Statements

In everyday speech and in mathematics you make inferences that adhere to common laws of logic. These methods of reasoning allow you to build an algebra of statements by using logical operations to form compound statements from simpler ones. A primary goal of logic is to determine the truth value (true or false) of a compound statement knowing the truth value of its simpler component. For instance, the compound statement “The temperature is below freezing and it is snowing” is true only if both component statements are true.

Definition of a Statement

1. A **statement** (or proposition) is a sentence to which only one truth value (either true or false) can be meaningfully assigned.

2. An **open statement** is a sentence that contains one or more variables and becomes a statement when each variable is replaced by a specific item from a designated set.

Example 1  
**Statements, Nonstatements, and Open Statements**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Truth Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>A square is a rectangle.</td>
<td>True</td>
</tr>
<tr>
<td>(-3) is less than (-5).</td>
<td>False</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Nonstatement</th>
<th>Truth Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Do your homework.</td>
<td>No truth value can be meaningfully assigned.</td>
</tr>
<tr>
<td>Did you call the police?</td>
<td>No truth value can be meaningfully assigned.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Open Statement</th>
<th>Truth Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x) is an irrational number.</td>
<td>A value of (x) is needed.</td>
</tr>
<tr>
<td>She is a computer science major.</td>
<td>A specific person is needed.</td>
</tr>
</tbody>
</table>
Symbolically, statements are represented by lowercase letters \( p \), \( q \), \( r \), and so on. Statements can be changed or combined to form compound statements by means of the three logical operations \( \text{and} \), \( \text{or} \), and \( \text{not} \), which are represented by \( \land \) (and), \( \lor \) (or), and \( \lnot \) (not). In logic, the word \( \text{or} \) is used in the inclusive sense (meaning “and/or” in everyday language). That is, the statement \( p \lor q \) is true if \( p \) is true, \( q \) is true, or both \( p \) and \( q \) are true. The following list summarizes the terms and symbols used with these three operations of logic.

### Operations of Logic

<table>
<thead>
<tr>
<th>Operation</th>
<th>Verbal Statement</th>
<th>Symbolic Form</th>
<th>Name of Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \land )</td>
<td>( p \text{ and } q )</td>
<td>( p \land q )</td>
<td>Conjunction</td>
</tr>
<tr>
<td>( \lor )</td>
<td>( p \text{ or } q )</td>
<td>( p \lor q )</td>
<td>Disjunction</td>
</tr>
<tr>
<td>( \lnot )</td>
<td>( \lnot p )</td>
<td>( \lnot p )</td>
<td>Negation</td>
</tr>
</tbody>
</table>

Compound statements can be formed using more than one logical operation, as demonstrated in Example 2.

### Example 2  
Forming Negations and Compound Statements

The statements \( p \) and \( q \) are as follows.

\[
p: \text{The temperature is below freezing.} \\
q: \text{It is snowing.}
\]

Write the verbal form for each of the following.

a. \( p \land q \)  
   b. \( \lnot p \)  
   c. \( \lnot (p \lor q) \)  
   d. \( \lnot p \land \lnot q \)

**Solution**

a. The temperature is below freezing and it is snowing.

b. The temperature is not below freezing.

c. It is not true that the temperature is below freezing or it is snowing.

d. The temperature is not below freezing and it is not snowing.

### Example 3  
Forming Compound Statements

The statements \( p \) and \( q \) are as follows.

\[
p: \text{The temperature is below freezing.} \\
q: \text{It is snowing.}
\]

a. Write the symbolic form for: The temperature is not below freezing or it is not snowing.

b. Write the symbolic form for: It is not true that the temperature is below freezing and it is snowing.

**Solution**

a. The symbolic form is: \( \lnot p \lor \lnot q \)

b. The symbolic form is: \( \lnot (p \land q) \)
Truth Tables

To determine the truth value of a compound statement, it is helpful to construct charts called truth tables. The following tables represent the three basic logical operations.

<table>
<thead>
<tr>
<th>Conjunction</th>
<th>Disjunction</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$q$</td>
<td>$p \land q$</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

For the sake of uniformity, all truth tables with two component statements will have T and F values for $p$ and $q$ assigned in the order shown in the first two columns of each of these three tables. Truth tables for several operations can be combined into one chart by using the same two first columns. For each operation, a new column is added. Such an arrangement is especially useful with compound statements that involve more than one logical operation and for showing that two statements are logically equivalent.

Logical Equivalence

Two compound statements are logically equivalent if they have identical truth tables. Symbolically, the equivalence of the statements $p$ and $q$ is denoted by writing $p \equiv q$.

Example 4 Logical Equivalence

Use a truth table to show the logical equivalence of the statements $\neg p \land \neg q$ and $\neg (p \lor q)$.

Solution

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$\neg p \land \neg q$</th>
<th>$p \lor q$</th>
<th>$\neg (p \lor q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
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<td>T</td>
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<td>F</td>
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<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Because the fifth and seventh columns in the table are identical, the two given statements are logically equivalent.
The equivalence established in Example 4 is one of two well-known rules in logic called DeMorgan’s Laws. Verification of the second of DeMorgan’s Laws is left as an exercise.

