utility with regression features or a computer software program to find a second-degree regression polynomial. Compare the results.

**43.** \(-2, 0), (-1, 2), (0, 3), (1, 2), and (3, 0)\)

**44.** \(-2, 2), (-1, 1), (0, -1), (1, -1), and (3, 0)\)

In Exercises 45–48, find the second-degree least squares regression polynomial for the data. Then use a graphing utility with regression features or a computer software program to find a second-degree regression polynomial. Compare the results.

**45.** The amounts spent for hospital care (in billions of dollars) in the United States from 2000 to 2004 are shown in the table. (Source: U.S. Centers for Medicare and Medicaid Services)

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount Spent</td>
<td>417.0</td>
<td>451.4</td>
<td>488.6</td>
<td>525.5</td>
<td>570.8</td>
</tr>
</tbody>
</table>

Find the second-degree least squares regression polynomial for the data. Then use a graphing utility with regression features or a computer software program to find a second-degree regression polynomial. Compare the results.
The populations (in millions) of the United States by decade from 1900 to 2000 are shown in the table.

<table>
<thead>
<tr>
<th>Year</th>
<th>1900</th>
<th>1910</th>
<th>1920</th>
<th>1930</th>
<th>1940</th>
<th>1950</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>76.2</td>
<td>92.2</td>
<td>106.0</td>
<td>123.2</td>
<td>132.2</td>
<td>151.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>179.3</td>
<td>203.3</td>
<td>226.5</td>
<td>248.7</td>
<td>281.4</td>
</tr>
</tbody>
</table>

Use a graphing utility or a computer software program to create a scatter plot of the data. Let $t = 0$ correspond to 1900, $t = 1$ correspond to 1910, and so on.

Using the scatter plot, describe any patterns in the data. Do the data appear to be linear, quadratic, or cubic in nature? Explain.

Use the techniques presented in the chapter to find
(a) a linear least squares regression equation.
(b) a second-degree least squares regression equation.
(c) a cubic least squares regression equation.

Graph each equation with the data. Briefly describe which of the regression equations best fits the data.

Use each model to predict the populations of the United States for the years 2010, 2020, 2030, 2040, and 2050. Which, if any, of the regression equations appears to be the best model for predicting future populations? Explain your reasoning.

The 2007 Statistical Abstract projected the populations of the United States for the same years, as shown in the table below. Do any of your models produce the same projections? Explain any possible differences between your projections and the Statistical Abstract projections.

<table>
<thead>
<tr>
<th>Year</th>
<th>2010</th>
<th>2020</th>
<th>2030</th>
<th>2040</th>
<th>2050</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>303.6</td>
<td>335.8</td>
<td>363.6</td>
<td>391.9</td>
<td>419.6</td>
</tr>
</tbody>
</table>