Fourier Series

Fourier Series with Period $2\pi$ • Fourier Series for Even and Odd Functions • Fourier Series with Period $2p$

**Fourier Series with Period $2\pi$**

In Chapter 10, you looked at several ways to represent functions. You will now study the problem of approximating periodic functions. Recall from Section 8.2 that a function $f$ is periodic if there exists a nonzero number $p$ such that $f(x + p) = f(x)$ for all $x$ in the domain of $f$. The smallest such value of $p$ is called the period of $f$. To approximate a periodic function with period $2\pi$, you can use a series made of cosine and sine terms called a Fourier series.

**Definition of Fourier Series with Period $2\pi$**

A Fourier series is an infinite series of cosine and sine terms of the following form.

$$g(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_n \cos nx + \cdots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_n \sin nx + \cdots$$

The coefficients are

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

where $n = 1, 2, 3, \ldots$. Note that the Fourier series is periodic with period $2\pi$ and the average value of $f$ on the interval $[-\pi, \pi]$ is $a_0$.

Note that you could use a Taylor series to approximate a periodic function $f$. The Taylor series would be a good approximation of $f$ near $x = c$, but not necessarily a good approximation over the entire period. A Fourier series for $f$ will often give a better approximation of the function over the entire period than a Taylor series.

**EXAMPLE 1 Finding a Fourier Series**

Find the Fourier series for $f(x) = 2x$, $-\pi \leq x < \pi$.

**Solution**

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2x \, dx = \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left( \frac{\pi^2}{2} - \frac{(-\pi)^2}{2} \right) = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \cos nx \, dx$$

$$= \frac{2}{n^2 \pi} \left[ \cos nx + nx \sin nx \right]_{-\pi}^{\pi} \quad \text{Formula 53, } u = nx$$
Because \( \sin n\pi = 0 \) for \( n = 1, 2, 3, \ldots \),

\[
    b_n = \frac{4}{n^2\pi} (0 - n\pi \cos n\pi) = \frac{-4\cos n\pi}{n}.
\]

For \( n = 1, 2, 3, \ldots \), the Fourier series for \( f \) is

\[
    g(x) = 0 + (0)(\cos x) + (0)(\cos 2x) + (0)(\cos 3x) + \cdots \\
    + \left(\frac{-4\cos \pi}{1}\right) \sin x + \left(\frac{-4\cos 2\pi}{2}\right) \sin 2x + \left(\frac{-4\cos 3\pi}{3}\right) \sin 3x + \cdots \\
    = 4 \sin x - 2 \sin 2x + \frac{4}{3} \sin 3x - \cdots.
\]

Notice that \( g \) has no constant term and no cosine terms. Figure 1 shows a graph of \( f \) compared to a curve made using the first several terms of \( g \). The greater the number of terms of \( g \) used to make the curve, the better the curve will fit \( f \).

Letting \( f(x + 2\pi) = f(x) \) for the function in Example 1, you can graph \( f \) for several periods and compare the graph to a curve made using the first several terms of \( g \). You can see that the Fourier series representation follows \( f \) over each period.
EXAMPLE 2 Finding a Fourier Series

Find the Fourier series for \( f(x) = \begin{cases} x, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases} \).

**Solution** Begin by noticing that \( f \) has a different definition for each interval of \( x \). So, you will need two integrals for each coefficient.

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} x \, dx + \frac{1}{2\pi} \int_{0}^{\pi} x \, dx = \frac{1}{2\pi} \left[ \frac{x^2}{2} \right]^{0}_{-\pi} + \frac{1}{2\pi} \left[ \frac{x}{2} \right]^{\pi}_{0} = \frac{-\pi}{4} + \frac{1}{2} = -\frac{2 - \pi}{4}
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{0} x \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} (1) \cos nx \, dx
\]

\[
= \frac{1}{n^2\pi} \left[ \cos nx + nx \sin nx \right]^{0}_{-\pi} + \frac{1}{n\pi} \left[ \sin nx \right]^{\pi}_{0}
\]

\[
= \frac{1 - \cos n\pi - n\pi \sin n\pi}{n^2\pi} + \sin n\pi
\]

Because \( \sin n\pi = 0 \) for \( n = 1, 2, 3, \ldots \), \( a_n = \frac{1 - \cos n\pi}{n^2\pi} \).