**DeMorgan’s Laws**

1. \( \neg(p \lor q) \equiv \neg p \land \neg q \)

2. \( \neg(p \land q) \equiv \neg p \lor \neg q \)

Compound statements that are true, no matter what the truth values of the component statements, are called tautologies. One simple example is the statement “\( p \) or not \( p \),” as shown in the table.

\( p \lor \neg p \) is a tautology

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
<th>( p \lor \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

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**E.1 Exercises**

In Exercises 1–12, classify the sentence as a statement, a nonstatement, or an open statement. See Example 1.

1. All dogs are brown.  
   2. Can I help you?  
   3. That figure is a circle.  
   4. Substitute 4 for \( x \).  
   5. \( x \) is larger than 4.  
   6. 8 is larger than 4.  
   7. \( x + y = 10 \)  
   8. 12 + 3 = 14  
   9. Hockey is fun to watch.  
   10. One mile is greater than 1 kilometer.  
   11. It is more than 1 mile to the school.  
   12. Come to the party.

In Exercises 13–20, determine whether the open statement is true for the given values of \( x \).

13. \( x^2 - 5x + 6 = 0 \)  
   (a) \( x = 2 \)  
   (b) \( x = -2 \)  
14. \( x^2 - x - 6 = 0 \)  
   (a) \( x = 2 \)  
   (b) \( x = -2 \)  
15. \( x^2 \leq 4 \)  
   (a) \( x = -2 \)  
   (b) \( x = 0 \)  
16. \( |x - 3| = 4 \)  
   (a) \( x = -1 \)  
   (b) \( x = 7 \)

17. \( 4 - |x| = 2 \)  
18. \( \sqrt{x^2} = x \)  
   (a) \( x = 0 \)  
   (b) \( x = 1 \)  
   (c) \( x = -1 \)  
   (d) \( x = -3 \)

19. \( \frac{x}{x} = 1 \)  
20. \( \frac{1}{x} = -2 \)  
   (a) \( x = -4 \)  
   (b) \( x = 0 \)  
   (c) \( x = 8 \)  
   (d) \( x = -8 \)

In Exercises 21–24, write the verbal form for each of the following. See Example 2.

21. \( p \): The sun is shining.  
   \( q \): It is hot.  
   \( r \): The sun is shining.  
   \( s \): It is hot.

22. \( p \): The car has a radio.  
   \( q \): The car is red.

23. \( p \): Lions are mammals.  
   \( q \): Lions are carnivorous.

24. \( p \): Twelve is less than 15.  
   \( q \): Seven is a prime number.
In Exercises 25–28, write the verbal form for each of the following.

(a) \( \neg p \land q \)  
(b) \( \neg p \lor q \) 
(c) \( p \land \neg q \)  
(d) \( p \lor \neg q \)

25. \( p \): The sun is shining.  
\( q \): It is hot.

26. \( p \): The car has a radio.  
\( q \): The car is red.

27. \( p \): Lions are mammals.  
\( q \): Lions are carnivorous.

28. \( p \): Twelve is less than 15.  
\( q \): Seven is a prime number.

In Exercises 29–32, write the symbolic form of the given compound statement. In each case let \( p \) represent the statement “It is four o’clock,” and let \( q \) represent the statement “It is time to go home.” See Example 3.

29. \( \neg p \land q \)  
30. \( p \lor \neg q \) 
31. \( p \lor q \)  
32. \( \neg p \land q \)

In Exercises 33–36, write the symbolic form of the given compound statement. In each case let \( p \) represent the statement “The dog has fleas,” and let \( q \) represent the statement “The dog is scratching.”

33. \( \neg p \lor q \)  
34. \( p \land q \) 
35. \( p \land \neg q \)  
36. \( p \lor \neg q \)

In Exercises 37–42, write the negation of the given statement.

37. \( \neg p \land p \)  
38. \( \neg p \lor q \) 
39. \( x \) is equal to 4.

40. \( x \) is not equal to 4.

41. Earth is not flat.

42. Earth is flat.

In Exercises 43–48, construct a truth table for the given compound statement.

43. \( \neg p \land q \)  
44. \( \neg p \lor q \) 
45. \( \neg p \lor q \)  
46. \( p \land \neg q \)

47. \( p \lor \neg q \)  
48. \( p \lor \neg q \)

In Exercises 49–54, use a truth table to determine whether the given statements are logically equivalent. See Example 4.

49. \( \neg p \land q \land p \lor \neg q \) 
50. \( (p \land \neg q) \lor (p \lor \neg q) \)

51. \( (p \lor \neg q) \land (p \land \neg q) \) 
52. \( (p \lor \neg q) \land (p \land \neg q) \)

53. \( (p \land \neg q) \lor (p \lor \neg q) \) 
54. \( p \lor \neg q \) 

In Exercises 55–58, determine whether the statements are logically equivalent.

55. (a) The house is red and it is not made of wood.

(b) The house is red or it is not made of wood.

56. (a) It is not true that the tree is not green.

(b) The tree is green.

57. (a) The statement that the house is white or blue is not true.

(b) The house is not white and it is not blue.

58. (a) I am not 25 years old and I am not applying for this job.

(b) The statement that I am 25 years old and I am applying for this job is not true.

In Exercises 59–62, use a truth table to determine whether the given statement is a tautology.

