Solutions for *Microeconomics: An Intuitive Approach with Calculus* (International Ed.)

Apart from end-of-chapter exercises provided in the student Study Guide, these solutions are provided for use by instructors. (End-of-Chapter exercises with solutions in the student Study Guide are so marked in the textbook.)

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- Each end-of-chapter exercise begins on a new page. This is to facilitate maximum flexibility for instructors who may wish to share answers to some but not all exercises with their students.

- If you are assigning only the A-parts of exercises in *Microeconomics: An Intuitive Approach with Calculus*, you may wish to instead use the solution set created for the companion book *Microeconomics: An Intuitive Approach*.

- *Solutions to Within-Chapter Exercises are provided in the student Study Guide.*
Strategic Thinking and Game Theory

24.1 In the Hollywood movie “A Beautiful Mind”, Russell Crowe plays John Nash who developed the Nash Equilibrium concept in his PhD thesis at Princeton University. In one of the early scenes of the movie, Nash finds himself in a bar with three of his fellow (male) mathematics PhD students when a group of five women enters the bar. The attention of the PhD students is focused on one of the five women, with each of the four PhD students expressing interest in asking her out. One of Nash’s fellow students reminds the others of Adam Smith’s insight that pursuit of self interest in competition with others results in the socially best outcome, but Nash — in what appears to be a flash of insight — claims “Adam Smith needs revision.”

A: In the movie, John Nash then explains that none of them will end up with the woman they are all attracted to if they all compete for her because they will block each other as they compete — and that furthermore they will not be able to go out with the other women in the group thereafter (because none of them will agree to a date once they know they are at best everyone’s second choice). Instead, he proposes, they should all ignore the woman they are initially attracted to and instead ask the others out — it’s the only way they will get a date. He quickly rushes off to write his thesis — with the movie implying that he had just discovered the concept of Nash Equilibrium.

(a) If each of the PhD students were to play the strategy John Nash suggests — i.e. each one selects a woman other than the one they are all attracted to, could this in fact be a pure strategy Nash Equilibrium?

Answer: No, it could not. In order for this to be a pure strategy Nash Equilibrium, it must be the case that everyone is playing a best response relative to everyone else. If everyone else is ignoring the woman that is deemed most attractive, it is not a best response for any single player to also ignore the woman. Thus, none of the PhD students is playing a best response strategy to the others if they are all ignoring the woman they find most attractive.

(b) Is it possible that any pure strategy Nash equilibrium could result in no one pursuing the woman they are all attracted to?

Answer: No, it is not possible — since this would imply everyone choosing to ignore the woman deemed most attractive — which would in turn imply that a single player’s best response is to approach this woman.

(c) Suppose we simplified the example to one in which it was only Nash and one other student encountering a group of two women. We then have two pure strategies to consider for each PhD student: Pursue woman A or pursue woman B. Suppose that each viewed a date with woman A as yielding a “payoff” of 10 and a date with woman B as yielding a payoff of 5. Each will in fact get a date with the woman that is approached if they approach different women, but neither will get a date if they approach the same woman in which case they both get a payoff of 0. Write down the payoff matrix of this game.

Answer: Table 24.1 gives the payoff matrix to this game.

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>0,0</td>
<td>10,5</td>
</tr>
<tr>
<td>B</td>
<td>5,10</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Table 24.1: Simplified Version of Game from “A Beautiful Mind”

(d) What are the pure strategy Nash Equilibria of this game?

Answer: The pure strategy equilibria of this game are \{A,B\} and \{B,A\} — where the first action in each pairing is player 1’s strategy and the second action is player 2’s strategy.

(e) Is there a mixed strategy Nash Equilibrium in this game?

Answer: Yes, there is a mixed strategy equilibrium. Consider first the probability \(\rho\) placed by player 1 on \(A\). We know that if \(\rho = 1\), player 2’s best response is to play \(B\). Letting \(\lambda\) denote

\[1\] Nash is actually with 4 others, but the rest of the scene unfolds as if there were 4 of them in total.
player 2’s probability of playing $A$, we therefore know that player 2’s best response to $ρ = 1$ is to play $λ = 0$ — which continues to be the case so long as $ρ > 2/3$. This is because player 2’s payoff from playing $A$ is $0(ρ) + 10(1 − ρ) = 10(1 − ρ)$ and player 2’s payoff from playing $B$ is $5ρ + 0(1 − ρ) = 5ρ$ — and these are equal when $ρ = 2/3$. Thus, when $ρ = 2/3$, player 2 will best respond with any $λ$ ranging from 0 and 1 (and when $ρ < 2/3$, player 2 best responds with $λ = 1$). This gives the solid best response function in Graph 24.1. Similarly, for player 1 the payoff from playing $A$ is $0(λ) + 10(1 − λ) = 10(1 − λ)$, and his payoff from playing $B$ is $5A + 0(1 − λ) = 5λ$. He is therefore indifferent between playing $A$ and $B$ when $λ = 2/3$ — implying he would best respond by setting $ρ$ anywhere from 0 to 1 when $λ = 2/3$ but best responds by playing $ρ = 0$ when $λ > 2/3$ and by playing $ρ = 1$ when $λ < 2/3$. This gives the dashed best response function for player 1 in Graph 24.1. The mixed strategy equilibrium occurs where the two best response functions intersect — i.e. at $(ρ = 2/3, λ = 2/3)$. Note that $(ρ = 0, λ = 1)$ and $(ρ = 1, λ = 0)$ are also at intersections of the two best response functions — and these are equivalent to the pure strategy Nash equilibria we found in part (d).

![Graph 24.1: Mixed Strategy Best Responses](image)

(f) Now suppose there is also a woman $C$ in the group of women — and a date with $C$ is viewed as equivalent to a date with $B$. Again, each PhD student gets a date if he is the only one approaching a woman, but if both approach the same woman, neither gets a date (and thus both get a payoff of zero). Now, however, the PhD students have 3 pure strategies: $A$, $B$ and $C$. Write down the payoff matrix for this game.

**Answer:** This is illustrated in Table 24.2.

|       | Player 2
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>$A$</td>
<td>$B$</td>
</tr>
<tr>
<td>$A$</td>
<td>0,0</td>
<td>10,5</td>
</tr>
<tr>
<td>$B$</td>
<td>5,10</td>
<td>0,0</td>
</tr>
<tr>
<td>$C$</td>
<td>5,10</td>
<td>5,5</td>
</tr>
</tbody>
</table>

Table 24.2: Simplified Version 2 of Game from “A Beautiful Mind”
(g) What are the pure strategy Nash Equilibria of this game? Does any of them involve woman A leaving without a date?

**Answer:** We now have the following Nash equilibria: \{(A,B), (A,C), (B,A), (C,A)\} (where the first action is again the strategy of player 1 and the second is the strategy of player 2). In each of these, one of the students goes out with woman A.

(h) In the movie, Nash then explains that "Adam Smith said the best result comes from everyone in the group doing what’s best for themselves." He goes on to say "...incomplete ... incomplete ... because the best result will come from everyone in the group doing what’s best for themselves and the group ... Adam Smith was wrong." Does the situation described in the movie illustrate any of this?

**Answer:** Not really. It is true that each Nash equilibrium is efficient in the sense that the sum of the payoffs is the largest it can possibly be — but that would in fact illustrate that the "best result comes from everyone in the group doing what’s best for themselves". No one in the game considers what’s best for the group.

(i) While these words have little to do with the concept of Nash Equilibrium, in what way does game theory — and in particular games like the Prisoners’ dilemma — challenge the inference one might draw from Adam Smith that self interest achieves the "best" outcome for the group?

**Answer:** It is true that game theory can illustrate how the “best” outcome for the group is not obtained by each individual doing what is best for himself. The prisoners’ dilemma is one game where this is the case — the Nash equilibrium is one where both the individuals and the group do worse than they could if they considered the group. Put differently, the Nash equilibrium in the prisoners’ dilemma is inefficient — and the individuals could in fact do better if they found a way to force each other to go against their self-interest. But that is not something that will happen as a result of Nash equilibrium behavior — the Nash equilibrium in that case simply illustrates that self-interest does not always serve the group (or the individual) when we leave the idealized setting of the first welfare theorem. The game depicted in the movie, however, does not illustrate anything like this when properly modeled and analyzed.

B: Consider the 2-player game described in part A(c). (Note: Part (a) and (b) below can be done without having read Section B of the Chapter.)

(a) Suppose that the players move sequentially — with player 1 choosing A or B first — and player 2 making his choice after observing player 1’s choice. What is the subgame perfect Nash equilibrium?

**Answer:** Solving from the end of the game, we first consider what player 2 will do if he observes player 1 having approached woman A or B. In both cases, it is optimal for player 2 to choose the other woman. Player 1 can predict this — and therefore will approach woman A. The subgame perfect equilibrium is therefore player 1 playing A and player 2 playing \((A,A)\) — with the first action indicating what player 2 will do if he observes A and the second indicating what he will do if he observes B on the part of player 1.

(b) Is there a Nash equilibrium in which player 2 goes out with woman A? If so, is there a non-credible threat that is needed to sustain this as an equilibrium?

**Answer:** Player 1 playing B and player 2 playing \((A,A)\) is a Nash equilibrium. (Given that player 1 plays B, it is a best response for player 2 to always choose A — and given that player 2 always chooses A, it is a best response by player 1 to choose B. But this Nash equilibrium requires that player 2 threatens to play A even if player 1 plays A in the first stage of the game — and that player 1 believes this threat. It is not, however, a credible threat since player 2 will optimally choose B if he observes A having been chosen in stage 1.

(c) Next, consider again the simultaneous move game from A(c). Draw a game tree for this simultaneous move game — with player 1’s decision on the top. (Hint: Use the appropriate information set for player 2 to keep this game a simultaneous move game). Can you state different beliefs for player 2 (when player 2 gets to his information set) such that the equilibria you derived in A(d) and A(e) arise?

**Answer:** This is illustrated in panel (a) of Graph 22.5 (next page). Player 1 starts by choosing A or B — and player 2 goes next but is unable to tell whether player 1 chose A or B. Thus, the two nodes for player 2 are contained in one information set. This is equivalent to the two

players moving simultaneously — because player 2 has no more information than he would if he were to move simultaneously. Let \( \rho \) be the probability with which player 2 thinks he is at the first node when it is his turn. His payoff from going A is then \( 0 \rho + 10(1 - \rho) = 10(1 - \rho) \), and his payoff from going B is \( 5 \rho + 0(1 - \rho) = 5 \rho \). Player 2 is therefore indifferent between moving A and moving B if \( 10(1 - \rho) = 5 \rho \) — i.e. if \( \rho = 2/3 \). Given such indifference, it would then be one (of many) best responses for player 2 to choose A with probability \( \gamma = 2/3 \) and B with probability \( (1 - \gamma) = 1/3 \) — and if player 2 plays this way, it is player 1’s best response to play A with probability \( \rho = 2/3 \). This is the mixed strategy equilibrium from part A(e) — and it requires player 2 to believe with probability 2/3 that player 1 went A — which in turn is consistent with individual 1’s strategy. If player 2 believes player 1 went A with probability greater than 2/3, it is a best response for individual 2 to choose B with probability 1 — and if player 2 chooses B with probability 1, player 1’s best response is to choose A with probability 1. Thus, the equilibrium in which player 1 chooses A and player 2 chooses B requires player 2 to believe individual 1 chooses A with probability 1. This is one of the two pure strategy equilibria found in A(d). The other is similar, with player 2 now believing that player 1 will play B with probability 1.