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{0} x \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} (1) \sin nx \, dx
\]

\[
= \frac{1}{n^2\pi} \left[ \sin nx + nx \cos nx \right]^{0}_{-\pi} + \frac{1}{n\pi} \left[ -\cos nx \right]^{\pi}_{0}
\]

\[
= \frac{\sin n\pi - n\pi \cos n\pi}{n^2\pi} - \cos n\pi + \frac{1}{n\pi}
\]

Because \( \sin n\pi = 0 \) for \( n = 1, 2, 3, \ldots \), \( b_n = \frac{-\cos n\pi - \cos n\pi}{n\pi} + \frac{1}{n\pi} = \frac{1 - \pi \cos (n\pi - \cos n\pi)}{n\pi} \).

For \( n = 1, 2, 3, \ldots \) the Fourier series for \( f \) is

\[
g(x) = \frac{2 - \pi}{4} + \left( \frac{1 - \cos \pi}{\pi} \right) \cos x + \left( \frac{1 - \cos 2\pi}{2^2\pi} \right) \cos 2x
\]

\[
+ \left( \frac{1 - \cos 3\pi}{3^2\pi} \right) \cos 3x + \cdots + \left( \frac{1 - \pi \cos (n\pi - \cos n\pi)}{n\pi} \right) \sin x
\]

\[
+ \left( \frac{1 - \pi \cos (2\pi - \cos 2\pi)}{2\pi} \right) \sin 2\pi + \left( \frac{1 - \pi \cos (3\pi - \cos 3\pi)}{3\pi} \right) \sin 3\pi + \cdots
\]

\[
= \frac{2 - \pi}{4} + \frac{2}{\pi} \cos x + \frac{2}{9\pi} \cos 3x + \cdots + \frac{2 + \pi}{3\pi} \sin x - \frac{1}{2} \sin 2x + \frac{2 + \pi}{3\pi} \sin 3x.
\]

Figure 3 shows a graph of \( f \) compared to a curve made using the first several terms of \( g \). Figure 3
For Fourier series coefficients that have integrals with forms such as $\sin au \sin bu$, $\sin au \cos bu$, and $\cos au \cos bu$, you can use the following formulas. Notice that the formulas are not valid when $a^2 = b^2$.

1. $\int \sin au \sin bu \, du = \frac{1}{2} \left( \sin((a - b)u) + \sin((a + b)u) \right)
2. \int \sin au \cos bu \, du = -\frac{1}{2} \left( \cos((a - b)u) + \cos((a + b)u) \right)
3. \int \cos au \cos bu \, du = \frac{1}{2} \left( \sin((a - b)u) + \sin((a + b)u) \right)$

**EXAMPLE 3 Finding a Fourier Series**

Find the Fourier series for $f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$

**Solution**

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} (0) \, dx + \frac{1}{2\pi} \int_{0}^{\pi} \sin x \, dx = 0 + \frac{1}{2\pi} \left[ -\cos x \right]_{0}^{\pi} = \frac{1}{\pi}$

For $n = 1, a_1$ is

$a_1 = \frac{1}{\pi} \int_{-\pi}^{0} (0) \cos x \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin x \cos x \, dx = 0 + \frac{1}{\pi} \left[ \frac{\sin^2 x}{2} \right]_{0}^{\pi} = 0.$

For $n = 2, 3, 4, \ldots$, you can find $a_n$ using the formula

$a_n = \frac{1}{\pi} \int_{-\pi}^{0} (0) \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin nx \cos nx \, dx$

$= 0 - \frac{1}{2\pi} \left[ \cos((1 - n)x) + \cos((1 + n)x) \right]_{0}^{\pi}$

$= -\frac{1}{2\pi} \left[ \cos((1 - n) \cdot \pi) + \cos((1 + n) \cdot \pi) \right] - \left( \cos((1 - n) \cdot 0) + \cos((1 + n) \cdot 0) \right)$