59. \( \neg p \land p \)  
60. \( \neg p \lor p \) 
61. \( \neg (p \lor \neg p) \)  
62. \( \neg (p \land \neg p) \)

63. Use a truth table to verify the second of DeMorgan’s Laws.

\( \neg (p \land q) \equiv \neg p \lor \neg q \)
Conditional Statements

A statement of the form “If \( p \), then \( q \),” is called a **conditional statement** (or an **implication**) and is denoted by

\[ p \rightarrow q. \]

The statement \( p \) is called the **hypothesis** and the statement \( q \) is called the **conclusion**. There are many different ways to write the conditional statement \( p \rightarrow q \), as shown in the following list.

<table>
<thead>
<tr>
<th>Different Ways of Writing Conditional Statements</th>
</tr>
</thead>
<tbody>
<tr>
<td>The conditional statement ( p \rightarrow q ) has the following equivalent verbal forms.</td>
</tr>
<tr>
<td>1. If ( p ), then ( q ).</td>
</tr>
<tr>
<td>2. ( p ) implies ( q ).</td>
</tr>
<tr>
<td>3. ( p ), only if ( q ).</td>
</tr>
<tr>
<td>4. ( q ) follows from ( p ).</td>
</tr>
<tr>
<td>5. ( q ) is necessary for ( p ).</td>
</tr>
<tr>
<td>6. ( p ) is sufficient for ( q ).</td>
</tr>
</tbody>
</table>

Normally, the conditional statement \( p \rightarrow q \) is thought of as having a cause-and-effect relationship between the hypothesis \( p \) and the conclusion \( q \). However, you should be careful not to confuse the truth value of the component statements with the truth value of the conditional statement. The following truth table should help you keep this distinction in mind.

<table>
<thead>
<tr>
<th>Conditional Statement: If ( p ), then ( q ).</th>
</tr>
</thead>
</table>
| \[ \begin{array}{ccc}
| p & q & p \rightarrow q \\
| T & T & T \\
| T & F & F \\
| F & T & T \\
| F & F & T \\
| \end{array} \] |

Note in the table that the conditional statement \( p \rightarrow q \) is false only when \( p \) is true and \( q \) is false. This is like a promise. Suppose you promise a friend that “If the sun shines, I will take you fishing.” The only way you can break your promise is if the sun shines (\( p \) is true) and you do not take your friend fishing (\( q \) is false). If the sun doesn’t shine (\( p \) is false), you have no obligation to go fishing, and so the promise cannot be broken.
Example 1  Finding Truth Values of Conditional Statements

Give the truth value of each conditional statement.

a. If 3 is odd, then 9 is odd.
b. If 3 is odd, then 9 is even.
c. If 3 is even, then 9 is odd.
d. If 3 is even, then 9 is even.

Solution

<table>
<thead>
<tr>
<th>Hypothesis</th>
<th>Conclusion</th>
<th>Conditional Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>b. T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>c. F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>d. F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

The next example shows how to write a conditional statement as a disjunction.

Example 2  Identifying Equivalent Statements

Use a truth table to show the logical equivalence of the following statements.

a. If I get a raise, I will take my family on a vacation.
b. I will not get a raise or I will take my family on a vacation.

Solution

Let \( p \) represent the statement “I will get a raise,” and let \( q \) represent the statement “I will take my family on a vacation.” Then, you can represent the statement in part (a) as \( p \rightarrow q \) and the statement in part (b) as \( \neg p \lor q \). The logical equivalence of these two statements is shown in the following truth table.

\[
p \rightarrow q = \neg p \lor q
\]

<table>
<thead>
<tr>
<th>( \neg p \lor q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \neg p \lor q )</td>
<td>( \neg p \lor q )</td>
</tr>
</tbody>
</table>

Because the fourth and fifth columns of the truth table are identical, you can conclude that the two statements \( p \rightarrow q \) and \( \neg p \lor q \) are logically equivalent.
From the table in Example 2 and the fact that \( (\neg p) \equiv p \), you can write the **negation of a statement**. That is, because \( p \to q \) is equivalent to \( \neg p \lor q \), it follows that the negation of \( p \to q \) must be \( \neg (\neg p \lor q) \), which by DeMorgan’s Laws can be written as follows.

\[
(\neg (p \to q)) \equiv p \land \neg q
\]

For the conditional statement \( p \to q \), there are three important associated conditional statements.

1. The **converse** of \( p \to q \): \( q \to p \)
2. The **inverse** of \( p \to q \): \( \neg p \to \neg q \)
3. The **contrapositive** of \( p \to q \): \( \neg q \to \neg p \)

From the table below, you can see that these four statements yield two pairs of logically equivalent conditional statements. The connective “\( \to \)” is used to determine the truth values in the last three columns of the table.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg q )</th>
<th>( p \to q )</th>
<th>( \neg q \to \neg p )</th>
<th>( q \to p )</th>
<th>( \neg p \to \neg q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<td>T</td>
</tr>
</tbody>
</table>

\[\uparrow\text{Identical} \quad \uparrow\text{Identical}\]

**Example 3**  
**Writing the Converse, Inverse, and Contrapositive**

Write the converse, inverse, and contrapositive of the conditional statement “If I get a B on my test, then I will pass the course.”

**Solution**

a. **Converse**: If I pass the course, then I got a B on my test.

b. **Inverse**: If I do not get a B on my test, then I will not pass the course.

c. **Contrapositive**: If I do not pass the course, then I did not get a B on my test.