Graph 24.2: Nash and Friend

(d) Continue to assume that both players get payoff of 0 if they approach the same woman. As before, player 1 gets a payoff of 10 if he is the only one to approach woman A and a payoff of 5 if he is the only one to approach woman B. But player 2 might be one of two possible types: if he is type 1, he has the same tastes as player 1, but if he is of type 2, he gets a payoff of only 5 if he is the only one to approach woman A and a payoff of 10 if he is the only one to approach women B. Prior to the beginning of the game, Nature assigns type 1 to player 2 with probability \( \delta \) (and thus assigns type 2 to player 2 with probability \( 1 - \delta \)). Graph the game tree for this game — using information sets to connect nodes where appropriate.

Answer: This is done in panel (b) of Graph 24.4.

(e) What are the pure strategy equilibria in this game? Does it matter what value \( \delta \) takes?

Answer: In any pure strategy equilibrium, player 1 either plays A or he plays B. If he plays A, then player 2 will best respond by playing B regardless of what type he was assigned. Similarly, if player 1 plays B, player 1 will best respond by playing A regardless of what type he was assigned. Thus, we have two pure strategy Nash equilibria: (1) Player 1 plays A and player 2 plays (B, B) — i.e. B if type 1 and B if type 2; and (2), player 1 plays B and player 1 plays (A, A). The value of \( \delta \) is irrelevant.
24.2 Consider a sequential game which is known as the Centipede Game. In this game, each of two players chooses between “Left” and “Right” each time he or she gets a turn. The game does not, however, automatically proceed to the next stage unless players choose to go “Right” rather than “Left”.

As Player 1 begins — and if he plays “Left”, the game ends with payoff of (1,0) (where here, and throughout this exercise, the first payoff refers to player 1 and the second to player 2). If however, he plays “Right”, the game continues and it’s player 2’s turn. If player 2 then plays “Left”, the game once again ends, this time with payoffs (0,2), but if she plays “Right”, the game continues and player 1 gets another turn. Once again, the game ends if player 1 decides to play “Left” — this time with payoffs of (3,1), but if he plays “Right” the game continues and it’s once again player 2’s turn. Now the game ends regardless of whether player 2 plays “Left” or “Right”, but payoffs are (2,4) if she plays “Left” and (3,3) if she plays “Right”.

(a) Draw out the game tree for this game. What is the subgame perfect Nash Equilibrium of this game.

Answer: The game tree for this Centipede Game is illustrated in Graph 24.3. To solve for the subgame perfect Nash equilibrium, we start at the bottom and work our way up. In the final stage, player 2 chooses between a payoff of 4 if she goes \(L\) and a payoff of 3 if she goes \(R\). She will therefore go \(L\). That means that, when player 1 chooses between \(L\) and \(R\) in the second to last stage, he is really choosing between a payoff of 3 and 2 (since he knows player 2 will go \(L\) if he goes \(R\)) — and he will therefore go \(L\). Player 2 knows this — and she therefore chooses between a payoff of 2 and a payoff of 1 when she chooses between \(L\) and \(R\) in the second stage of the game — which implies she will go \(L\). Finally, player 1 knows all this in the very first stage — and thus knows that, in choosing between \(L\) and \(R\) at the beginning of the game, he is really choosing between a payoff of 1 and 0. He therefore chooses \(L\). The subgame perfect equilibrium therefore involves both players playing \(L\) at every point of the game — i.e. the equilibrium is \(\{(L, L), (L, L)\}\), where the first pair indicates player 1’s strategy and the second pay indicates player 2’s strategy.

(b) Write down the 4 by 4 payoff matrix for this game. What are the pure strategy Nash Equilibria in this game? Is the subgame perfect Nash Equilibrium you derived in (a) among these?

Answer: The payoff matrix is written in Table 24.3. There are four pure strategy Nash equilibria in this game: \(\{(L, L), (L, L)\}\), \(\{(L, L), (R, R)\}\), \(\{(L, R), (L, L)\}\) and \(\{(L, R), (R, R)\}\) — with each resulting in payoffs (1,0). The first of these is the subgame perfect Nash Equilibrium.
(c) Why are the other Nash Equilibria in the game not subgame perfect?

Answer: The other Nash equilibria are not subgame perfect because they have at least one player playing a non-credible strategy — i.e. a strategy that involves taking an action at a point in the game that is not in that player’s interest.

(d) Suppose you changed the (2,4) payoff pair to (2,3). Do we now have more than 1 subgame perfect Nash Equilibrium?

Answer: Yes. Beginning again in the last stage, player 2 would end up with a payoff of 3 regardless of which way she chose — and thus both going L and R in the last stage can be part of a subgame perfect strategy for player 2. If she goes L, then the subgame perfect equilibrium will be exactly the same as the one we derived in part (a) — i.e. (L, L), (L, L)).

If, however, player 2 plays R in the final stage, then player 1 ends up with a payoff of 3 regardless of which way he goes in the third stage — making both L and R subgame perfect when player 2 plays R in the final stage. But if player 1 plays L in the third stage, we again get L played in the first two stages — giving us the subgame perfect equilibrium (L, L), (L, R)) (which has an outcome no different from the equilibrium we previously identified.) But if player 1 goes R in the second to last stage (when player 2 goes R in the last stage), player 2 is best off going R in the second stage and player 1 is best off going R in the first stage. Thus, ((R, R), (R, R)) is also a subgame perfect equilibrium.

(e) How does your answer to (b) change?

Answer: The new payoff matrix would now be the one given in Table 24.4. The same four equilibria as in (b) are still Nash equilibria now — including the first subgame perfect Nash equilibrium (L, L), (L, L)). However, there is now one additional Nash Equilibrium — ((R, R), (R, R)). This was not a Nash Equilibrium before because, when player 1 plays (R, R), player 2’s best response was to play (R, L) before we changed the payoffs — but now it is also a best response to play (R, R).

(f) Consider again the original game but suppose I came as an outsider and offered to change the payoff pairs in the final stage from (2,4) and (3,3) to (2,2) and (4,4). How much would each of the two players be willing to pay me to change the game in this way (assuming we know that players always play subgame perfect equilibria)?
Answer: By changing the payoffs in the final stage in this way, only \((R, R), (R, R)\) is a subgame perfect equilibrium when before only \((L, L), (L, L)\) was a subgame perfect equilibrium. Thus, the equilibrium payoff outcome changes from \((1, 0)\) to \((4, 4)\) — which implies that player 1 would be willing to pay 3 and player 2 would be willing to pay 4 in order for the payoffs in the final stage to change in this way.

B: Consider the original centipede game described in part A. Suppose that, prior to the game being played, Nature moves and assigns a type to player 2, with type 1 being assigned with probability \(\rho\) and type 2 with probability \((1 - \rho)\). Throughout, type 1 is a rational player who understands subgame perfection.

(a) Suppose type 2 is a super-naive player that simply always goes “Right” whenever given a chance. For what values of \(\rho\) will player 1 go “Right” in the first stage?

Answer: In this case, the expected payoff from going \(L\) in the first stage is 1 while the expected payoff from going \(R\) is \(0\rho + 3(1 - \rho)\). Player 1 will therefore go \(R\) if \(3(1 - \rho) \geq 1\) or just \(\rho \leq 2/3\).

(b) Suppose instead that type 2 always goes “Right” the first time and “Left” the second time. How does your answer change?

Answer: The answer does not change because player 1 will still be able to get a payoff of 3 by going \(R\) first — followed by \(L\) the second time he moves — if player 2 goes \(R\) the first time.

(c) (Note: This (and the next) part requires that you have read Chapter 17.) We have not explicitly mentioned this in the chapter — but game theorists often assume that payoffs are given in utility terms, with utility measured by a function \(u\) that allows gambles to be represented by an expected utility function. Within the context of this exercise, can you see why?

Answer: In our answers above, we compared a sure thing — i.e. a payoff of 1 if player 1 goes \(L\) in the first stage, to an expected payoff \(0\rho + 3(1 - \rho)\) resulting from the fact that going \(R\) implies getting 0 with probability \(\rho\) and 3 with probability \((1 - \rho)\). If the outcomes are expressed in utility terms that allow for the use of an expected utility function, then the expected utility of going \(R\) is in fact just the probability weighted average of the utilities associated with the two possible outcomes.

(d) Suppose the payoffs in the centipede game are in dollar terms, not in utility terms. What do your answers to (a) and (b) assume about the level of risk aversion of player 1?

Answer: Our answers then assumed risk neutrality — i.e. the payoff of 1 for sure was just as good as a risky gamble with the same expected value.
24.3 Consider a simultaneous game in which both players choose between the actions “Cooperate”, denoted by C, and “Defect”, denoted by D.

A: Suppose that the payoffs in the game are as follows: If both players play C, each gets a payoff of 1; if both play D, both players get 0; and if one player plays C and the other plays D, the cooperating player gets α while the defecting player gets β.

(a) Illustrate the payoff matrix for this game.

Answer: This is illustrated in Table 24.5.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1, 1</td>
<td>β, α</td>
</tr>
<tr>
<td>D</td>
<td>β, α</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Table 24.5: Potentially a Prisoners’ Dilemma Game

(b) What restrictions on α and β would you have to impose in order for this game to be a Prisoners’ Dilemma? Assume from now on that these restrictions are in fact met.

Answer: In order for this to be a Prisoners’ Dilemma game, it must be that D is a dominant strategy for both players. Consider player 2 first: If player 1 plays C, it must be that it is optimal for player 2 to play D — which implies it must be that β > 1. Similarly, if player 1 plays D, it must be that D is optimal for player 2 — which implies it must be that α < 0. Player 1’s situation is symmetric — so the same restrictions will work to make D a dominant strategy for player 1. This is therefore a prisoners’ dilemma game if α < 0 and β > 1.

B: Now consider a repeated version of this game in which players 1 and 2 meet 2 times. Suppose you were player 1 in this game, and suppose that you knew that player 2 was a “Tit-for-Tat” player — i.e. a player that does not behave strategically but rather is simply programmed to play the Tit-for-Tat strategy.