$= -\frac{1}{2\pi} \left( \cos((1 - n) \pi) + \cos((1 + n) \pi) \right) - \frac{1}{1 - n} - \frac{1}{1 + n}$

For $n = 1, b_1$ is

$b_1 = \frac{1}{\pi} \int_{-\pi}^{0} (0) \sin x \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin x \sin x \, dx$

$= 0 + \frac{1}{\pi} \left[ \sin^2 x \right]_{0}^{\pi}$

$= \frac{1}{2\pi} \left[ x - \sin x \cos x \right]_{0}^{\pi}$

$= \frac{1}{2}$

For $n = 2, 3, 4, \ldots$, you can find $b_n$ using the formula

$\int \sin au \sin bu \, du = \frac{1}{2} \left( \sin((a - b)u) - \sin((a + b)u) \right)$. 
The first several terms of the graph of Figure 4 compared to a curve made using \( \pi \cdot x \) of Figure 4 shows a graph of \( f \) by noticing if the function is even (symmetric with respect to the origin). You can determine which terms to eliminate anytime you find a Fourier series for a function that is symmetric with respect to the origin. If a function is symmetric with respect to the \( y \)-axis, the Fourier series for the function will have no sine terms. You can determine which terms to eliminate by noticing if the function is even (symmetric with respect to the \( y \)-axis) or odd (symmetric with respect to the origin). Recall that a function is even if \( f(-x) = f(x) \) and a function is odd if \( f(-x) = -f(x) \).

**Fourier Series for Even and Odd Functions**

In Example 1, the Fourier series you found had no cosine terms. This will occur anytime you find a Fourier series for a function that is symmetric with respect to the origin. If a function is symmetric with respect to the \( y \)-axis, the Fourier series for the function will have no sine terms. You can determine which terms to eliminate by noticing if the function is even (symmetric with respect to the \( y \)-axis) or odd (symmetric with respect to the origin). Recall that a function is even if \( f(-x) = f(x) \) and a function is odd if \( f(-x) = -f(x) \).

**Fourier Series for Even and Odd Functions**

A Fourier series for an even function has cosine terms and may have a constant term, but the series will not have any sine terms.

A Fourier series for an odd function has sine terms, but the series will not have any cosine terms nor a constant term.

The function in Example 1, \( f(x) = 2x, -\pi \leq x < \pi \), is odd because the graph of \( f \) is symmetric with respect to the origin. You also know \( f \) is odd because \( f(-x) = -f(x) \). So, the Fourier series for \( f \) has only sine terms.
EXAMPLE 4  Finding a Fourier Series for an Even Function

Find the Fourier series for \( f(x) = \begin{cases} 
-x, & -\pi \leq x < 0 \\
x, & 0 \leq x < \pi 
\end{cases} \).

Solution  In Figure 5, you can see that the function \( f \) is symmetric with respect to the \( y \)-axis, so \( f \) is an even function. Also, you know \( f \) is an even function because \( f(-x) = f(x) \). This means that the Fourier series for \( f \) will have cosine terms and maybe a constant term, but no sine terms. So you do not have to find \( b_n \).

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{0} (-x) \, dx + \frac{1}{2\pi} \int_{0}^{\pi} x \, dx = -\frac{1}{2\pi} \left[ \frac{x^2}{2} \right]_{-\pi}^{0} + \frac{1}{2\pi} \left[ \frac{x^2}{2} \right]_{0}^{\pi} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{0} (-x) \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \cos nx \, dx
\]

\[
= -\frac{1}{n^2} \left[ \cos nx + nx \sin nx \right]_{-\pi}^{0} + \frac{1}{n^2} \left[ \cos nx + nx \sin nx \right]_{0}^{\pi}
\]

\[
= -1 - \cos n\pi - n\pi \sin n\pi + \left( -1 - \cos n\pi - n\pi \sin n\pi \right)
\]

\[
= -2 \left( 1 - \cos n\pi \right) \frac{n\pi}{n^2} \sin n\pi
\]

Because \( \sin n\pi = 0 \) for \( n = 1, 2, 3, \ldots \), \( a_n = -2 \left( \frac{1 - \cos n\pi}{n^2} \right) \).