In Example 3, be sure you see that neither the converse nor the inverse is logically equivalent to the original conditional statement. To see this, consider that the original conditional statement simply states that if you get a B on your test, then you will pass the course. The converse is not true because knowing that you passed the course does not imply that you got a B on the test. After all, you might have gotten an A on the test.

A **biconditional statement**, denoted by \( p \iff q \), is the conjunction of the conditional statement \( p \to q \) and its converse \( q \to p \). Often a biconditional statement is written as “\( p \) if and only if \( q \)” or in shorter form as “\( p \text{ iff } q \)” A biconditional statement is true when both components are true and when both components are false, as shown in the following truth table.
Biconditional Statement: \( p \) if and only if \( q \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
<th>( q \rightarrow p )</th>
<th>( p \leftrightarrow q )</th>
<th>( (p \rightarrow q) \land (q \rightarrow p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

The following list summarizes some of the laws of logic that have been discussed up to this point.

**Laws of Logic**

1. For every statement \( p \), either \( p \) is true or \( p \) is false. \( \text{Law of Excluded Middle} \)
2. \( \neg
\neg p \equiv p \)
3. \( \neg (p \lor q) \equiv \neg p \land \neg q \) \( \text{DeMorgan’s Law} \)
4. \( \neg (p \land q) \equiv \neg p \lor \neg q \) \( \text{DeMorgan’s Law} \)
5. \( p \rightarrow q \equiv \neg p \lor q \) \( \text{Law of Implication} \)
6. \( p \rightarrow q \equiv \neg q \rightarrow \neg p \) \( \text{Law of Contraposition} \)

**Logical Quantifiers**

Logical quantifiers are words such as *some*, *all*, *every*, *each*, *one*, and *none*. Here are some examples of statements with quantifiers.

- Some isosceles triangles are right triangles.
- Every painting on display is for sale.
- Not all corporations have male chief executive officers.
- All squares are parallelograms.

Being able to recognize the negation of a statement involving a quantifier is one of the most important skills in logic. For instance, consider the statement “All dogs are brown.” In order for this statement to be false, you do not have to show that *all* dogs are not brown, you must simply find at least one dog that is not brown. So, the negation of the statement is “Some dogs are brown.”

Some of the more common negations involving quantifiers are listed below.

**Negating Statements with Quantifiers**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. All ( p ) are ( q ).</td>
<td>Some ( p ) are not ( q ).</td>
</tr>
<tr>
<td>2. Some ( p ) are ( q ).</td>
<td>No ( p ) is ( q ).</td>
</tr>
<tr>
<td>3. Some ( p ) are not ( q ).</td>
<td>All ( p ) are ( q ).</td>
</tr>
<tr>
<td>4. No ( p ) is ( q ).</td>
<td>Some ( p ) are ( q ).</td>
</tr>
</tbody>
</table>
When using logical quantifiers, the word *all* can be replaced by the word *each* or the word *every*. For instance, the following are equivalent.

All $p$ are $q$. Each $p$ is $q$. Every $p$ is $q$.

Similarly, the word *some* can be replaced by the words *at least one*. For instance, the following are equivalent.

Some $p$ are $q$. At least one $p$ is $q$.

### Example 4  
**Negating Quantifying Statements**

Write the negation of each statement.

a. All students study.
b. Not all prime numbers are odd.
c. At least one mammal can fly.
d. Some bananas are not yellow.

**Solution**

a. Some students do not study.
b. All prime numbers are odd.
c. No mammal can fly.
d. All bananas are yellow.

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### Venn Diagrams

**Venn diagrams** are figures that are used to show relationships between two or more sets of objects. They can help you to interpret quantifying statements. Study the following Venn diagrams in which the circle marked $A$ represents people over 6 feet tall and the circle marked $B$ represents the basketball players.

1. All basketball players are over 6 feet tall.
2. Some basketball players are over 6 feet tall.
In Exercises 1–4, write the verbal form for each of the following.

(a) $p \rightarrow q$  
(b) $q \rightarrow p$  
(c) $\sim q \rightarrow \sim p$  
(d) $p \rightarrow \sim q$

1. $p$: The engine is running.  
$q$: The engine is wasting gasoline.

2. $p$: The student is at school.  
$q$: It is nine o’clock.

3. $p$: The integer is even.  
$q$: It is divisible by 2.

4. $p$: The person is generous.  
$q$: The person is rich.

In Exercises 5–10, write the symbolic form of the compound statement. Let $p$ represent the statement “The economy is expanding,” and let $q$ represent the statement “Interest rates are low.”

5. If interest rates are low, then the economy is expanding.

6. If interest rates are not low, then the economy is not expanding.

7. An expanding economy implies low interest rates.

8. Low interest rates are sufficient for an expanding economy.

9. Low interest rates are necessary for an expanding economy.

10. The economy will expand only if interest rates are low.

In Exercises 11–20, give the truth value of the conditional statement. See Example 1.

11. If 4 is even, then 12 is even.

12. If 4 is even, then 2 is odd.

13. If 4 is odd, then 3 is odd.

14. If 4 is odd, then 2 is odd.

15. If $2n$ is even, then $2n + 2$ is odd.

16. If $2n + 1$ is even, then $2n + 2$ is odd.

17. $3 + 11 > 16$ only if $2 + 3 = 5$.