(a) Assuming you do not discount the future, would you ever cooperate with this player?

Answer: You know that player 2 will play C the first time and will then mimic what you do the first time when you meet the second time. You have four possible pure strategies to play: (C, C), (C, D), (D, C) and (D, D). Assuming you do not discount the future, your payoffs from these will be (1 + α), (1 + β), (β + α) and (β + 0) respectively. Since β > 1 and α < 0, we know that

\[(1 + \beta) > (1 + \alpha), \quad (\beta + 1) > (\beta + \alpha) \quad \text{and} \quad (1 + \beta) > (\beta + 0).\]  \hspace{1cm} (24.1)

The first inequality implies that (C, D) strictly dominates (C, C) and you will therefore not play (C, C). The second inequality implies that (D, D) strictly dominates (D, C) and you will therefore not play (D, C). And the third equality implies that (C, D) strictly dominates (D, D) and you will therefore not play (D, D). Thus, you will play (C, D) — and you will therefore cooperate in the first stage.

(b) Suppose you discount a dollar in period 2 by δ where 0 < δ < 1. Under what condition will you cooperate in this game?

Answer: Your payoffs from strategies (C, C), (C, D), (D, C) and (D, D) will now be (1 + δ), (1 + δβ), (β + δα) and (β + 0) respectively. We can then say unambiguously that

\[(1 + \delta \beta) > (1 + \delta), \quad (\beta + 1) > (\beta + \delta \alpha),\]  \hspace{1cm} (24.2)

which implies that (C, D) strictly dominates (C, C) and (D, D) strictly dominates (D, C). You will therefore definitely never play (C, C) or (D, C). This leaves only (C, D) and (D, D), with payoffs of (1 + δβ) and (β + 0) respectively. The former is larger than the latter so long as

\[\beta < \frac{1}{(1 - \delta)}.\]  \hspace{1cm} (24.3)
Strategic Thinking and Game Theory

which implies you will play \((C, D)\) when this condition holds and \((D, D)\) otherwise. (Both are possible if the equation holds with equality.) Thus, if \(\beta\) is sufficiently large relative to the discount factor \(\delta\), you will still cooperate in the first stage.

(c) Suppose instead that the game was repeated 3 rather than 2 times. Would you ever cooperate with this player (assuming again that you don’t discount the future)? (Hint: Use the fact that you should know the best action in period 3 to cut down on the number of possibilities you have to investigate.)

Answer: In the third encounter, it will always be best to play \(D\) since \(D\) is a dominant strategy in the single shot game. Thus, the only question is what you would do in the first and second encounter. We therefore have 4 possible strategies to consider: \((C, C, D)\), \((C, D, D)\), \((D, C, D)\) and \((D, D, D)\). The payoffs (given that the other player plays Tit-for-Tat) for these strategies are: \((1 + 1 + \beta)\), \((1 + \beta + 0)\), \((\beta + 0 + \beta)\) and \((\beta + 0 + 0)\) respectively. Since

\[
(1 + 1 + \beta) > (1 + \beta + 0) > (\beta + 0 + \beta),
\]

we know that \((C, C, D)\) dominates \((C, D, D)\) which dominates \((D, D, D)\) and thus neither \((C, D, D)\) nor \((D, D, D)\) get played. That leaves only \((C, C, D)\) and \((D, C, D)\) to consider, and \((C, C, D)\) dominates \((D, C, D)\) so long as \((1 + 1 + \beta) > (\beta + 0 + 0)\). This simplifies to

\[
\beta < 2 - \alpha,
\]

and as long as this condition holds, you will play \((C, C, D)\). If, however, \(\beta > 2 - \alpha\), you will play \((D, C, D)\) (and if \(\beta = 2 - \alpha\), either of the two strategies will maximize your payoff.) In either case, you will cooperate at some point, though only under \((C, C, D)\) do both players ever cooperate at the same time.

(d) In the repeated game with 3 encounters, what is the intuitive reason why you might play \(D\) in the first stage?

Answer: If the reward from defecting when your opponent cooperates is sufficiently large relative to the loss one takes when cooperating in the face of the opponent defecting, it makes sense to take advantage of the Tit-for-Tat opponent right away, and then trick him into cooperating again in the last stage.

(e) If player 2 is strategic, would he ever play the “Tit-for-Tat” strategy in either of the two repeated games?

Answer: No, it would not make sense because player 2 should realize that player 1 will play \(D\) in the last encounter. This implies that the last encounter does not matter for what action will be played in the second to last encounter — which again implies player 1 will play \(D\). Thus, the logic of subgame perfection should imply that player 2 will play \(D\) always.

(f) Suppose that each time the two players meet, they know they will meet again with probability \(\gamma > 0\). Explain intuitively why “Tit-for-Tat” can be an equilibrium strategy for both players if \(\gamma\) is relatively large (i.e. close to 1) but not if it is relatively small (i.e. close to 0).

Answer: If \(\gamma\) is close to 1, the probability of meeting again is large. Thus, the short term payoff for player 1 from deviating from “Tit-for-Tat” and playing \(D\) without provocation is outweighed by the fact that the opponent will now play \(D\) until player 1 unilaterally starts playing \(C\) again. Put differently, rather than getting a payoff of 1 by playing \(C\) against the “Tit-for-Tat” strategy this period, player 1 can get \(\beta > 1\), but it also implies that player 1 will face payoffs of 0 (rather than 1) from now on as both players switch to \(D\), or player 1 will have to incur a payoff of \(\alpha < 1\) (rather than 1) in a future period in order to get his opponent to cooperate again. If the chance of meeting again is sufficiently large, that’s not worth it. If it is sufficiently small, however, it makes sense to graph \(\beta\) while you can. Thus, “Tit-for-Tat” can be a best response to “Tit-for-Tat” only if the chance of another encounter is large enough.
24.4 Interpreting Mixed Strategies in the Battle of the Sexes. One of the most famous games treated in early game theory courses is known as the "Battle of the Sexes" — and it bears close resemblance to the game in which you and I choose sides of the street when you are British and I am American. In the "Battle of the Sexes" game, two partners in a newly blossoming romance have different preferences for what to do on a date, but neither can stand the thought of not being with the other. Suppose we are talking about you and your partner. You love opera and your partner loves football.

Both you and your partner can choose to go to the opera and today's football game, with each of you getting 0 payoff if you aren't at the same activity as the other, 10 if you are at your favorite activity with your partner, and 5 if you are at your partner's favorite activity with him/her.

A: In this exercise, we will focus on mixed strategies.

(a) Begin by depicting the game in the form of a payoff matrix.

Answer: This is depicted in Table 24.6.

<table>
<thead>
<tr>
<th></th>
<th>Partner Opera</th>
<th>Partner Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>10,5</td>
<td>0,0</td>
</tr>
<tr>
<td>Opera</td>
<td>0,0</td>
<td>5,10</td>
</tr>
<tr>
<td>Football</td>
<td>0,0</td>
<td>5,10</td>
</tr>
</tbody>
</table>

Table 24.6: Battle of the Sexes

(b) Let $\rho$ be the probability you place on going to the opera, and let $\delta$ be the probability your partner places on going to the opera. For what value of $\delta$ are you indifferent between showing up at the opera or showing up at the football game?

Answer: Your payoff from going to the opera is $10\delta + (1 - \delta)0 = 10\delta$. Your payoff from going to the football game is $0\delta + 5(1 - \delta) = 5(1 - \delta)$. You are therefore indifferent if $10\delta = 5(1 - \delta)$ which solves to give us $\delta = 1/3$.

(c) For what values of $\rho$ is your partner indifferent between these two actions?

Answer: Your partner's payoff from going to the opera is $5\rho + 0(1 - \rho) = 5\rho$, and the payoff from going to the football game is $0\rho + 10(1 - \rho) = 10(1 - \rho)$. Your partner is therefore indifferent between going to the opera and going to the football game if $10(1 - \rho) = 5\rho$ — which solves to $\rho = 2/3$.

(d) What is the mixed strategy equilibrium to this game?

Answer: We just concluded that you are indifferent between playing opera and football if $\delta = 1/3$ — and your partner is indifferent if $\rho = 2/3$. Thus, if your partner plays $\delta = 1/3$, it is a best response for you to play any mixed strategy — including $\rho = 2/3$; and if you play $\rho = 2/3$, it is a best response for your partner to play any mixed strategy including $\delta = 1/3$. The mixed strategy Nash equilibrium therefore is $\rho = 2/3$ and $\delta = 1/3$.

(e) What are the expected payoffs for you and your partner in this game?

Answer: The probability of both showing up at the opera is $2/3(1/3) = 2/9$, and the probability of both showing up to the football game is also $1/3(2/3) = 2/9$. This implies that both of you get a payoff of $10(2/9) + 5(2/9) = 30/9 = 10/3$.

B: In the text, we indicated that mixed strategy equilibria in complete information games can be interpreted as pure strategy equilibria in a related incomplete information game. We will illustrate this here. Suppose that you and your partner know each other's ordinal preferences over opera and football — but you are not quite sure just how much the other values the most preferred outcome. In particular, your partner knows your payoff from both showing up at the football game is 5, but he thinks your payoff from both showing up at the opera is $10 + \alpha$ with some uncertainty about what

---

2Since this game dates back quite a few decades, you can imagine which of the two players was referred to as the “husband” and which as the “wife” in early incarnation. I will attempt to write this problem without any such gender (or other) bias and apologize to the reader if he/she is not a fan of opera.
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exactly \( a \) is. Similarly, you know your partner gets a payoff of 5 if both of you show up at the opera, but you think his/her payoff from both showing up at the football game is \((10 + \beta)\), with you unsure of what exact value \( \beta \) takes. We will assume that both \( a \) and \( \beta \) are equally likely to take any value in the interval from 0 to \( x \); i.e. \( a \) and \( \beta \) are drawn randomly from a uniform distribution on \([0,x]\).

We have thus turned the initial complete information game into a related incomplete information game in which your type is defined by the randomly drawn value of \( a \) and your partner's type is defined by the randomly drawn value of \( \beta \), with \([0,x]\) defining the set of possible types for both of you.

(a) Suppose that your strategy in this game is to go to the opera if \( a > a \) (and to go to the football game otherwise), with \( a \) falling in the interval \([0,x]\). Explain why the probability (evaluated in the absence of knowing \( a \)) that you will go to the opera is \((x - a)/x\). What is the probability you will go to the football game?