For \( n = 1, 2, 3, \ldots \), the Fourier series for \( f \) is

\[
g(x) = \frac{\pi}{2} - \frac{4}{\pi} \cos x - \frac{4}{9\pi} \cos 3x - \frac{4}{25\pi} \cos 5x - \cdots
\]

\[
= \frac{\pi}{2} - \frac{4}{\pi} \left( \cos x + \frac{1}{9\pi} \cos 3x + \frac{1}{25\pi} \cos 5x + \cdots \right)
\]

Figure 6 shows a graph of \( f \) compared to a curve made using the first several terms of \( g \).
EXAMPLE 5  Finding a Fourier Series for an Odd Function

Find the Fourier series for \( f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x < \pi \end{cases} \).

Solution  In Figure 7, you can see that the function \( f \) is symmetric with respect to the origin, so \( f \) is an odd function. Also, you know \( f \) is an odd function because \( f(-x) = -f(x) \). This means that the Fourier series for \( f \) will have sine terms, but will not have a constant term nor any cosine terms. So you do not have to find \( a_0 \) or \( a_n \).

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (-1) \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} (1) \sin nx \, dx
\]

\[
= -\frac{1}{\pi} \left[ -\cos nx \right]_{-\pi}^{0} + \frac{1}{\pi} \left[ -\cos nx \right]_{0}^{\pi}
\]

\[
= -\frac{1}{\pi} (-1 + \cos n\pi) + \frac{1}{\pi} (-\cos n\pi + 1)
\]

\[
= 2\left( \frac{1 - \cos n\pi}{n\pi} \right)
\]

For \( n = 1, 2, 3, \ldots \), the Fourier series for \( f \) is

\[
g(x) = \frac{4}{\pi} \sin x + (0)\sin 2x + \frac{4}{3\pi} \sin 3x + (0)\sin 4x + \frac{4}{5\pi} \sin 5x + \cdots
\]

\[
= \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)
\]

Figure 8 shows a graph of \( f \) compared to a curve made using the first several terms of \( g \). The function \( f \) is often referred to as a square wave function.

The function \( f \) is odd.

Figure 7

Graph of \( f \) compared to a curve made using the first several terms of \( g \).

Figure 8
Fourier Series with Period $2p$

You can also use a Fourier series for a function with a period other than $2\pi$. For a function with period $2p$, the coefficients are found as follows.

**Definition of Fourier Series with Period $2p$**

A Fourier series is an infinite series of cosine and sine terms of the following form.

$$g(x) = a_0 + a_1 \cos \frac{\pi x}{p} + a_2 \cos \frac{2\pi x}{p} + a_3 \cos \frac{3\pi x}{p} + \cdots + a_n \cos \frac{n\pi x}{p} + \cdots$$

$$+ b_1 \sin \frac{\pi x}{p} + b_2 \sin \frac{2\pi x}{p} + b_3 \sin \frac{3\pi x}{p} + \cdots + b_n \sin \frac{n\pi x}{p} + \cdots$$

The coefficients of a Fourier series with period $2p$ are

$$a_0 = \frac{1}{2p} \int_{-p}^{p} f(x) \, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p} \, dx$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p} \, dx$$

where $n = 1, 2, 3, \ldots$. The Fourier series is periodic with period $2p$.