18. $\frac{1}{6} < \frac{1}{2}$ is necessary for $\frac{1}{2} > 0$.

19. $x = -2$ follows from $2x + 3 = x + 1$.

20. If $2x = 224$, then $x = 10$.

In Exercises 21–28, use a truth table to show the logical equivalence of the two statements. See Example 2.

21. $q \rightarrow p$  
   $\sim p \rightarrow \sim q$

22. $\sim p \rightarrow q$  
   $p \lor q$

23. $\sim (p \rightarrow q)$  
   $p \land \sim q$

24. $(p \lor q) \rightarrow q$  
   $p \rightarrow q$

25. $(p \rightarrow q) \lor \sim q$  
   $p \lor \sim p$

26. $q \rightarrow (\sim p \lor q)$  
   $q \lor \sim q$

27. $p \rightarrow (\sim p \land q)$  
   $\sim p$

28. $\sim (p \land q) \rightarrow \sim q$  
   $p \lor \sim q$

29. Select the statement that is logically equivalent to the statement “If a number is divisible by 6, then it is divisible by 2.”

   (a) If a number is divisible by 2, then it is divisible by 6.

   (b) If a number is not divisible by 6, then it is not divisible by 2.

   (c) If a number is not divisible by 2, then it is not divisible by 6.

   (d) Some numbers are divisible by 6 and not divisible by 2.
30. Select the statement that is logically equivalent to the statement “It is not true that Pam is a conservative and a Democrat.”
   (a) Pam is a conservative and a Democrat.
   (b) Pam is not a conservative and not a Democrat.
   (c) Pam is not a conservative or she is not a Democrat.
   (d) If Pam is not a conservative, then she is a Democrat.

31. Select the statement that is not logically equivalent to the statement “Every citizen over the age of 18 has the right to vote.”
   (a) Some citizens over the age of 18 have the right to vote.
   (b) Each citizen over the age of 18 has the right to vote.
   (c) All citizens over the age of 18 have the right to vote.
   (d) No citizen over the age of 18 can be restricted from voting.

32. Select the statement that is not logically equivalent to the statement “It is necessary to pay the registration fee to take the course.”
   (a) If you take the course, then you must pay the registration fee.
   (b) If you do not pay the registration fee, then you cannot take the course.
   (c) If you pay the registration fee, then you may take the course.
   (d) You may take the course only if you pay the registration fee.

In Exercises 33–38, write the converse, inverse, and contrapositive of the statement. See Example 3.

33. If the sky is clear, then you can see the eclipse.
34. If the person is nearsighted, then he is ineligible for the job.
35. If taxes are raised, then the deficit will increase.
36. If wages are raised, then the company’s profits will decrease.
37. It is necessary to have a birth certificate to apply for the visa.
38. The number is divisible by 3 only if the sum of its digits is divisible by 3.

In Exercises 39–46, construct a truth table for the compound statement.

39. \( \neg (p \rightarrow q) \)  
40. \( \neg q \rightarrow (p \rightarrow q) \)
41. \( \neg (q \rightarrow p) \land q \)  
42. \( p \rightarrow (\neg p \lor q) \)
43. \( [(p \lor q) \land (\neg p)] \rightarrow q \)  
44. \( [(p \rightarrow q) \land (\neg q)] \rightarrow p \)
45. \( (p \leftrightarrow \neg q) \rightarrow \neg p \)  
46. \( (p \lor \neg q) \leftrightarrow (q \rightarrow \neg p) \)

In Exercises 47–60, write the negation of the statement. See Example 4.

47. Paul is a junior or a senior.
48. Jack is a senior and he plays varsity basketball.
49. If the temperature increases, then the metal rod will expand.
50. If the test fails, then the project will be halted.
51. We will go to the ocean only if the weather forecast is good.
52. Completing the pass on this play is necessary if we are going to win the game.
53. Some students are in extracurricular activities.
54. Some odd integers are not prime numbers.
55. All contact sports are dangerous.
56. All members must pay their dues prior to June 1.
57. No child is allowed at the concert.
58. No contestant is over the age of 12.
59. At least one of the $20 bills is counterfeit.
60. At least one unit is defective.

In Exercises 61–70, sketch a Venn diagram and shade the region that illustrates the given statement. Let A be a circle that represents people who are happy, and let B be a circle that represents college students.

61. All college students are happy.
62. All happy people are college students.
63. No college students are happy.
64. No happy people are college students.
65. Some college students are not happy.
66. Some happy people are not college students.
67. At least one college student is happy.
68. At least one happy person is not a college student.
69. Each college student is sad.
70. Each sad person is not a college student.
In Exercises 71–74, determine whether the statement follows from the given Venn diagram. Assume that each area shown in the Venn diagram is non-empty. (Note: Use only the information given in the diagram. Do not be concerned with whether the statement is actually true or false.)

71. (a) All toads are green.
   (b) Some toads are green.

72. (a) All men are company presidents.
   (b) Some company presidents are women.

73. (a) All blue cars are old.
   (b) Some blue cars are not old.

74. (a) No football players are over 6 feet tall.
   (b) Every football player is over 6 feet tall.