**Answer:** We are assuming that all values between 0 and \( x \) are equally likely for \( a \) and that you will go to the opera only if \( a \) falls in the interval from \( a \) to \( x \). The probability that \( a \) falls between \( a \) and \( x \) then depends on how large this interval is relative to the whole interval from 0 to \( x \). The length of the interval from \( a \) to \( x \) is \((x - a)\) while the length of the interval from 0 to \( x \) is \( x \) — implying that \( a \) will fall in between \( a \) and \( x \) with probability \((x - a)/x\) — which is then the probability that you will go to the opera. The probability that \( a \) falls in the interval from 0 to \( a \), on the other hand, is equal to the size of that interval divided by the size of the interval from 0 to \( x \) — i.e. the probability that \( a \) falls between 0 and \( a \) is \( a/x \). (Note that you can also derive this by simply subtracting the probability that \( a \) falls between \( a \) and \( x \) — which is \((x - a)/x \) — from 1.)

(b) Suppose your partner plays a similar strategy: go to the football game if \( \beta > b \) and otherwise go to the opera. What is the probability that your partner will go to the football game? What is the probability that he/she will go to the opera?

**Answer:** By reasoning exactly as in part (a), we get that the probability that your partner goes to the football game is \((x - b)/x\) and the probability that your partner goes to the opera is \(b/x\).

(c) Given you know the answer to (b), what is your expected payoff from going to the opera for a given \( a \)? What is your expected payoff from going to the football game?

**Answer:** Your expected payoff from going to the opera depends on the probability that your partner goes to the opera. It is

\[
\frac{b}{x} (10 + a) + \left(\frac{x - b}{x}\right) 0 = \frac{(10 + a)b}{x}.
\]

Your expected payoff from going to the football game is

\[
\frac{b}{x} (0) + \left(\frac{x - b}{x}\right) 5 = \frac{5(x - b)}{x}.
\]

(d) Given your partner knows the answer to (a), what is your partner's expected payoff from going to the opera? What about the expected payoff from going to the football game?

**Answer:** Your partner's expected payoff from going to the opera depends on the probability that you go to the opera. It is

\[
\left(\frac{x - a}{x}\right) 5 + \left(\frac{a}{x}\right) 0 = \frac{5(x - a)}{x}.
\]

Your partner's expected payoff from going to the football game is

\[
\left(\frac{x - a}{x}\right) 0 + \left(\frac{a}{x}\right) (10 + \beta) = \frac{(10 + \beta)a}{x}.
\]

(e) Given your answer to (c), for what value of \( a \) (in terms of \( b \) and \( x \)) are you indifferent between going to the opera and going to the football game?

**Answer:** Setting the right hand sides of equations (24.6) and (24.7) equal to one another, we get
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\[(10 + a)b \quad \frac{x}{x} = 5(x - b) \quad \frac{x}{x} \quad (24.10)\]

which solves to

\[a = \frac{5x}{b} - 15. \quad (24.11)\]

(f) Given your answer to (d), for what value of \(\beta\) (in terms of \(a\) and \(x\)) is your partner indifferent between going to the opera and going to the football game?

Answer: Setting the right hand sides of equations (24.8) and (24.9) equal to one another, we get

\[\frac{5(x - a)}{x} = \frac{(10 + \beta)a}{x} \quad (24.12)\]

which solves to

\[\beta = \frac{5x}{a} - 15. \quad (24.13)\]

(g) Let \(a\) be equal to the value of \(\alpha\) you calculated in (e), and let \(b\) be equal to the value of \(\beta\) you calculated in (f). Then solve the resulting system of two equations for \(a\) and \(b\) (using the quadratic formula).

Answer: The two equations are

\[a = \frac{5x}{b} - 15 \quad \text{and} \quad b = \frac{5x}{a} - 15. \quad (24.14)\]

Plugging the second of these into the first and collecting terms to one side, we get \(15a^2 + 225a - 75x\). Applying the quadratic formula and picking the solution that is positive, we get

\[a = \frac{-225 + \sqrt{50625 + 4500x}}{30}. \quad (24.15)\]

Given the symmetry between \(a\) and \(b\), we then also know that

\[b = \frac{-225 + \sqrt{50625 + 4500x}}{30}. \quad (24.16)\]

(h) Why do these values for \(a\) and \(b\) make the strategies defined in (a) and (b) pure (Bayesian Nash) equilibrium strategies?

Answer: We have chosen \(a\) such that, when your type turns out to be \(a = a\), you are indifferent between going to the opera and going to the football game given your expected payoffs that arise from your uncertainty about your partner’s type and given your partner’s strategy as articulated in part (b). For any \(a > a\), the expected payoff from opera is greater for you given your partner’s strategy. Thus, your strategy of going to the opera if your type \(a\) turns out to be higher than \(a\) is a best response to your partner’s strategy. Similarly, we have chosen \(b\) so that your partner is indifferent between his/her two actions given his/her uncertainty about your type and given the strategy you adopt (as articulated in (a)). For any \(\beta\) greater than \(b\), your partner is better off going to the football game (given your strategy in (a)). Thus, your partner is best responding to your strategy from part (a) by playing the strategy outlined in part (b). These strategies are pure equilibrium strategies because they involve no randomization by either player — both you and your partner will definitely settle on one action once you find out your type.

(i) How likely is it in this equilibrium that you will go to the opera? How likely is it that your partner will go to the football game? How do your answers change as \(x\) approaches zero — and how does this compare to the probabilities you derived for the mixed strategy equilibrium in part A of the exercise? (Hint: Following the rules of taking limits, you will in this case have to take the derivative of a numerator and a denominator before taking the limit.)
Answer: We concluded in (a) that the probability of you going to the opera under this strategy is \( \frac{(x - a)}{x} \). Having now calculated \( a \), we can write this probability as

\[
\frac{x - a}{x} = 1 - \frac{a}{x} = 1 - \frac{-225 + \sqrt{225^2 + 4500x}}{30x}.
\]  

(24.17)

Just letting \( x \) go to zero causes the fraction above to be undefined. The rules of taking limits imply that we need to first take the derivative of the numerator and denominator of the fraction with respect to \( x \) — which gives us

\[
\frac{75}{\sqrt{225^2 + 4500x}}
\]

(24.18)

which has a limit of 1/3 as \( x \) approaches 0. Thus, as \( x \) approaches zero, \( (x - a)/x \) approaches 2/3. Put differently, as the range of uncertainty in our Bayesian game decreases (with lower \( x \)), the probability that you will go to the opera as well as the probability that your partner will go to the football game both approach 2/3 — which is exactly the probability placed on those actions in the mixed strategy equilibrium in part A of the exercise.

(j) True or False: The mixed strategy equilibrium to the complete information Battle of the Sexes game can be interpreted as a pure strategy Bayesian equilibrium in an incomplete information game that is almost identical to the original complete information game — allowing us to interpret the mixed strategies in the complete information game as arising from uncertainty that players have about the other player.

Answer: This is true. We have shown that adding a slight bit of uncertainty about the opponent’s payoffs gives us a Bayesian game with a pure strategy equilibrium that mimics the mixed strategy equilibrium in the complete information game that we analyzed in part A of the exercise.
24.5 Everyday Application: Splitting the Pot. Suppose two players are asked to split $100 in a way that is agreeable to both.

As The structure for the game is as follows: Player 1 moves first — and he is asked to simply state some number between zero and 100. This number represents his “offer” to player 2 — the amount player 1 offers for player 2 to keep, with player 1 keeping the rest. For instance, if player 1 says “30”, he is offering player 2 a split of the $100 that gives $70 to player 1 and $30 to player 2. After an offer has been made by player 1, player 2 simply chooses from two possible actions: either “Accept” the offer or “Reject” it. If player 2 accepts, the $100 is split in the way proposed by player 1; if player 2 rejects, neither player gets anything. (A game like this is often referred to as an ultimatum game.)

(a) What are the subgame perfect equilibria in this game assuming that player 1 is restricted to making his “offer” in integer terms — i.e. assuming that player 1 has to state a whole number.

Answer: As always, we begin by thinking about player 2 who gets zero by playing “Reject” and whatever was offered by playing “Accept”. For any positive amount x, the payoff from accepting therefore strictly dominated the payoff from rejecting — implying that, in any subgame perfect equilibrium, player 2 should always “Accept” any positive offer. If the offer is zero, player 2 is indifferent between accepting and rejecting — so either “Accept” or “Reject” could then be subgame perfect when zero is offered. Player 1 knows this — and thus knows he is playing someone whose strategy is either “Always Accept” or “Accept unless the offer is zero”. The best response to the first (“Always Accept”) strategy is to offer zero — giving us the subgame perfect equilibrium in which player 1 plays “Zero” and player 2 plays “Always Accept”. The best response to the second strategy by player 2 (i.e. “Accept unless the offer is zero”) is to offer $1 (since only integer offers are allowed). Thus, we have two possible subgame perfect equilibria — one under which player 1 walks away with $100, the other under which player 1 walks away with $99.

(b) Now suppose that offers can be made to the penny — i.e. offers like $31.24 are acceptable. How does that change the subgame perfect equilibria? What if we assumed dollars could be divided into arbitrarily small quantities (i.e. fractions of pennies)?

Answer: If offers can be made to the penny, the two subgame perfect equilibria described in (a) still exist — except that the latter involves an offer of a penny instead of an offer of a dollar by player 1. Thus, player 1 walks away with either $100 or $99.99. The more divisible we make a dollar, the smaller the quantity that is offered in the second subgame perfect equilibrium — with that quantity converging to zero. If we remove all assumptions about money being indivisible beyond a certain amount, the second subgame perfect equilibrium vanishes entirely — because for any positive amount that player 1 might offer, it is in fact better (for player 1) to offer half that amount. Thus, when we assume a dollar can be divided into arbitrarily small quantities, the only subgame perfect equilibrium is one in which player 2 plays “Accept any offer” and player 1 offers zero.

(c) It turns out that there are at most two subgame perfect equilibria to this game (and only 1 if dollars are assumed to be fully divisible) — but there is a very large number of Nash equilibria regardless of exactly how player 1 can phrase his offer (and an infinite number when dollars are assumed fully divisible). Can you, for instance, derive Nash equilibrium strategies that result in player 2 walking away with $80? Why is this not subgame perfect?

Answer: The following would be a Nash equilibrium in which player 2 gets $80: Player 1 offers $80, and player 2 plays the strategy ”Reject all offers below $80, accept all others.” Given this strategy by player 2, it is a best response for player 1 to offer $80, and given the offer of $80, player 2’s strategy is a best response as well. Thus we have a Nash Equilibrium. However, this Nash equilibrium involves player 2 playing a strategy that involves a non-credible threat — the threat to reject all offers below $80. For this reason, this Nash Equilibrium is not subgame perfect.

