Chapter 2  Differentiation

Section 2.1  The Derivative and the Tangent Line Problem

Objective: In this lesson you learned how to find the derivative of a function using the limit definition and understand the relationship between differentiability and continuity.

I. The Tangent Line Problem  (Pages 96–99)

Essentially, the problem of finding the tangent line at a point \( P \) boils down to \( \frac{f(c + \Delta x) - f(c)}{\Delta x} \). You can approximate this slope using \( \frac{f(c + \Delta x) - f(c)}{\Delta x} \) through the point of tangency \( (c, f(c)) \) and a second point on the curve \( (c + \Delta x, f(c + \Delta x)) \). The slope of the secant line through these two points is \( m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x} \).

The right side of this equation for the slope of a secant line is called a \( \frac{f(c + \Delta x) - f(c)}{\Delta x} \). The denominator \( \Delta x \) is the \( \frac{f(c + \Delta x) - f(c)}{\Delta x} \), and the numerator \( \Delta y = f(c + \Delta x) - f(c) \) is the \( \frac{f(c + \Delta x) - f(c)}{\Delta x} \).

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by \( \frac{f(c + \Delta x) - f(c)}{\Delta x} \).

If \( f \) is defined on an open interval containing \( c \), and if the limit

\[
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m
\]

exists, then the line passing
through \((c, f(c))\) with slope \(m\) is \[
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}
\]

The slope of the tangent line to the graph of \(f\) at the point \((c, f(c))\) is also called \[
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}
\]

**Example 1:** Find the slope of the graph of \(f(x) = 9 - \frac{x}{2}\) at the point \((4, 7)\).

**Example 2:** Find the slope of the graph of \(f(x) = 2 - 3x^2\) at the point \((-1, -1)\).

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. If \(f\) is continuous at \(c\) and

\[
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty,
\]

the vertical line \(x = c\) passing through \((c, f(c))\) is \[
\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}
\]

to the graph of \(f\).

**II. The Derivative of a Function** (Pages 99–101)

The \[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

is given by \(f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\), provided the limit exists. For all \(x\) for which this limit exists, \(f'\) is \[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

The derivative of a function of \(x\) gives the \[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

to the graph of \(f\) at the point \((x, f(x))\), provided that the graph has a tangent line at this point.

A function is **differentiable on an open interval** \((a, b)\) if ____

---

*What you should learn*

How to use the limit definition to find the derivative of a function.
Example 3: Find the derivative of \( f(t) = 4t^2 + 5 \).

III. Differentiability and Continuity (Pages 101–103)

Name some situations in which a function will not be differentiable at a point.

If a function \( f \) is differentiable at \( x = c \), then ________________
_____________________________.

Complete the following statements.

1. If a function is differentiable at \( x = c \), then it is continuous at \( x = c \). So, differentiability ________________ continuity.

2. It is possible for a function to be continuous at \( x = c \) and not be differentiable at \( x = c \). So, continuity ________________ ________________ differentiability.
Additional notes

Homework Assignment

Page(s)

Exercises
Section 2.2 Basic Differentiation and Rates of Change

Objective: In this lesson you learned how to find the derivative of a function using basic differentiation rules.

I. The Constant Rule (Page 107)

The derivative of a constant function is ____________.

If $c$ is a real number, then $\frac{d}{dx}[c] = ____________.

II. The Power Rule (Pages 108–109)

The Power Rule states that if $n$ is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = ________________.$$ For $f$ to be differentiable at $x = 0$, $n$ must be a number such that $x^{n-1}$ is ________________

______________________________.

Also, $\frac{d}{dx} x = ________________.$

Example 1: Find the derivative of the function $f(x) = \frac{1}{x^3}$.

Example 2: Find the slope of the graph of $f(x) = x^2$ at $x = 2$.

III. The Constant Multiple Rule (Pages 110–111)

The Constant Multiple Rule states that if $f$ is a differentiable function and $c$ is a real number then $cf$ is also differentiable and

$$\frac{d}{dx} cf(x) = ________________.$$ Informally, the Constant Multiple Rule states that __________

______________________________.
Example 3: Find the derivative of \( f(x) = \frac{2x}{5} \)

The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is

\[
\frac{d}{dx}[cx^n] = \text{______________________}.
\]

Example 4: Find the derivative of \( y = \frac{2}{5x^3} \)

IV. The Sum and Difference Rules (Page 111)

The Sum and Difference Rules of Differentiation state that the sum (or difference) of two differentiable functions \( f \) and \( g \) is itself differentiable. Moreover, the derivative of \( f + g \) (or \( f - g \)) is the sum (or difference) of the derivatives of \( f \) and \( g \).