**EXAMPLE 6  Finding a Fourier Series with Period $2p$**

Find the Fourier series for $f(x) = \begin{cases} 0, & 0 \leq x < 4 \\ 5, & 4 \leq x < 8 \end{cases}$

**Solution** The period of $f$ is $2p = 8$. So, $p = 4$.

$$a_0 = \frac{1}{2(4)} \int_{0}^{4} f(x) \, dx + \frac{1}{2(4)} \int_{4}^{8} f(x) \, dx = \frac{1}{2} \left[ x \right]_{0}^{4} + \frac{5}{8} \left[ x \right]_{4}^{8} = \frac{5}{2}$$

$$a_n = \frac{1}{4} \int_{0}^{4} f(x) \cos \frac{n\pi x}{4} \, dx + \frac{1}{4} \int_{4}^{8} f(x) \cos \frac{n\pi x}{4} \, dx$$

$$= \frac{5}{n\pi} \left[ \sin \frac{n\pi x}{4} \right]_{0}^{4}$$

$$= \frac{5}{n\pi} (\sin 2n\pi - \sin n\pi)$$

Because $\sin 2n\pi = 0$ and $\sin n\pi = 0$ for $n = 1, 2, 3, \ldots$, $a_n = 0$.

$$b_n = \frac{1}{4} \int_{0}^{4} f(x) \sin \frac{n\pi x}{4} \, dx + \frac{1}{4} \int_{4}^{8} f(x) \sin \frac{n\pi x}{4} \, dx$$

$$= -\frac{5}{n\pi} \left[ \cos \frac{n\pi x}{4} \right]_{0}^{4}$$

$$= -\frac{5}{n\pi} (\cos 2n\pi - \cos n\pi)$$
For $n = 1, 2, 3, \ldots$, the Fourier series for $f$ is

$$g(x) = \frac{5}{2} + (0)\left(\cos \frac{\pi x}{4}\right) + (0)\left(\cos \frac{2\pi x}{4}\right) + (0)\left(\cos \frac{3\pi x}{4}\right) + \cdots$$

$$+ \left(-\frac{5}{\pi}\right)\left(2\sin \frac{\pi x}{4}\right) + \left(-\frac{5}{2\pi}\right)(0)\left(\sin \frac{2\pi x}{4}\right) + \left(-\frac{5}{3\pi}\right)(2)\left(\sin \frac{3\pi x}{4}\right) + \cdots$$

$$= \frac{5}{2} - \frac{10}{\pi}\left(\sin \frac{\pi x}{4} + \frac{1}{3}\sin \frac{3\pi x}{4} + \cdots\right).$$

Figure 9 shows a graph of $f$ compared to a curve made using the first several terms of $g$.

**EXERCISES FOR FOURIER SERIES**

In Exercises 1–4, (a) use $a_n$, $a_0$, and $b_n$ to write a Fourier series for the function where $n = 1, 2, \text{and } 3$. (b) Use a graphing utility to graph $f(x)$ and the Fourier series in the same window. (c) Show by graphing that the Fourier series approximation improves as $n$ increases.

1. $f(x) = -x$, $-\pi \leq x < \pi$
   
   $a_0 = 0$
   
   $a_n = 0$
   
   $b_n = \frac{2\cos n\pi}{n}$

2. $f(x) = x^2$, $-\pi \leq x < \pi$
   
   $a_0 = \frac{\pi^2}{3}$
   
   $a_n = \frac{4\cos n\pi}{n^2}$
   
   $b_n = 0$

3. $f(x) = \begin{cases} 0, & -2 \leq x < 0 \\ 1, & 0 \leq x < 2 \end{cases}$
   
   $a_0 = \frac{1}{2}$
   
   $a_n = \frac{\sin n\pi}{n\pi}$
   
   $b_n = \frac{1 - \cos n\pi}{n\pi}$

4. $f(x) = \begin{cases} x, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \end{cases}$
   
   $a_0 = \frac{1}{4}$
   
   $a_n = \frac{1 - \cos n\pi}{n^2\pi}$
   
   $b_n = \frac{1 - 2\cos n\pi}{n\pi}$
In Exercises 5–14, write a Fourier series for the function. Let $n = 1, 2,$ and 3. Use a graphing utility to verify your answer.