(d) This game has been played in experimental settings in many cultures — and, while the average amount that is “offered” differs somewhat between cultures, it usually falls between $25 and $50, with players often rejecting offers below that. One possible explanation for this is that individuals across different cultures have somewhat different notions of “fairness” — and that they get utility from “standing up for what’s fair”. Suppose player 2 is willing to pay $30 to stand up to “injustice” of any kind, and anything other than a 50-50 split is considered by player 2 to be unjust. What is now the subgame perfect equilibrium if dollars are viewed
as infinitely divisible? What additional subgame perfect equilibrium arises if offers can only be made in integer amounts?

**Answer:** This sense of fairness has changed the payoffs of the game from what we have assumed so far — with player 2 getting a payoff of 30 for rejecting any offer of less than 50. Thus, it is now credible for player 2 to play the strategy “Reject any offer below 30” — because we know player 2 gets the equivalent of $30 in utility from rejecting unfair offers. It is not, however, credible for player 2 to play a strategy that rejects offers above $30 — because accepting such offers yields more than rejecting them (which only yields $30). It would therefore be a subgame perfect equilibrium for player 1 to offer $30 and for player 2 to play the strategy “Reject any offer below $30, otherwise accept”. In the case where player 1 is limited to making offers in integer terms, we would also have a subgame perfect equilibrium in which player 1 offers $31 and player 2 plays the strategy “Accept only offers greater than $30”.

(e) Suppose instead that player 2 is outraged at “unfair” outcomes in direct proportion to how far the outcome is removed from the “fair” outcome, with the utility player 2 gets from rejecting an offer equal to the difference between the amount offered and the “fair” amount. Suppose player 2 believes the “fair” outcome is splitting the $100 equally. Thus, if the player faces an offer $x < 50$, the utility she gets from rejecting the offer is $(50 - x)$. What are the subgame perfect equilibria of this game now under the assumption of infinitely divisible dollars and under the assumption of offers having to be made in integer terms?

**Answer:** For any offer of less than $25, the payoff from rejecting is greater than the payoff from accepting. Thus, subgame perfection requires that player 2 reject any offer less than $25. For any offer greater than $25, the payoff from accepting is greater than the payoff from rejecting — thus subgame perfection implies player 2 will accept all offers greater than $25. Offers of $25 leave player 2 indifferent between accepting and rejecting the offer. In the case of infinitely divisible dollars, the only subgame perfect equilibrium therefore involves player 1 offering $25 and player 2 playing the strategy “Reject all offers below $25, accept all others” — which results in player 1 walking away with $75 and player 2 walking away with $25. In the case of offers restricted to integer amounts, the following additional subgame perfect equilibrium is possible: Player 1 offers $26, and player 2 plays the strategy “Accept offers greater than $25”, which results in player 1 getting $74 and player 2 getting $26.

B: Consider the same game as that outlined in A and suppose you are the one that splits the $100 and I am the one who decides to accept or reject. You think there is a pretty good chance that I am the epitome of a rational human being who cares only about walking away with the most I can from the game. But you don’t know me that well — you think there is some chance $\rho$ that I am a self-righteous moralist who will reject any offer that is worse for me than a 50-50 split. (Assume throughout that dollars can be split into infinitesimal parts.)

(a) Structure this game as an incomplete information game.

**Answer:** The game has 3 stages: In stage 1, Nature moves and assigns me type 1 with probability $\rho$ and type 2 with probability $(1 - \rho)$. In stage 2, you split the $100 without knowing what type I am — i.e. both nodes in the game tree lie in the same information set for you. In stage 3, I (knowing my type) decide whether or not to accept the offer you have made.

(b) There are two types of pure strategy equilibria to this game (depending on what value $\rho$ takes). What are they?

**Answer:** You will either decide to offer the minimum to make the moralist happy — i.e. $50 — or you will take your chances and offer zero (or very close to zero). It makes no sense to offer something between 0 and 50 — because your offer will then be rejected by the moralist while leaving you with less than a zero offer if you face a non-moralist. It also makes no sense to offer more than $50 — since we know even the moralist accepts the $50 offer. If you offer 0, you expected payoff is then $100 with probability $(1 - \rho)$ and 0 with probability $\rho$ — for a total expected payoff of $100(1 - \rho)$. If, on the other hand, you offer $50, you know you will walk away with $50 because the offer will be accepted for sure. You will therefore offer 0 so long as $100(1 - \rho) > 50$ — i.e. so long as $\rho < 0.5$. If $\rho > 0.5$, you will offer $50. In all cases, the non-moralist will accept what is offered. (If $\rho = 0.5$, both offers are consistent with equilibrium behavior.)
(c) How would your answer change if I, as a self-righteous moralist (which I am with probability $\rho$) reject all offers that leave me with less than $10$?

Answer: In this case, you will either offer 0 or 10. Your expected payoff from offering 0 is $100(1 - \rho)$ while your expected payoff from offering 10 is 90. Thus, you will offer 0 if $\rho < 0.1$ and 10 if $\rho > 0.1$ (with both offers possible when $\rho = 0.1$).

(d) What if it's only less than $1$ that is rejected by self-righteous moralists?

Answer: The cut-off for when you will offer $1$ rather than 0 is now $\rho = 0.01$.

(e) What have we implicitly assumed about risk aversion?

Answer: We have implicitly assumed that you are indifferent between gambles that leave you with the same expected value — i.e. we have assumed risk neutrality.
24.6 Everyday Application: Another Way to Split the Pot. Suppose again, as in exercise 24.5, that two players have $100 to split between them.

A: But now, instead of one player proposing a division and the other accepting or rejecting it, suppose that player 1 divides the $100 into two piles and player 2 then selects his preferred pile.

(a) What is the subgame perfect equilibrium of this game?

**Answer:** Consider stage 2 when player 2 has to decide which pile to pick up and keep. Naturally, player 2 will pick up the larger of the two piles. Player 1 knows this — and thus knows he will end up keeping the smaller of the two piles he creates. Thus, he will create two equal piles — both with $50 — and both players end up walking away with $50.

(b) Can you think of a Nash equilibrium (with an outcome different than the subgame perfect outcome) that is not subgame perfect?

**Answer:** No. An outcome different from the subgame perfect outcome would have player 2 walk away with an amount different from $50 — which implies it would require player 1 to split the $100 into unequal piles. Suppose that player 1 in fact does create unequal piles. Then player 2 must end up picking the larger pile in equilibrium — otherwise he is not best-responding to seeing unequal piles — which in turn implies that player 1 walks away with less than $50. But player 1 can guarantee himself $50 by just splitting the pot equally — which means he cannot be best responding to player 2 by creating unequal piles. So there is no way to get the two players to best-respond to one another without having each walk away with half the pot. (The underlying reason is that there is no non-credible off-the-equilibrium path threat that player 2 can use to change the outcome from the subgame perfect equilibrium outcome.)

(c) In exercise 24.5, we considered the possibility of restricting offers to be in integer amounts, to be in pennies, etc. Would our prediction differ here if we made different such assumptions?

**Answer:** No, since the equilibrium has player 1 splitting the pot exactly in 2, the piles will be exactly $50 under all of these assumptions.

(d) Suppose that the pot was $99 and player 1 can only create piles in integer (i.e. full dollar) amounts. Who would you prefer to be: player 1 or 2?

**Answer:** In this case, the subgame perfect equilibrium is for player 1 to create one pile with $50 and one with $49 — and player 2 will pick the larger pile. You would therefore prefer to be player 2.

(e) Suppose that player 2 has three possible actions: Pick up the smaller pile, pick up the larger pile, and set all of it on fire. Can you now think of Nash equilibria that are not subgame perfect?

**Answer:** Now we can get lots of different Nash Equilibria. Suppose, for instance, player 2 adopts a strategy of “Pick up $90 if one of the piles has $90; otherwise burn everything.” Given this strategy, player 1 best responds by creating a pile of $90 and a pile of $10. Given these two piles, the strategy “Pick up $90 if one of the piles has $90; otherwise burn everything” is a best response for player 2. We therefore have a Nash equilibrium. The problem, of course, is that the equilibrium is supported by the non-credible threat that player 2 will burn all the money if he does not see a $90 pile.

B: In exercise 24.5, we next considered an incomplete information game in which you split the $100 and I was a self-righteous moralist with some probability $\rho$. Assuming that the opposing player is some strange type with some probability can sometimes allow us to reconcile experimental data that differs from game theory predictions.

(a) Why might this be something we introduce into the game from exercise 24.5 but not here?

**Answer:** In exercise 24.5, the subgame perfect prediction was extreme — player 1 offers zero (or very close to zero). As you might guess (and as indicated in the exercise), this is not the outcome we typically see in experiments where considerably more than 0 is offered by player 1. The introduction of a “non-rational” player — or a player that evaluates pay-offs differently — can therefore help create subgame perfect equilibria in which non-zero amounts are offered by player 1 who is uncertain about what type of player he is facing. In the example here, of course, the prediction is already that the pot will be split 50-50 — which is probably what one would find in experiments. There is therefore no need to introduce anything further to reconcile the data from experiments with the prediction from the game.
(b) If we were to introduce the possibility that player 2 plays a strategy other than the “rational” strategy with probability $\rho$, is there any way that this will result in player 1 getting less than $50 in this game?

Answer: No — because player 1 can always guarantee himself $50 by simply creating equal piles.
24.7 Everyday Application: Real World Mixed Strategies: In the text, we discussed the “Matching Pennies” game and illustrated that such a game only has a mixed strategy equilibrium.

As: Consider each of the following and explain (unless you are asked to do something different) how you might expect there to be no pure strategy equilibrium — and how a mixed strategy equilibrium might make sense.

(a) A popular children’s game, often played on long road trips, is “Rock, Paper, Scissors.” The game is simple: Two players simultaneously signal through a hand gesture one of three possible actions: Rock, Paper or Scissors. If the two players signal the same, the game is a tie. Otherwise, Rock beats Scissors, Scissors beats Paper and Paper beats Rock.

Answer: Suppose player 1 plays anything other than a mixed strategy with equal probabilities (of 1/3) on each of the three actions. Then we know that he places probability greater than 1/3 on at least one of his actions. Choose the action that he plays with highest probability — for illustration, suppose it is Rock. Then player 2’s best response is to play the action that beats the most played action by player 1 — Paper, in this case. But if player 2 plays Paper, player 1’s best response does not include any strategy that involves positive probabilities of playing Rock. So it cannot be that a strategy that places probabilities other than equal probabilities of 1/3 on each action is part of an equilibrium. This leaves us with mixed strategies that place equal probability on each action — which is the mixed strategy equilibrium. That seems reasonable — as we play this game, the best we can do is to pick randomly between the three options in a way that does not permit our opponent to predict that we might be favoring one option.

(b) One of my students objects: “I understand that Scissors can beat Paper, and I get how Rock can beat Scissors, but there is no way Paper should beat Rock. Why can’t Paper do this to Scissors? For that matter, why can’t Paper do this to people?... I’ll tell you why: Because Paper can’t beat anybody!”