Arguments

An argument is a collection of statements, listed in order. The last statement is called the conclusion and the other statements are called the premises. An argument is valid if the conjunction of all the premises implies the conclusion. The most common type of argument takes the following form.

Premise #1: \( p \rightarrow q \)
Premise #2: \( p \)
Conclusion: \( q \)

This form of argument is called the Law of Detachment or Modus Ponens. It is illustrated in the following example.
Example 1  A Valid Argument

Show that the following argument is valid.

Premise #1: If Sean is a freshman, then he is taking algebra.
Premise #2: Sean is a freshman.
Conclusion: So, Sean is taking algebra.

Solution

Let $p$ represent the statement “Sean is a freshman,” and let $q$ represent the statement “Sean is taking algebra.” Then the argument fits the Law of Detachment, which can be written as $[(p \rightarrow q) \land p] \rightarrow q$. The validity of this argument is shown in the following truth table.

Law of Detachment

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
<th>$(p \rightarrow q) \land p$</th>
<th>$[(p \rightarrow q) \land p] \rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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<td>T</td>
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</tbody>
</table>

Keep in mind that the validity of an argument has nothing to do with the truthfulness of the premises or conclusion. For instance, the following argument is valid—the fact that it is fanciful does not alter its validity.

Premise #1: If I snap my fingers, elephants will stay out of my house.
Premise #2: I am snapping my fingers.
Conclusion: So, elephants will stay out of my house.

Four Types of Valid Arguments

<table>
<thead>
<tr>
<th>Name</th>
<th>Pattern</th>
</tr>
</thead>
</table>
| 1. Law of Detachment or Modus Ponens | Premise #1: $p \rightarrow q$
|                                     | Premise #2: $p$
|                                     | Conclusion: $q$
| 2. Law of Contraposition or Modus Tollens | Premise #1: $p \rightarrow q$
|                                     | Premise #2: $\neg q$
|                                     | Conclusion: $\neg p$
| 3. Law of Transitivity or Syllogism | Premise #1: $p \rightarrow q$
|                                     | Premise #2: $q \rightarrow r$
|                                     | Conclusion: $p \rightarrow r$
| 4. Law of Disjunctive Syllogism      | Premise #1: $p \lor q$
|                                     | Premise #2: $\neg p$
|                                     | Conclusion: $q$
**Example 2** An Invalid Argument

Determine whether the following argument is valid.

Premise #1: If John is elected, the income tax will be increased.
Premise #2: The income tax was increased.
Conclusion: So, John was elected.

**Solution**

This argument has the following form.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>Premise #1: $p \rightarrow q$</td>
<td>[ (p \rightarrow q) \land q \rightarrow p ]</td>
</tr>
<tr>
<td>Premise #2: $q$</td>
<td>$p$</td>
</tr>
</tbody>
</table>

This is not one of the four valid forms of arguments that were listed. You can construct a truth table to verify that the argument is invalid, as follows.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
<th>$(p \rightarrow q) \land q$</th>
<th>$[(p \rightarrow q) \land q] \rightarrow p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

An invalid argument, such as the one in Example 2, is called a **fallacy**. Other common fallacies are given in the following example.

**Example 3** Common Fallacies

Each of the following arguments is invalid.

a. **Arguing from the Converse**: If the football team wins the championship, then students will skip classes. The students skipped classes. So, the football team won the championship.

b. **Arguing from the Inverse**: If the football team wins the championship, then students will skip classes. The football team did not win the championship. So, the students did not skip classes.

c. **Arguing from False Authority**: Wheaties are best for you because Joe Montana eats them.

d. **Arguing from an Example**: Beta brand products are not reliable because my Beta brand snowblower does not start in cold weather.
e. **Arguing from Ambiguity:** If an automobile carburetor is modified, the automobile will pollute. Brand X automobiles have modified carburetors. So, Brand X automobiles pollute.

f. **Arguing by False Association:** Joe was running through the alley when the fire alarm went off. So, Joe started the fire.

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**Example 4**  
**A Valid Argument**

Determine whether the following argument is valid.

Premise #1: You like strawberry pie or you like chocolate pie.

Premise #2: You do not like strawberry pie.

Conclusion: So, you like chocolate pie.

**Solution**

This argument has the following form.

Premise #1: \( p \lor q \)

Premise #2: \( \neg p \)

Conclusion: \( q \)

This argument is a disjunctive syllogism, which is one of the four common types of valid arguments.

---

In a valid argument, the conclusion drawn from the premise is called a **valid conclusion**.

**Example 5**  
**Making Valid Conclusions**

Given the following two premises, which of the conclusions are valid?

Premise #1: If you like boating, then you like swimming.

Premise #2: If you like swimming, then you are a scholar.

a. Conclusion: If you like boating, then you are a scholar.

b. Conclusion: If you do not like boating, then you are not a scholar.

c. Conclusion: If you are not a scholar, then you do not like boating.

**Solution**

a. This conclusion is valid. It follows from the Law of Transitivity or Syllogism.

b. This conclusion is invalid. The fallacy stems from arguing from the inverse.

c. This conclusion is valid. It follows from the Law of Contraposition.
Venn Diagrams and Arguments

Venn diagrams can be used to test informally the validity of an argument. For instance, a Venn diagram for the premises in Example 5 is shown in Figure E.1. In this figure, the validity of conclusion (a) is seen by choosing a boater \( x \) in all three sets. conclusion (b) is seen to be invalid by choosing a person \( y \) who is a scholar but does not like boating. Finally, person \( z \) indicates the validity of conclusion (c).