5. $f(x) = x + \pi, -\pi \leq x < \pi$

6. $f(x) = \pi x, -\pi \leq x < \pi$

7. $f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ x/2, & 0 \leq x < \pi \end{cases}$

8. $f(x) = \begin{cases} -\sin x, & -\pi \leq x < 0 \\ \sin x, & 0 \leq x < \pi \end{cases}$

9. $f(x) = \begin{cases} -2, & -\pi \leq x < 0 \\ 2, & 0 \leq x < \pi \end{cases}$

10. $f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ \pi, & 0 \leq x < \pi \end{cases}$

11. $f(x) = \frac{x}{\pi}, -\pi \leq x < \pi$

12. $f(x) = x - 1, -\pi \leq x < \pi$

13. $f(x) = \begin{cases} -1, & -\pi \leq x < -\pi/2 \\ 0, & -\pi/2 \leq x < \pi/2 \\ 1, & \pi/2 \leq x < \pi \end{cases}$

14. $f(x) = \begin{cases} \cos x, & -\pi/2 \leq x < -\pi/2 \\ 0, & \pi/2 \leq x < \pi \end{cases}$

In Exercises 15–18, decide whether the function is odd, even, or neither. Explain your answer analytically and graphically.

15. $f(x) = x^3, -1 \leq x < 1$

16. $f(x) = x^4, -2 \leq x < 2$

17. $f(x) = \begin{cases} 0, & -3 \leq x < 0 \\ x, & 0 \leq x < 3 \end{cases}$

18. $f(x) = \begin{cases} x, & -1 \leq x < 0 \\ -x, & 0 \leq x < 1 \end{cases}$

In Exercises 19–22, the function $f(x)$ is either even or odd. Write a Fourier series for the function. Let $n = 1, 2,$ and 3. Use a graphing utility to verify your answer.

19. $f(x) = \begin{cases} -3, & -\pi \leq x < 0 \\ -x, & 3 \leq x < \pi \end{cases}$

20. $f(x) = \begin{cases} 0, & -\pi \leq x < -\pi/2 \\ 1, & -\pi/2 \leq x < -\pi/2 \\ 0, & \pi/2 \leq x < \pi \end{cases}$

21. $f(x) = \begin{cases} \cos x, & -\pi/2 \leq x < -\pi/2 \\ -\cos x, & \pi/2 \leq x < \pi \end{cases}$

22. $f(x) = \begin{cases} x/\pi + 1, & -\pi \leq x < 0 \\ x/\pi - 1, & 0 \leq x < \pi \end{cases}$

23. Describe how a Fourier series representation of a function is different from a Taylor series representation of a function.

24. Describe the differences between Fourier series for functions with periods of $2\pi$ and Fourier series for functions with periods other than $2\pi$. 
In Exercises 25–32, function \( f \) is defined for one period. (a) Identify the period and (b) write a Fourier series for the function. Let \( n = 1, 2, \) and 3. Use a graphing utility to verify your answer.

25. \( f(x) = \frac{x}{4} + 1, -4 \leq x < 4 \)

26. \( f(x) = -\frac{x}{2}, -5 \leq x < 5 \)

27. \( f(x) = \begin{cases} 2, & -3 \leq x < 0 \\ 0, & 0 \leq x < 3 \end{cases} \)

28. \( f(x) = \begin{cases} 0, & -2 \leq x < 0 \\ x - 2, & 0 \leq x < 2 \end{cases} \)

29. \( f(x) = \begin{cases} x, & -6 \leq x < 0 \\ 1, & 0 \leq x < 6 \end{cases} \)

30. \( f(x) = \begin{cases} -\frac{2x}{3}, & -5 \leq x < 0 \\ \frac{2x}{3}, & 0 \leq x < 5 \end{cases} \)

31. \( f(x) = \begin{cases} -2, & -4 \leq x < -2 \\ 2, & -2 \leq x < 2 \\ -2, & 2 \leq x < 4 \end{cases} \)

32. \( f(x) = \begin{cases} 0, & -2 \leq x < -1 \\ -x^2 + 1, & -1 \leq x < 1 \\ 0, & 1 \leq x < 2 \end{cases} \)

In Exercises 33–36, the graph of a function \( f \) is shown. Match each Fourier series to the graph that it represents without graphing. Explain your reasoning.