Answer: In this case, Paper does not beat anything — and so it should never be played in equilibrium. Effectively, this implies that we can treat the game is if there were in fact only two actions — Scissors and Rock. Rock beats Scissors — so it is a dominant strategy to play Rock — and all games are then ties as both players play Rock in equilibrium.

(c) In soccer, penalty kicks often resolve ties. The kicker has to choose which side of the goal to aim for, and, because the ball moves so fast, the goalie has to decide simultaneously which side of the goal to defend.

Answer: This is exactly the matching pennies game: The goalie would like to match the side that the kicker is kicking to, and the kicker would like to kick to the side opposite the one being defended by the goalie. From our analysis of the matching pennies game, we know that we only have a mixed strategy equilibrium in which the goalie defends each side with probability 0.5 and the kicker kicks to each side with probability 0.5. This makes a reasonable amount of sense for the example. (Some economists have actually tried to see whether the goal kicking in professional soccer games — or the serve patterns by professional tennis players (see the next part of the exercise) conforms with mixed strategy equilibrium behavior. It turns out, that it does not quite — players are not sufficiently randomizing their actions to qualify them as true mixed strategies. But it may be that, while technically not playing mixed strategies, players are playing strategies sufficiently complex so that the opponent cannot predict the next move — which is similar to what happens under mixed strategies.)

(d) How is the soccer example similar to a situation encountered by a professional tennis player whose turn it is to serve?

Answer: The exact same issue comes up — do you serve to the right or to the left — with the defending player having to decide which side to defend in the instant that the serve happens. If the serve is sufficiently fast, the game is a simultaneous matching pennies game.

---

3My student continues (with some editing on my part to make it past the editorial censors): “When I play “Rock, Paper, Scissors”, I always choose Rock. Then, when someone claims to have beaten me with Paper, I can punch them in the face with my already clenched fist and say — oh, sorry — I thought paper would protect you, moron.”
For reasons I cannot explain, teenagers in the 1950's sometimes played a game called "chicken". Two teenagers in separate cars drove at high speed in opposite directions on a collision course — and whoever swerved to avoid a crash lost the game. Sometimes, the cars crashed and both teenagers were severely injured (or worse). If we think behavior in these games arose within an equilibrium, could that equilibrium be in pure strategies?

**Answer:** No, we could not have a pure strategy equilibrium that involved crashes. If player 1 knew that player 2 would not swerve, his best response would presumably be to swerve. There are therefore two pure strategy Nash equilibria — one in which player 1 swerves and one in which player 2 swerves. But there is also a mixed strategy equilibrium in which the two players randomize between swerving and not swerving — which can lead to equilibrium crashes.

**B:** If you have done part B of exercise 24.4, appeal to incomplete information games with almost complete information to explain intuitively how the mixed strategy equilibrium in the chicken game of A(e) can be interpreted.

**Answer:** The argument in exercise 24.4 is that mixed strategy behavior might emerge from uncertainty over the payoffs of the opponent. The chicken game is in fact quite similar to the Battle of the Sexes game in exercise 24.4 — with both cases being examples of coordination games. (In the Battle of the Sexes, the players try to coordinate on the same action — in the chicken game, they try to coordinate on opposing actions.) So the same argument can be made here: A player might not be certain of just how much the opponent values being the one that does not swerve. We would model such a game as a game of imperfect information in which Nature assigns a type to one (or both) players before the game begins — with each player only knowing his own type and the probability with which Nature assigned the different types to the opponent. Even if we make the range of possible payoff differences across types narrow, we showed that such uncertainty can lead to exactly the behavior that looks like mixed strategies in the complete information game.
24.8 Everyday Application: Burning Money, Iterated Dominance and the Battle of the Sexes. Consider again the “Battle of the Sexes” game described in exercise 24.4. Recall that you and your partner have to decide whether to show up at the opera or a football game for your date — with both of you getting a payoff of 0 if you show up at different events and therefore aren’t together. If both of you show up at the opera, you get a payoff of 10 and your partner gets a payoff of 5, with these reversed if you both show up at the football game.

As in this part of the exercise, you will have a chance to test your understanding of some basic building blocks of complete information games whereas in part B we introduce a new concept related to dominant strategies. Neither part requires any material from Section B of the chapter.

(a) Suppose your partner works the night shift and you work during the day — and, as a result, you miss each other in the morning as you leave for work just before your partner gets home. Neither of you is reachable at work — and you come straight from work to your date. Unable to consult one another before your date, each of you simply has to individually decide whether to show up at the opera or at the football game. Depict the resulting game in the form of a payoff matrix.

Answer: This is depicted in Table 24.7.

<table>
<thead>
<tr>
<th></th>
<th>Partner Opera</th>
<th>Partner Football</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Opera</td>
<td>10,5</td>
<td>0,0</td>
</tr>
<tr>
<td>Football</td>
<td>0,0</td>
<td>5,10</td>
</tr>
</tbody>
</table>

Table 24.7: Battle of the Sexes

(b) In what sense is this an example of a “coordination game”?

Answer: It is a coordination game in the sense that both of you would prefer to coordinate to be in the same place even though you disagree which is the better place. As illustrated in the next part, there are therefore two pure strategy Nash equilibria corresponding to the two ways we can get to the same place.

(c) What are the pure strategy Nash equilibria of the game.

Answer: The pure strategy equilibria are {Opera, Opera} and {Football, Football} — where the first item in each set stands for your strategy and the second stands for your partner’s strategy. (Given you go to the Opera, it is a best response for your partner to go to the Opera and vice versa; similarly with Football.)

(d) After missing each other on too many dates, you come up with a clever idea: Before leaving for work in the morning, you can choose to burn $5 on your partner’s nightstand — or you can decide not to. Your partner will observe whether or not you burned $5. So we now have a sequential game where you first decide whether or not to burn $5, and you and your partner then simultaneously have to decide whether to show up at the opera or at the football game. Depict the resulting game in the form of a payoff matrix.

Answer: Your four strategies in this game are then (Burn, Opera), (Burn, Football), (Don’t Burn, Opera) and (Don’t Burn, Football).

(e) What are your partner’s four strategies in this new game (given that your partner may or may not observe the evidence of the burnt money depending on whether or not you chose to burn the money.)

Answer: Your partner observes either Burn or Don’t Burn from you. A strategy is then a complete plan of action — a plan for what to do after observing Burn and a plan for what to do after observing Don’t Burn. Let O stand for Opera and F for Football. With the first action in each parenthesis indicating the plan in the event that Burn has been observed and the second indicating the plan in the event that Don’t Burn has been observed, your partner’s strategies are then (O, O), (O, F), (F, O) and (F, F). The last strategy, for instance, says “always go to the football game regardless of what happened” and the first strategy says “always go to the opera”. But the second strategy says “go to the opera if there is burnt
money on the night stand but go to the football game if there isn’t”, and the third strategy says “go to the football game if there is burnt money and to the opera if there isn’t.”

(f) Illustrate the payoff matrix of the new game assuming that the original payoffs were denominated in dollars. What are the pure strategy Nash Equilibria?

Answer: The payoff matrix is given in Table 24.8.

<table>
<thead>
<tr>
<th>You</th>
<th>Partner (O, O)</th>
<th>(O, F)</th>
<th>(F, O)</th>
<th>(F, F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Burn, Opera)</td>
<td>5, 5</td>
<td>5, 5</td>
<td>-5, 0</td>
<td>-5, 0</td>
</tr>
<tr>
<td>(Burn, Football)</td>
<td>-5, 0</td>
<td>-5, 0</td>
<td>0, 10</td>
<td>0, 10</td>
</tr>
<tr>
<td>(Don’t Burn, Opera)</td>
<td>10, 5</td>
<td>0, 0</td>
<td>10, 5</td>
<td>0, 0</td>
</tr>
<tr>
<td>(Don’t Burn, Football)</td>
<td>0, 0</td>
<td>5, 10</td>
<td>0, 0</td>
<td>5, 10</td>
</tr>
</tbody>
</table>

Table 24.8: Battle of the Sexes with Money Burning

B: In the text, we defined a dominant strategy as a strategy under which a player does better no matter what his opponent does than he does under any other strategy he could play. Consider now a weaker version of this: We will say that a strategy B is weakly dominated by a strategy A for a player if the player does at least as well playing A as he would playing B regardless of what the opponent does.

(a) Are there any weakly dominated strategies for you in the payoff matrix you derived in A(f)? Are there any such weakly dominated strategies for your partner?

Answer: For you, the strategy (Burn, Football) is weakly dominated by the strategy (Don’t Burn, Opera) — because the first number in each cell of the row (Don’t Burn, Opera) is greater than or equal to the first number in the cell immediately above it. There are no weakly dominated strategies for your partner.

(b) It seems reasonable that neither of you expects the other to play a weakly dominated strategy. So take your payoff matrix and strike out all weakly dominated strategies. The game you are left with is called a reduced game. Are there any strategies for either you or your partner that are weakly dominated in this reduced game? If so, strike them out and derive an even more reduced game. Keep doing this until you can do it no more — what are you left with in the end?

Answer: After eliminating (Burn, Football), we are left with the game in Table 24.9. In this game, there are no weakly dominated strategies for you. But (F, O) is weakly dominated for your partner — because he would do at least as well and sometimes better by playing (O, O).

<table>
<thead>
<tr>
<th>You</th>
<th>Partner (O, O)</th>
<th>(O, F)</th>
<th>(F, O)</th>
<th>(F, F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Burn, Opera)</td>
<td>5, 5</td>
<td>5, 5</td>
<td>-5, 0</td>
<td>-5, 0</td>
</tr>
<tr>
<td>(Don’t Burn, Opera)</td>
<td>10, 5</td>
<td>0, 0</td>
<td>10, 5</td>
<td>0, 0</td>
</tr>
<tr>
<td>(Don’t Burn, Football)</td>
<td>0, 0</td>
<td>5, 10</td>
<td>0, 0</td>
<td>5, 10</td>
</tr>
</tbody>
</table>

Table 24.9: Reduced Game 1

Eliminating (F, O) from the matrix, we get the game in Table 24.10. In this new game, (F, F) is weakly dominated by (O, F) for your partner.

Eliminating (F, F), we get a further reduced game depicted in Table 24.11. Now your strategy (Don’t Burn, Football) is weakly dominated by (Burn, Opera).

