CHAPTER 8 Online Supplement

Coherence and the Dutch Book

Subjective probabilities must obey the same postulates and laws as so-called objective probabilities. That is, probabilities must be between 0 and 1, probabilities for mutually exclusive and collectively exhaustive sets of events must add to 1, and the probability of any event occurring from among a set of mutually exclusive events must be the sum of the probabilities of the individual events. All of the other properties of probabilities follow from these postulates, and subjective probabilities thus also must obey all of the probability “laws” and formulas that we studied in Chapter 7.

If an individual’s subjectively assessed probabilities obey the probability postulates, the person is said to be coherent. What benefit is there in being coherent? The mathematician de Finetti proved a famous theorem, the Dutch Book Theorem, which says that if a person is not coherent, then it is possible to set up a Dutch Book against him or her. A Dutch Book is a series of bets, each one of which is attractive on its own merits, but when taken together always results in a losing wager. That is, taken together, the bets guarantee your opponent will lose and you will win.

To illustrate that violating one of the three probability postulates results in a Dutch Book, suppose your friend Joe insisted on the following probabilities in the final championship game between the Lakers and Celtics:

- 0.4 chance that the Lakers win
- 0.5 chance that the Celtics win

You point out to Joe that his probabilities only sum to 0.9, but he insists that his probabilities are fine just the way they are. Perhaps Joe thinks another outcome could occur, but he acknowledges that either the Lakers will win or the Celtics will win. Let’s turn Joe’s misjudgment or incoherence into a Dutch Book.

We want to create a series of bets; in this case two, both of which Joe would take given his stated probabilities, but when taken together guarantee Joe losses money. To create these two bets, we will manipulate the dollar amounts (wagers) so that both bets are attractive to Joe. The two bets are:

**Bet 1** $(B_1)$

- Joe losses $L_1$ if Lakers win (probability: $p$)
- Joe wins $W_1$ if Lakers lose (probability: $1 - p$)
Bet 2 ($B_2$)  Joe losses $L_2$ if Celtics win  (probability: $q$)

Joe wins $W_2$ if Celtics lose  (probability: $1 - q$)

Note that for Joe $p = 0.4$ and $q = 0.5$.

We choose the values $W_1$, $W_2$, $L_1$, and $L_2$ to satisfy two conditions: (1) Joe finds both $B_1$ and $B_2$ attractive, and (2) taken together $B_1$ and $B_2$ guarantee Joe losses money. To satisfy condition (1), Joe would find the bets attractive if their expected values were positive, or at least nonnegative, i.e., $E(B_1) \geq 0$ and $E(B_2) \geq 0$. To simplify the mathematics, we assume the expected payoffs of $B_1$ and $B_2$ are equal to zero. Thus:

$$E(B_1) = p(-L_1) + (1 - p)W_1 = 0$$

$$E(B_2) = q(-L_2) + (1 - q)W_2 = 0$$

Using these two equations, we can solve for two of our four unknowns:

$$L_1 = \frac{1 - p}{p} W_1$$

$$L_2 = \frac{1 - q}{q} W_2$$

As long as $L_1$, and $L_2$ satisfy the above relationships, then Joe will take the bets. (Actually, Joe would be indifferent, because the expected value of each bet equals zero.)

To satisfy condition (2), we need Joe to lose money no matter what happens. As there are only two outcomes (Lakers win or Celtics win), we have:

$Lakers$ Win $\rightarrow$ Joe wins $W_1$ and loses $L_2$

$Celtics$ Win $\rightarrow$ Joe wins $W_2$ and loses $L_1$

Thus, to guarantee Joe loses money, we must have:

$$W_1 + (-L_2) < 0 \text{ or } W_1 < L_2$$

$$W_2 + (-L_1) < 0 \text{ or } W_2 < L_1.$$
Now, we are ready to create a Dutch Book based on Joe’s assessed probability values of $p = 0.4$ and $q = 0.5$. For example, if we choose $W_1 = $120, then we have:

$$L_1 = \frac{1 - p}{p} W_1 = \frac{0.6}{0.4} ($120) = $180,$$

$$L_2 = \frac{1 - q}{q} W_2 = \frac{0.5}{0.5} W_2 = W_2,$$

$$W_1 = $120 < L_2,$$ and

$$W_2 < L_1 = $180.$$

Hence, $W_1 = $120 implies $L_1 = $180, $L_2 = W_2$, and $120 < W_2 <$180. Thus, as long as we choose $W_2$ to be between $120$ and $180$, we will have constructed a Dutch Book. Choosing $W_2 = $150 implies $L_2 =$ $150 and our Dutch Book is:

**Bet 1 ($B_1$)**
- Joe losses $180 if Lakers win (probability: $p = 0.4$)
- Joe wins $120 if Lakers lose (probability: $1 - p = 0.6$)

**Bet 2 ($B_2$)**
- Joe wins $150 if Celtics lose (probability: $q = 0.5$)
- Joe losses $150 if Celtics win (probability: $1 - q = 0.5$)

It is easy to now check that the expected value of each bet equals zero and no matter who wins (Lakers or Celtics), Joe loses $30. We can do even better: if we add $1 to of Joe’s winnings, $121 if Lakers lose and $151 if Celtics lose, then the expected value of both bets is strictly greater than zero, and Joe will lose $29 no matter who wins.