Venn diagrams work well for testing arguments that involve quantifiers, as shown in the next two examples.

Example 6 Using a Venn Diagram to Show That an Argument Is Not Valid

Use a Venn diagram to test the validity of the following argument.

Premise #1: Some plants are green.
Premise #2: All lettuce is green.
Conclusion: So, lettuce is a plant.

Solution

From the Venn diagram shown in Figure E.2, you can see that this is not a valid argument. Remember that even though the conclusion is true (lettuce is a plant), this does not imply that the argument is true.

When you are using Venn diagrams, you must remember to draw the most general case. For example, in Figure E.2 the circle representing plants is not drawn entirely within the circle representing green things because you are told that only some plants are green.

Example 7 Using a Venn Diagram to Show That an Argument Is Valid

Use a Venn diagram to test the validity of the following argument.

Premise #1: All good tennis players are physically fit.
Premise #2: Some golfers are good tennis players.
Conclusion: So, some golfers are physically fit.

Solution

Because the set of golfers intersects the set of good tennis players, as shown in Figure E.3, the set of golfers must also intersect the set of physically fit people. So, the argument is valid.
Proofs

What does the word proof mean to you? In mathematics, the word proof is used to mean simply a valid argument. Many proofs involve more than two premises and a conclusion. For instance, the proof in Example 8 involves three premises and a conclusion.

Example 8 A Proof by Contraposition

Use the following three premises to prove that “It is not snowing today.”

Premise #1: If it is snowing today, Greg will go skiing.
Premise #2: If Greg is skiing today, then he is not studying.
Premise #3: Greg is studying today.

Solution

Let \( p \) represent the statement “It is snowing today,” let \( q \) represent “Greg is skiing,” and let \( r \) represent “Greg is studying today.” So, the given premises have the following form.

Premise #1: \( p \rightarrow q \)
Premise #2: \( q \rightarrow \neg r \)
Premise #3: \( r \)

By noting that \( r = \neg (\neg r) \), reordering the premises, and writing the contrapositives of the first and second premises, you obtain the following valid argument.

Premise #3: \( r \)
Contrapositive of Premise #2: \( r \rightarrow \neg q \)
Contrapositive of Premise #1: \( \neg q \rightarrow \neg p \)
Conclusion: \( \neg p \)

So, you can conclude \( \neg p \). That is, “It is not snowing today.”

E.3 Exercises

In Exercises 1–4, use a truth table to show that the given argument is valid. See Example 1.

1. Premise #1: \( p \rightarrow \neg q \)
   Premise #2: \( q \)
   Conclusion: \( \neg p \)

2. Premise #1: \( p \leftrightarrow q \)
   Premise #2: \( p \)
   Conclusion: \( q \)

3. Premise #1: \( p \lor q \)
   Premise #2: \( \neg p \)
   Conclusion: \( q \)

4. Premise #1: \( p \land q \)
   Premise #2: \( \neg p \)
   Conclusion: \( q \)

In Exercises 5–8, use a truth table to show that the given argument is invalid. See Example 2.

5. Premise #1: \( \neg p \rightarrow q \)
   Premise #2: \( p \)
   Conclusion: \( q \)

6. Premise #1: \( p \rightarrow q \)
   Premise #2: \( \neg p \)
   Conclusion: \( \neg q \)

7. Premise #1: \( p \lor q \)
   Premise #2: \( q \)
   Conclusion: \( p \)

8. Premise #1: \( \neg (p \land q) \)
   Premise #2: \( q \)
   Conclusion: \( p \)
In Exercises 9–22, determine whether the argument is valid or invalid. See Example 4.

9. Premise #1: If taxes are increased, then businesses will leave the state.
   Premise #2: Taxes are increased.
   Conclusion: So, businesses will leave the state.

10. Premise #1: If a student does the homework, then a good grade is certain.
    Premise #2: Liza does the homework.
    Conclusion: So, Liza will receive a good grade for the course.

11. Premise #1: If taxes are increased, then businesses will leave the state.
    Premise #2: Businesses are leaving the state.
    Conclusion: So, taxes were increased.

12. Premise #1: If a student does the homework, then a good grade is certain.
    Premise #2: Liza received a good grade for the course.
    Conclusion: So, Liza did her homework.

13. Premise #1: If the doors are kept locked, then the car will not be stolen.
    Premise #2: The car was stolen.
    Conclusion: So, the car doors were unlocked.

14. Premise #1: If Jan passes the exam, she is eligible for the position.
    Premise #2: Jan is not eligible for the position.
    Conclusion: So, Jan did not pass the exam.

15. Premise #1: All cars manufactured by the Ford Motor Company are reliable.
    Premise #2: Lincolns are manufactured by Ford.
    Conclusion: So, Lincolns are reliable cars.

16. Premise #1: Some cars manufactured by the Ford Motor Company are reliable.
    Premise #2: Lincolns are manufactured by Ford.
    Conclusion: So, Lincolns are reliable.

17. Premise #1: All federal income tax forms are subject to the Paperwork Reduction Act of 1980.
    Premise #2: The 1040 Schedule A form is subject to the Paperwork Reduction Act of 1980.
    Conclusion: So, the 1040 Schedule A form is a federal income tax form.