That is, \[
\frac{d}{dx} f(x) + g(x) = \text{______________________}
\]

and \[
\frac{d}{dx} f(x) - g(x) = \text{______________________}
\]

Example 5: Find the derivative of \( f(x) = 2x^3 - 4x^2 + 3x - 1 \)

V. Derivatives of Sine and Cosine Functions (Page 112)

\[
\frac{d}{dx} \sin x = \text{______________________}
\]

\[
\frac{d}{dx} \cos x = \text{______________________}
\]

Example 6: Differentiate the function \( y = x^2 - 2\cos x \).
VI. Rates of Change (Pages 113–114)

The derivative can also be used to determine ______________
__________________________.

Give some examples of real-life applications of rates of change.

The function $s$ that gives the position (relative to the origin) of an object as a function of time $t$ is called a ______________.

The average velocity of an object that is moving in a straight line is found as follows.

Average velocity $=$ ______________ $=$ _______

Example 7: If a ball is dropped from the top of a building that is 200 feet tall, and air resistance is neglected, the height $s$ (in feet) of the ball at time $t$ (in seconds) is given by $s = -16t^2 + 200$. Find the average velocity of the object over the interval $[1, 3]$.

If $s = s(t)$ is the position function for an object moving along a straight line, the (instantaneous) velocity of the object at time $t$ is

$\nu(t) =$ ______________ $=$ ______________.

In other words, the velocity function is the ______________ the position function. Velocity can be ______________ ______________. The ______________ of an object is the absolute value of its velocity. Speed cannot be ______________.
Example 8: If a ball is dropped from the top of a building that is 200 feet tall, and air resistance is neglected, the height \( s \) (in feet) of the ball at time \( t \) (in seconds) is given by \( s(t) = -16t^2 + 200 \). Find the velocity of the ball when \( t = 3 \).

The position function for a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation \( \text{______________________________} \), where \( s_0 \) is the initial height of the object, \( v_0 \) is the initial velocity of the object, and \( g \) is the acceleration due to gravity. On Earth, the value of \( g \) is \( \text{______________________________} \).

<table>
<thead>
<tr>
<th>Homework Assignment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page(s)</td>
</tr>
<tr>
<td>Exercises</td>
</tr>
</tbody>
</table>
Section 2.3  Product and Quotient Rules and Higher-Order Derivatives

Objective: In this lesson you learned how to find the derivative of a function using the Product Rule and Quotient Rule.

I. The Product Rule  (Pages 119–120)

The product of two differentiable functions $f$ and $g$ is itself differentiable. The Product Rule states that the derivative of the function $fg$ is equal to

$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + g(x)f'(x)$.

Example 1: Find the derivative of $y = (4x^2 + 1)(2x - 3)$.

The Product Rule can be extended to cover products that have more than two factors. For example, if $f$, $g$, and $h$ are differentiable functions of $x$, then

$\frac{d}{dx} f(x)g(x)h(x) = \frac{d}{dx} f(x)g(x) + f(x)g'(x)h(x)$.

Explain the difference between the Constant Multiple Rule and the Product Rule.
II. The Quotient Rule (Pages 121–123)

The quotient \( f / g \) of two differentiable functions \( f \) and \( g \) is itself differentiable at all values of \( x \) for which \( g(x) \neq 0 \). The derivative of \( f / g \) is given by

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}, \quad g(x) \neq 0.
\]

**Example 2:** Find the derivative of \( y = \frac{2x+5}{3x} \).

With the Quotient Rule, it is a good idea to enclose all factors and derivatives and to pay special attention to.

III. Derivatives of Trigonometric Functions (Pages 123–124)

\[
\frac{d}{dx} \tan x = \quad \frac{d}{dx} \cot x = \quad \frac{d}{dx} \sec x = \quad \frac{d}{dx} \csc x =
\]

**Example 3:** Differentiate the function \( f(x) = \sin x \sec x \).
IV. Higher-Order Derivatives (Page 125)

The derivative of \( f'(x) \) is the second derivative of \( f(x) \) and is denoted by \( f''(x) \). The derivative of \( f''(x) \) is the \( f'''(x) \) of \( f(x) \) and is denoted by \( f''' \).

These are examples of \( f''(x) \) of \( f(x) \).

The following notation is used to denoted the \( f^{(n)}(x) \) of the function \( y = f(x) \):

\[
\frac{d^6 y}{dx^6} \quad D^6_y[y] \quad y^{(6)} \quad \frac{d^6}{dx^6}[f(x)] \quad f^{(6)}(x)
\]

Example 4: Find \( y^{(5)} \) for \( y = 2x^7 - x^5 \).

Example 5: On the moon, a ball is dropped from a height of 100 feet. Its height \( s \) (in feet) above the moon’s surface is given by \( s = -\frac{27}{10} t^2 + 100 \). Find the height, the velocity, and the acceleration of the ball when \( t = 5 \) seconds.
Example 6: Find $y''$ for $y = \sin x$.

Additional notes

<table>
<thead>
<tr>
<th>Homework Assignment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page(s)</td>
</tr>
<tr>
<td>Exercises</td>
</tr>
</tbody>
</table>
Section 2.4 The Chain Rule

Objective: In this lesson you learned how to find the derivative of a function using the Chain Rule and General Power Rule.