Eliminating (Don’t Burn, Football), we get a further reduced game depicted in Table 24.12. But now (O, F) is weakly dominated by (O, O) for your partner.
Strategic Thinking and Game Theory

<table>
<thead>
<tr>
<th>You</th>
<th>Partner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(O, O)</td>
</tr>
<tr>
<td>(Burn, Opera)</td>
<td>5,5</td>
</tr>
<tr>
<td>(Don’t Burn, Opera)</td>
<td>10,5</td>
</tr>
<tr>
<td>(Don’t Burn, Football)</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Table 24.10: Reduced Game 2

<table>
<thead>
<tr>
<th>You</th>
<th>Partner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(O, O)</td>
</tr>
<tr>
<td>(Burn, Opera)</td>
<td>5,5</td>
</tr>
<tr>
<td>(Don’t Burn, Opera)</td>
<td>10,5</td>
</tr>
<tr>
<td>(Don’t Burn, Football)</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Table 24.11: Reduced Game 3

<table>
<thead>
<tr>
<th>You</th>
<th>Partner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(O, O)</td>
</tr>
<tr>
<td>(Burn, Opera)</td>
<td>5,5</td>
</tr>
<tr>
<td>(Don’t Burn, Opera)</td>
<td>10,5</td>
</tr>
</tbody>
</table>

Table 24.12: Reduced Game 4

Eliminating (O, F), we get the further reduced game in Table 24.13. And now (Burn, Opera) is strictly dominated by (Don’t Burn, Opera).

<table>
<thead>
<tr>
<th>You</th>
<th>Partner</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(O, O)</td>
</tr>
<tr>
<td>(Burn, Opera)</td>
<td>5,5</td>
</tr>
<tr>
<td>(Don’t Burn, Opera)</td>
<td>10,5</td>
</tr>
</tbody>
</table>

Table 24.13: Reduced Game 5

Eliminating (Burn, Opera), we are left with the trivial game in Table 24.14.

(c) After repeatedly eliminating weakly dominated strategies, you should have ended up with a single strategy left for each player. Are these strategies an equilibrium in the game from A(f) that you started with?

Answer: Yes, they are. But there are lots of other Nash equilibria in the original game as well. A list of all the pure strategy Nash equilibria is as follows: (1) ((Don’t Burn, Opera), (O, O)); (2) ((Don’t Burn, Opera), (F, O)); (3) ((Don’t Burn, Football), (O, F)); (4) ((Don’t Burn, Football), (F, F)); and (1) ((Burn, Opera), (O, F)).

(d) Selecting among multiple Nash equilibria to a game by repeatedly getting rid of weakly dominated strategies is known as applying the idea of iterative dominance. Consider the initial game from A(a) (before we introduced the possibility of you burning money). Would applying the same idea of iterative dominance narrow the set of Nash equilibria in that game?

Answer: No, there are no weakly dominated strategies in the initial game (without money burning).
Strategic Thinking and Game Theory

You (Don't Burn, Opera) 10,5
Partner (O, O)

Table 24.14: Reduced Game 6

(e) True or False: By introducing an action that ends up not being used, you have made it more likely that you and your partner will end up at the opera.

Answer: If we believe that eliminating Nash equilibria through iterative dominance gives us the right prediction of what will be played, then the statement is true. The action — money burning — that was introduced does not get used in the Nash equilibrium that survives iterated dominance. But it’s introduction allowed the application of iterated dominance. In the context of this game, by introducing the idea of burning money, you are able to get to your most preferred Nash Equilibrium (that results in both you and your partner getting to the opera) — without actually having to burn the money. (Strange, isn’t it?)
24.9 Everyday and Business Application: Bargaining over a Fixed Amount: Consider a repeated version of the game in exercise 24.3. In this version, we do not give all the proposal power to one person but rather imagine that the players are bargaining by making different proposals to one another until they come to an agreement. In part A of the exercise we analyze a simplified version of such a bargaining game, and in part B we use the insights from part A to think about an infinitely repeated bargaining game. (Note: Part B of the exercise, while conceptually building on part A, does not require any material from Section B of the Chapter.)

A: We begin with a 3-period game in which $100 gets split between the two players. It begins with player 1 stating an amount $x_1$ that proposes she should receive $x_1$ and player 2 should receive $(100 - x_1)$. Player 2 can then accept the offer — in which case the game ends with payoff $x_1$ for player 1 and $(100 - x_1)$ for player 2; or player 2 can reject the offer, with the game moving on to period 2. In period 2, player 2 now has a chance to make an offer $x_2$ which proposes player 1 gets $x_2$ and player 2 gets $(100 - x_2)$. Now player 1 gets a chance to accept the offer — and the proposed payoffs — or to reject it. If the offer is rejected, we move on to period 3 where player 1 simply receives $x$ and player 2 receives $(100 - x)$. Suppose throughout that both players are somewhat impatient — and they value $1 a period from now at $\delta < 1$. Also suppose throughout that each player accepts an offer whenever he/she is indifferent between accepting and rejecting the offer.

(a) Given that player 1 knows she will get $x$ in period 3 if the game continues to period 3, what is the lowest offer she will accept in period 2 (taking into account that she discounts the future as described above)?

Answer: The amount $x$ that player 1 is guaranteed in period 3 is worth $\delta x$ in period 2. Thus, player 1 will accept any offer that is at least equal to $\delta x$ — i.e. any offer $x_2 \geq \delta x$.

(b) What payoff will player 2 get in period 2 if he offers the amount you derived in (a)? What is the present discounted value (in period 2) of what he will get in this game if he offers less than that in period 2?

Answer: If player 2 offers $x_2 = \delta x$, he will get a payoff of $(100 - \delta x)$ in period 2. If player 2 makes an offer $x_2 < \delta x$, the offer will be rejected and he will get $(100 - x)$ one period later. Such an amount is worth $\delta(100 - x)$ now — so the present discounted value of making an offer that gets rejected is $\delta(100 - x)$.

(c) Based on your answer to (b), what can you conclude player 2 will offer in period 2?

Answer: We determined that player 2 will get $(100 - \delta x)$ if he makes the lowest possible offer that is accepted in period 2 — while he makes $\delta(100 - x)$ if he makes an offer that gets rejected. Since $0 < \delta < 1$, the former is larger than the latter — which implies that player 2 will make an offer of $x_2 = \delta x$ in period 2, with player 1 accepting the offer.

(d) When the game begins, player 2 can look ahead and know everything you have thus far concluded. Can you use this information to derive the lowest possible period 1 offer that will be accepted by player 2 in period 2?

Answer: Player 2 knows that he will get $(100 - \delta x)$ in period 2 if the game continues to period 2 — and this amount is worth $\delta(100 - \delta x)$ in period 1. Thus, player 2 will accept any offer that allocates at least this amount to player 2 — i.e. any $x_1$ such that

$$x_1 \geq 100 - \delta(100 - \delta x) \tag{24.19}$$

which can also be written as

$$x_1 \geq 100 - \delta(100 - \delta x). \tag{24.20}$$

(e) What payoff will player 1 get in period 1 if she offers the amount you derived in (d)? What will she get (in present value terms) if she offers an amount higher for her (and lower for player 2)?

Answer: If player 1 offers $x_1 = 100 - \delta(100 - \delta x)$, this is the amount she gets (because the offer will be accepted). If player 1 offers $x_1 > 100 - \delta(100 - \delta x)$ (which means player 2 gets less), then the offer is rejected and player 1 gets $\delta x$ in period 2 — which is worth $\delta^2 x$ in period 1.

(f) Based on your answer to (e), can you conclude how much player 1 offers in period 1 — and what this implies for how the game unfolds in subgame perfect equilibrium?
Answer: Player 1 will therefore offer \( x_1 = 100 - \delta (100 - \delta x) \) so long as it is greater than \( \delta^2 x \) — which it is for any \( \delta < 1 \). Thus, player 1 offers \( x_1 = 100 - \delta (100 - \delta x) \) in period 1 — and player 2 accepts. In the subgame perfect equilibrium, the game therefore ends in period 1.

(g) True or False: The more player 1 is guaranteed to get in the third period of the game, the less will be offered to player 2 in the first period (with player 2 always accepting what is offered at the beginning of the game).

Answer: This is true — because \( x_1 = 100 - \delta (100 - \delta x) = 100(1 - \delta) + \delta^2 x \) — with \( x_1 \) increasing as \( x \) increases. Thus, the greater the guaranteed amount \( x \) for player 1 in period 3, the more player 1 will propose for herself in period 1 — with less going to player 2.

B: Now consider an infinitely repeated version of this game; i.e. suppose that in odd-numbered periods — beginning with period 1 — player 1 gets to make an offer that player 2 can accept or reject, and in even-numbered periods the reverse is true.

(a) True or False: The game that begins in period 3 (assuming that period is reached) is identical to the game beginning in period 1.

Answer: This is true — in period 3 we begin with player 1 making a proposal, just as we do in period 1, and in the game that begins in period 3, the proposals and counter-proposals continue indefinitely until someone accepts, just as is the case in the game that begins in period 1.

(b) Suppose that, in the game beginning in period 3, it is part of an equilibrium for player 1 to offer \( x \) and player 2 to accept it at the beginning of that game. Given your answer to (a), is it also part of an equilibrium for player 1 to begin by offering \( x \) and for player 2 to accept it in the game that begins with period 1?

Answer: Yes, this must be the case, since the two games — i.e. the one that begins in period 1 and the one that begins in period 3 are identical. Any equilibrium in one game must therefore also be an equilibrium in the other.

(c) In part A of the exercise, you should have concluded that — when the game was set to artificially end in period 3 with payoffs \( x \) and \((100 - x)\), player 1 ends up offering \( x_1 = 100 - \delta (100 - \delta x) \) in period 1, with player 2 accepting. How is our infinitely repeated game similar to what we analyzed in part A when we suppose, in the infinitely repeated game beginning in period 3, the equilibrium has player 1 offer \( x \) and player 2 accepting the offer?

Answer: If the players know that, in the game that begins in period 3, player 1 will offer \( x \) (to herself) and player 2 will accept (leaving her with \((100 - x)\)), then, when viewed from period 1, it is just as if period 3 simply gives \( x \) to player 1 and \((100 - x)\) to player 2 if we end up reaching that period.

(d) Given your answers above, why must it be the case that \( x = 100 - \delta (100 - \delta x) \)?

Answer: In part A, we concluded that it is subgame perfect for player 1 to offer \( x_1 = 100 - \delta (100 - \delta x) \) in period 1 (and for player to to accept in period 1) when the guaranteed period 3 payoffs are \( x \) and \((100 - x)\). If \( x \) and \((100 - x)\) is the allocation that happens in the game that begins in period 3, the same must therefore hold in period 1 for the infinitely repeated game. But we also concluded that, if \( x \) and \((100 - x)\) is the equilibrium allocation in the game that begins in period 3, it must also be the equilibrium allocation in the game that begins in period 1 — i.e. it must be that \( x_1 = x \). Combining this with our conclusion that \( x_1 = 100 - \delta (100 - \delta x) \), we can conclude that \( x = 100 - \delta (100 - \delta x) \).

(e) Use this insight to derive how much player 1 offers in period 1 of the infinitely repeated game. Will player 2 accept?