33. \( g(x) = 1 + \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots \)

34. \( g(x) = \frac{4}{\pi} \cos x - \frac{4}{3\pi} \cos 3x + \frac{4}{5\pi} \cos 5x - \cdots \)

35. \( g(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots \)

36. \( g(x) = \frac{4}{\pi} \sin \frac{\pi x}{4} + \frac{4}{3\pi} \sin \frac{3\pi x}{4} + \frac{4}{5\pi} \sin \frac{5\pi x}{4} + \cdots \)

True or False? In Exercises 37–40, decide whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

37. The Fourier series for an odd function will consist only of sine terms and possibly a constant term.

38. The graph of an even function is always symmetric to the \( y \)-axis.

39. If \( g \) represents a Fourier series for a periodic function \( f \), then \( g \) also represents the Fourier series for \( f(x + 2\pi) \).

40. If \( g \) is a Fourier series for a function \( f \) where \( a_0 = \frac{1}{2} \), then it is possible that \( f(x) = \begin{cases} 1, & -\pi \leq x < 0 \\ 0, & 0 \leq x < \pi \end{cases} \).

41. Electrical Circuits A half-wave rectifier is an electrical device that converts AC voltage to DC voltage. A model for the output voltage \( V \) at time \( t \) is

\[
V(t) = \begin{cases} 12 \sin t, & 0 \leq t < \pi \\ 0, & \pi \leq t < 2\pi \end{cases}
\]

(a) Use a graphing utility to graph at least 3 periods of \( V(t) \).

(b) Write a Fourier series for \( V(t) \).

(c) Graph \( V(t) \) and the Fourier series in the same window.
42. **Electrical Circuits**  A full-wave rectifier is an electrical device that converts AC voltage into a pulsating DC voltage using both half cycles of the AC voltage. A model for the output voltage $V$ at time $t$ is

$$V(t) = \begin{cases} 
9 \sin t, & 0 \leq t < \pi \\
-9 \sin t, & \pi \leq t < 2\pi 
\end{cases}$$

(a) Use a graphing utility to graph at least 3 periods of $V(t)$.

(b) Write a Fourier series for $V(t)$.

(c) Graph $V(t)$ and the Fourier series in the same window.

43. **Sound**  The loudness $A$, in decibels, of a pulsating alarm at time $t$ can be modeled by the function

$$A(t) = \begin{cases} 
100, & 0 \leq t < \pi \\
0, & \pi \leq t < 2\pi 
\end{cases}$$

(a) Use a graphing utility to graph at least 3 periods of $A(t)$.

(b) Write a Fourier series for $A(t)$.

(c) Graph $A(t)$ and the Fourier series in the same window.

44. **Sound**  The loudness $A$, in decibels, of an alarm at time $t$ can be modeled by the function

$$A(t) = \begin{cases} 
75t, & 0 \leq t < \frac{\pi}{2} \\
75(\pi - t), & \frac{\pi}{2} \leq t < \pi \\
0, & \pi \leq t < 2\pi 
\end{cases}$$

(a) Use a graphing utility to graph at least 3 periods of $A(t)$.

(b) Write a Fourier series for $A(t)$.

(c) Graph $A(t)$ and the Fourier series in the same window.

45. **Gibbs Phenomenon**  Gibbs phenomenon centers on the behavior, primarily the overshoots, of a Fourier series approximation of a function near points of discontinuity. Consider the function

$$f(x) = \begin{cases} 
-1, & -\pi \leq x < 0 \\
1, & 0 \leq x < \pi 
\end{cases}$$

and its corresponding Fourier series

$$g(x) = \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right).$$

The overshoots for this function are shown below.

(a) Use a graphing utility to graph $f$ and the first three terms of $g$ in the same window.

(b) Estimate the height and width of the overshoots.

(c) What happens to the height and width of the overshoots as the number of terms of $g$ increase?