I. The Chain Rule (Pages 130–132)

The Chain Rule, one of the most powerful differentiation rules, deals with \( y = f(u) \) functions.

Basically, the Chain Rule states that if \( y \) changes \( dy/du \) times as fast as \( u \), and \( u \) changes \( du/dx \) times as fast as \( x \), then \( y \) changes \( \frac{dy}{dx} \) times as fast as \( x \).

The Chain Rule states that if \( y = f(u) \) is a differentiable function of \( u \), and \( u = g(x) \) is a differentiable function of \( x \), then \( y = f(g(x)) \) is a differentiable function of \( x \), and

\[
\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du} \quad \text{or, equivalently,} \quad \frac{d}{dx} f(g(x)) = \frac{d}{du} f(u) \cdot \frac{du}{dx}.
\]

When applying the Chain Rule, it is helpful to think of the composite function \( f \circ g \) as having two parts, an inner part and an outer part. The Chain Rule tells you that the derivative of \( y = f(u) \) is the derivative of the \( \frac{dy}{du} \) (at the inner function \( u \)) times the derivative of the \( \frac{d}{dx} f(g(x)) \). That is, \( y' = \frac{dy}{du} \cdot \frac{du}{dx} \).

Example 1: Find the derivative of \( y = (3x^2 - 2)^5 \).
II. The General Power Rule (Pages 132–133)

The General Power Rule is a special case of the _________ _________.

The General Power Rule states that if $y = u(x)^n$, where $u$ is a differentiable function of $x$ and $n$ is a rational number, then

$$
\frac{dy}{dx} = \quad \text{or, equivalently,}
$$

$$
\frac{d}{dx} [u^n] = \quad \text{Example 2:} \quad \text{Find the derivative of} \quad y = \frac{4}{(2x-1)^3}.
$$

III. Simplifying Derivatives (Page 134)

Example 3: Find the derivative of $y = \frac{3x^2}{(1-x^3)^2}$ and simplify.

What you should learn
How to find the derivative of a function using the General Power Rule

What you should learn
How to simplify the derivative of a function using algebra
IV. Trigonometric Functions and the Chain Rule
(Pages 135–136)

Complete each of the following “Chain Rule versions” of the derivatives of the six trigonometric functions.

\[
\frac{d}{dx} \sin u = \]

\[
\frac{d}{dx} \cos u = \]

\[
\frac{d}{dx} \tan u = \]

\[
\frac{d}{dx} \cot u = \]

\[
\frac{d}{dx} \sec u = \]

\[
\frac{d}{dx} \csc u = \]

**Example 4:** Differentiate the function \( y = \sec 4x \).

**Example 5:** Differentiate the function \( y = x^2 - \cos(2x + 1) \).
Additional notes

<table>
<thead>
<tr>
<th>Homework Assignment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page(s)</td>
</tr>
<tr>
<td>Exercises</td>
</tr>
</tbody>
</table>
Section 2.5 Implicit Differentiation

Objective: In this lesson you learned how to find the derivative of a function using implicit differentiation.

I. Implicit and Explicit Functions (Page 141)

Up to this point in the text, most functions have been expressed in explicit form \( y = f(x) \), meaning that \( \ldots \). However, some functions are only \( \ldots \) by an equation.

Give an example of a function in which \( y \) is implicitly defined as a function of \( x \).

Implicit differentiation is a procedure for taking the derivative of an implicit function when you are unable to \( \ldots \). Failed to parse (unknown function 'dydx').

To understand how to find \( \frac{dy}{dx} \) implicitly, realize that the differentiation is taking place \( \ldots \). This means that when you differentiate terms involving \( x \) alone, \( \ldots \). However, when you differentiate terms involving \( y \), you must apply \( \ldots \) because you are assuming that \( y \) is defined \( \ldots \) as a differentiable function of \( x \).

Example 1: Differentiate the expression with respect to \( x \):

\[ 4x + y^3 \]
II. Implicit Differentiation (Pages 142–145)

Consider an equation involving \( x \) and \( y \) in which \( y \) is a differentiable function of \( x \). List the four guidelines for applying implicit differentiation to find \( dy/dx \).

1. 

2. 

3. 

4. 

Example 2: Find \( dy/dx \) for the equation \( 4y^2 - x^2 = 1 \).

Homework Assignment

Page(s)

Exercises
Section 2.6 Related Rates

Objective: In this lesson you learned how to find a related rate.

I. Finding Related Variables (Page 149)

Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to ________________.

Example 1: The variables $x$ and $y$ are differentiable functions of $t$ and are related by the equation $y = 2x^3 - x + 4$.

When $x = 2$, $dx/dt = -1$. Find $dy/dt$ when $x = 2$.

II. Problem Solving with Related Rates (Pages 150–153)

List the guidelines for solving a related-rate problems.

1. 

2. 

3. 

4. 

Example 2: Write a mathematical model for the following related-rate problem situation:

The population of a city is decreasing at the rate of 100 people per month.
Additional notes

Homework Assignment

Page(s)

Exercises