Can we create a Dutch Book if the probabilities sum to one, i.e., $p + q = 1$? The short answer is no. To see why, we return to our four equations, substituting $1 - p = q$.

$$L_1 = \frac{1 - p}{p} W_1 = \frac{q}{p} W_1,$$

$$L_2 = \frac{1 - q}{q} W_2 = \frac{p}{q} W_2,$$

$$W_1 + (-L_2) < 0 \text{ or } W_1 < \frac{p}{q} W_2,$$ and

$$W_2 + (-L_1) < 0 \text{ or } W_2 < \frac{q}{p} W_1.$$
Thus, $W_1 < \frac{p}{q} W_2 < \frac{p}{q} (\frac{q}{p} W_1) = W_1$, which is impossible because $W_1$ cannot be strictly less than $W_1$. Thus, there is no way to construct a Dutch Book for Joe if he is coherent with his probability assessments.

This example points out how incoherence can be exploited. In fact, if a person’s probabilities do not conform to the probability laws, it is always possible to create a Dutch Book, no matter which probability law is violated. The contribution of de Finetti was to prove that it is only possible not to be exploited if subjective probabilities obey the probability postulates and laws. The practical importance of this is not that you can set up Dutch Books against incoherent probability assessors—because no one in his or her right mind would agree to such a series of bets—but to provide insight into why subjective probability should work the same way as “long-run frequency” probability. Because no one would make decisions in a way that would guarantee their losing money, coherence is a reasonable condition to guide our assessments of probabilities for decision-making purposes. When we assess probabilities for use in a decision tree or influence diagram, those probabilities should obey all the normal probability properties.

The idea of coherence has an important implication for assessment when a decision analysis involves assessing several probabilities. Once all probabilities have been assessed, it is important to check for coherence. If the assessments are not coherent, the decision maker should be made aware of this and given the chance to adjust the assessments until they are coherent.

Problems

8S.1 It is not necessary to have someone else set up a series of bets against you in order for incoherence to take its toll. It is conceivable that one could inadvertently get oneself into a no-win situation through inattention to certain details and the resulting incoherence, as this problem shows.

Suppose that an executive of a venture-capital investment firm is trying to decide how to allocate his funds among three different projects, each of which requires a $100,000 investment. The projects are such that one of the three will definitely succeed, but it is not possible for more than one to succeed. Looking at each project as an investment, the anticipated payoff is good, but not wonderful. If a project succeeds, the payoff will be a net gain of $150,000. Of course, if the project fails, he loses all of the money invested in that project. Because he feels as though he knows nothing about whether a project will succeed or fail, he assigns a probability of 0.5 that each project will succeed, and he decides to invest in
each project.

a According to his assessed probabilities, what is the expected profit for each project?

b What are the possible outcomes of the three investments, and how much will he make in each case?

c Do you think he invested wisely? Can you explain why he is in such a predicament?

d If you feel you know nothing about some event, is it reasonable to assess equal probabilities for the outcomes? Give an example where this might be reasonable and another where it might not.

8S.2 Could you ever set up a Dutch book against a bookie who places bets for a living? Work through the following problem and think about this question.

Consider a baseball season when the Chicago Cubs and the New York Yankees meet in the World Series. A friend of yours is a Cubs fan, and you are trying to find out how confident he is about the Cubs’ chances of winning. He tells you that he would bet on the Cubs at odds of 3:2 or better and on the Yankees at odds of 1:2 or better. [Odds of $a:b$ for an event means the probability of the event is $\frac{a}{a + b}$.] This means that he would be happy with either of the following bets:

**Bet 1** He wins $20 if the Cubs win.
He loses $30 if the Yankees win.

**Bet 2** He loses $20 if the Cubs win.
He wins $40 if the Yankees win.

a Explain why he would be happy to accept any modification of these bets as long as the amount he would win increases and the amount he would lose stays the same or is lower. Explain in terms of his expected bet value.

b Show mathematically that $0.60 \leq P(\text{Cubs Win}) \leq 0.67$. (Hint: If he is willing to accept either bet, what does that imply about his expected value for each bet?)

c Would it be possible to set up a Dutch book against this individual? If your answer is yes, what bets would you place, and how much would you be sure to win? If your answer is no, explain why not. (Be careful! Make sure he is willing to accept the bets you propose.)

d Most individuals have a hard time assessing subjective probabilities with a very high degree of precision. (See Problem 8.23 n the text.) Does this problem shed any light on the issue?