18. Premise #1: All integers divisible by 6 are divisible by 3.
    Premise #2: Eighteen is divisible by 6.
    Conclusion: So, 18 is divisible by 3.

19. Premise #1: Eric is at the store or the handball court.
    Premise #2: He is not at the store.
    Conclusion: So, he must be at the handball court.

20. Premise #1: The book must be returned within 2 weeks or you pay a fine.
    Premise #2: The book was not returned within 2 weeks.
    Conclusion: So, you must pay a fine.

21. Premise #1: It is not true that it is a diamond and it sparkles in the sunlight.
    Premise #2: It does sparkle in the sunlight.
    Conclusion: So, it is a diamond.

22. Premise #1: Either I work tonight or I pass the mathematics test.
    Premise #2: I’m going to work tonight.
    Conclusion: So, I will fail the mathematics test.

In Exercises 23–30, determine which conclusion is valid from the given premises. See Example 5.

23. Premise #1: If 7 is a prime number, then 7 does not divide evenly into 21.
    Premise #2: Seven divides evenly into 21.
    (a) Conclusion: So, 7 is a prime number.
    (b) Conclusion: So, 7 is not a prime number.
    (c) Conclusion: So, 21 divided by 7 is 3.

24. Premise #1: If the fuel is shut off, then the fire will be extinguished.
    Premise #2: The fire continues to burn.
    (a) Conclusion: So, the fuel was not shut off.
    (b) Conclusion: So, the fuel was shut off.
    (c) Conclusion: So, the fire becomes hotter.

25. Premise #1: It is necessary that interest rates be lowered for the economy to improve.
    Premise #2: Interest rates were not lowered.
    (a) Conclusion: So, the economy will improve.
    (b) Conclusion: So, interest rates are irrelevant to the performance of the economy.
    (c) Conclusion: So, the economy will not improve.

26. Premise #1: It will snow only if the temperature is below 32° at some level of the atmosphere.
    Premise #2: It is snowing.
    (a) Conclusion: So, the temperature is below 32° at ground level.
    (b) Conclusion: So, the temperature is above 32° at some level of the atmosphere.
    (c) Conclusion: So, the temperature is below 32° at some level of the atmosphere.
27. Premise #1: Smokestack emissions must be reduced or acid rain will continue as an environmental problem.
Premise #2: Smokestack emissions have not decreased.
(a) Conclusion: So, the ozone layer will continue to be depleted.
(b) Conclusion: So, acid rain will continue as an environmental problem.
(c) Conclusion: So, stricter automobile emission standards must be enacted.

28. Premise #1: The library must upgrade its computer system or service will not improve.
Premise #2: Service at the library has improved.
(a) Conclusion: So, the computer system was upgraded.
(b) Conclusion: So, more personnel were hired for the library.
(c) Conclusion: So, the computer system was not upgraded.

29. Premise #1: If Rodney studies, then he will make good grades.
Premise #2: If he makes good grades, then he will get a good job.
(a) Conclusion: So, Rodney will get a good job.
(b) Conclusion: So, if Rodney doesn’t study, then he won’t get a good job.
(c) Conclusion: So, if Rodney doesn’t get a good job, then he didn’t study.

30. Premise #1: It is necessary to have a ticket and an ID card to get into the arena.
Premise #2: Janice entered the arena.
(a) Conclusion: So, Janice does not have a ticket.
(b) Conclusion: So, Janice has a ticket and an ID card.
(c) Conclusion: So, Janice has an ID card.

In Exercises 31–34, use a Venn diagram to test the validity of the argument. See Examples 6 and 7.

31. Premise #1: All numbers divisible by 10 are divisible by 5.
Premise #2: Fifty is divisible by 10.
Conclusion: So, 50 is divisible by 5.

32. Premise #1: All human beings require adequate rest.
Premise #2: All infants are human beings.
Conclusion: So, all infants require adequate rest.

33. Premise #1: No person under the age of 18 is eligible to vote.
Premise #2: Some college students are eligible to vote.
Conclusion: So, some college students are under the age of 18.

34. Premise #1: Every amateur radio operator has a radio license.
Premise #2: Jackie has a radio license.
Conclusion: So, Jackie is an amateur radio operator.

In Exercises 35–38, use the premises to prove the given conclusion. See Example 8.

35. Premise #1: If Sue drives to work, then she will stop at the grocery store.
Premise #2: If she stops at the grocery store, then she will buy milk.
Premise #3: Sue drove to work today.
Conclusion: So, Sue will buy milk.

36. Premise #1: If Bill is patient, then he will succeed.
Premise #2: Bill will get bonus pay if he succeeds.
Premise #3: Bill did not get bonus pay.
Conclusion: So, Bill is not patient.

37. Premise #1: If this is a good product, then we should buy it.
Premise #2: Either it was made by XYZ Corporation, or we will not buy it.
Premise #3: It is not made by XYZ Corporation.
Conclusion: So, it is not a good product.

38. Premise #1: If it is raining today, Pam will clean her apartment.
Premise #2: If Pam is cleaning her apartment today, then she is not riding her bike.
Premise #3: Pam is riding her bike today.
Conclusion: It is not raining today.