Answer: We then have to solve \( x = 100 - \delta (100 - \delta x) \) for \( x \). The equation can also be written as \( x = 100(1 - \delta) + \delta^2 x \), collecting the \( x \) terms on the left hand side, as

\[ (1 - \delta^2) x = 100(1 - \delta). \]  

(24.21)

Since \( (1 - \delta^2) = (1 + \delta)(1 - \delta) \), this reduces to \( x = 100/(1 + \delta) \) — implying that the $100 is split into

\[ x = \frac{100}{1 - \delta} \]  

for player 1, and \( (100 - x) = \frac{100 \delta}{1 - \delta} \) for player 2.  

(24.22)

Player 2 will accept this offer.
Does the first mover have an advantage in this infinitely repeated bargaining game? If so, why do you think this is the case?

Answer: Yes, the first player has an advantage because she gets $100/(1-\delta)$ compared to $(100\delta/(1-\delta))$ for the second player. For any $0 < \delta < 1$, the former is larger than the latter. Thus, the first mover advantage derives from the fact that our players are impatient — that they do not value a dollar in the future the same as a dollar now. Player 1 can exploit this and therefore offer player 2 less than half of the $100 and still get him to accept — because a little less now is better than a little more in the future.
24.10 Everyday and Business Application: Auctions: Many items are sold not in markets but in auctions where bidders do not know how much others value the object that is up for bid. We will analyze a straightforward setting like this here—which technologically means we are analyzing (for much of this exercise) an incomplete information game of the type covered in Section B of the chapter. The underlying logic of the exercise is, however, sufficiently transparent for you to be able to attempt the exercise even if you have not read Section B of the chapter. Consider the following—a second-price sealed bid auction. In this kind of auction, all people who are interested in an item x submit sealed bids (simultaneously). The person whose bid is the highest then gets the item x at a price equal to the second highest bid.

A: Suppose there are n different bidders who have different marginal willingness to pay for the item x. Player i’s marginal willingness to pay for x is denoted v_i. Suppose initially that this is a complete information game—i.e., everyone knows everyone’s marginal willingness to pay for the item that is auctioned.

(a) Is it a Nash equilibrium in this auction for each player i to bid v_i?

Answer: Yes, this is a Nash equilibrium. Consider first the person j whose willingness to pay is the highest. If everyone other than j submits v_j as their bid, j wins the auction and has to pay a price p below his marginal willingness to pay. He therefore gets surplus of v_j − p > 0. He could bid something higher—in which case the outcome would still be the same. He could bid something lower—and, as long as he still bids something higher than the second highest bid, the outcome is still the same. Any other bid he submits would result in him not winning the auction—and getting zero instead of v_j − p > 0. Thus, by bidding v_j, he is best responding. Now consider any i ≠ j who ends up getting 0 (because he doesn’t win the auction and therefore doesn’t have to pay anything) if he plays v_j. Any other bid by i will also get him a payoff of 0 so long as the bid is below v_j. Any bid above v_j would cause i to win the auction and have to pay v_j—yielding a payoff of v_i − v_j < 0. Thus, by submitting the bid v_j, person i is best responding.

(b) Suppose individual j has the highest marginal willingness to pay. Is it a Nash equilibrium for all players other than j to bid zero and player j to bid v_j?

Answer: Yes, this is also a Nash Equilibrium. If everyone plays this strategy, individual j wins and has to pay p = 0—thus getting a payoff of v_j. But the outcome will be identical for any other bid j makes so long as he bids anything greater than zero. Thus, j is best responding. Now consider i ≠ j. Individual i does not win and does not have to pay anything—thus getting a payoff of 0 if he submits a bid of zero. This will be unchanged for any bid i makes so long as the bid is less than v_j. If he bids something greater than v_j, he would win but would have to pay v_j—thus getting a payoff of v_i − v_j < 0. Thus, he is best responding by submitting a bid of zero.

(c) Can you think of another Nash equilibrium to this auction?

Answer: There exist many other Nash Equilibria to this game. So long as j submits a bid of v_j and all others submit bids less than that, everyone is best responding to everyone else.

(d) Suppose that players are not actually sure about the marginal willingness to pay of all the other players—only about their own. Can you think of why the Nash equilibrium in which all players bid their marginal willingness to pay is now the most compelling Nash equilibrium?

Answer: If players are unsure of other players’ marginal willingness to pay, each player i bidding v_i will insure that no player will ever get a negative payoff and, if i wins, he will get a positive payoff (because he only has to pay the second highest bid). Bidding below v_i does not affect i’s payoff if he wins the auction—because it does not affect the price he will have to pay and thus will get a payoff of v_i − p > 0. It also does not affect player i’s payoff if he loses the auction—he will just get zero. It does, however, affect whether player i wins the auction. If player i bids below v_i, he may end up not winning the auction and therefore getting a payoff of zero when he could have gotten a positive payoff by winning the auction. If he bids above v_i, he might end up getting a negative payoff if he wins the auction—because then the second highest bid might be above his marginal willingness to pay and would therefore result in a price higher than v_i. Thus, if players are uncertain about other peoples’ willingness to pay, each player i will do at least as well if he plays v_i (for any set of bids by others) as he would by bidding differently—and he may do worse if he does not play v_i.
(e) Now consider a sequential first price auction in which an auctioneer keeps increasing the price of \( x \) in small increments and any potential bidder signals the auctioneer whether she is willing to pay that price. (Assume that the signal from bidders to auctioneer is not observable by other bidders.) The auction ends when only a single bidder signals a willingness to pay the price — and the winner then buys the item \( x \) for the price equal to his winning bid. Assuming the increments the auctioneer uses to raise the price during the auction are sufficiently small, approximately what will each player’s final bid be?

Answer: Every time the auctioneer shouts out a price, there is no reason for each bidder \( i \) not to signal he is willing to pay that price so long as the price is below \( v_i \). (If he does not, he may lose the auction and get zero — if he does signal and wins, he gets \( v_i - p > 0 \).) The last time player \( i \) signals will then occur approximately when \( p = v_i \).

(f) In equilibrium, approximately what price will the winner of the sequential auction pay?

Answer: As the auction progresses, fewer and fewer players will signal a willingness to pay — until only two bidders still send a positive signal. Suppose those players are \( i \) and \( j \) — and suppose \( i \)’s marginal willingness to pay \( v_i \) is smaller than \( j \)’s marginal willingness to pay \( v_j \). As the price continues to increase in small increments, player \( i \) will eventually stop signaling when it is approximately the case that \( p = v_i \). At that point, the auction ends, and player \( j \) pays approximately \( p = v_j \) to get the item \( x \). His payoff is then approximately \( v_j - p = v_j - v_i > 0 \), and everyone else’s payoff is zero.

(g) True or False: The outcome of the sealed bid second price auction is approximately equivalent to the outcome of the sequential (first price) auction.

Answer: This is true assuming the Nash equilibrium to the second price sealed bid auction is the one in which all players bid their marginal willingness to pay. In (d), we argued that this is indeed the most compelling Nash equilibrium if we think that individuals are not certain of anyone’s marginal willingness to pay other than their own. The outcome is then that the person with the highest marginal willingness to pay wins the auction and pays a price equal to the second highest marginal willingness to pay. This is the same outcome the sequential (first bid) auction produces.

B: This part provides a real-world example of how an auction of the type analyzed in part A can be used. When I became Department Chair in our economics department at Duke, the chair was annually deciding how to assign graduate students to faculty to provide teaching and research support. Students were paid a stipend by the department but their services were free to the faculty member to whom they were assigned.

(a) Under this system, faculty complained perpetually of a “teaching assistant shortage”. Why do you think this was?

Answer: The department was giving away something (i.e. graduate student assistance) that is valued by faculty for free. It is then not surprising that the quantity demanded was greater than the quantity available — i.e. there was a shortage created by the fact that a zero price was charged.

(b) I replaced the system with the following: Aside from some key assignments of graduate students as TAs to large courses, I no longer assigned any students to faculty. Instead, I asked the faculty to submit dollar bids for the right to match with a graduate student. If we had \( N \) graduate students available, I then took the top \( N \) bids, let those faculty know they had qualified for the right to match with a student and then let the matches take place (with students and faculty seeking each other out to create matches). Every faculty member who had a successful bid was then charged (to his/her research account) a price equal to the lowest winning bid which we called the “market price”. (Students were still paid by the department as before — the charges to faculty accounts simply came into the chair discretionary account and were then redistributed in a lump sum way to all faculty.) Given that we have a large number of faculty, should any individual faculty member think that his/her bid would appreciably impact the “market price”?

Answer: One bid out of many was unlikely to affect the lowest winning bid a whole lot — and only the lowest winning bid mattered in terms of determining the market price.

(c) In my annual e-mail to the faculty at the beginning of the auction for rights to match with students, I included the following line: “For those of you who are not game theorists, please
note that it is a dominant strategy for you to simply bid the actual value you place on the right to match with a student. “Do you agree or disagree with this statement? Why?

Answer: Since a person’s bid is unlikely to have an appreciable impact on the “market price” that is charged to those who bid successfully, the primary impact of an individual bid is whether or not the person gets the right to match with a student. Thus, if a faculty member bids his/her true valuation of the right to match with a student — i.e. his/her true marginal willingness to pay, then he/she will win the right to match with a student if his/her marginal willingness to pay is greater than the market price that is charged but not otherwise. If the faculty member were to bid higher than his/her true marginal willingness to pay, then he/she may end up getting the right to match at a “market price” that is higher than the faculty member’s actual marginal willingness to pay. If the faculty member were to bid lower than his/her true marginal willingness to pay, then he/she may end up not getting the right to match with a student even though he/she would have been willing to pay more than the “market price”. Thus, by over- or under-bidding, the faculty member risks getting an outcome that is not desirable, whereas by truthfully bidding the faculty member always gets the right to match when he/she values that right at or above the “market price”. And since the bid has no real impact on the market price, that is the only factor to really consider. Thus, my statement was roughly correct.

(d) Would it surprise you to discover that, for the rest of my term as chair, I never again heard complaints that we had a “TA shortage”? Why or why not?

Answer: That should not surprise you. Through the auction, we essentially established a market clearing (or competitive) price — a price at which everyone who wanted the assistance of a student at that price was able to get one, and no one that was not willing to pay that price received one. At the market clearing price, there is thus no one that wants a match with a student but does not get one, and there is no one that doesn’t want a match at that price that gets one. No shortage, and no surplus.

(e) Why do you think I called the lowest winning bid the “market price”? Can you think of several ways in which the allocation of students to faculty might have become more efficient as a result of the implementation of the new way of allocating students?

Answer: I called it a “market price” because it is exactly the price that lies at the intersection of the faculty demand curve for rights to match with students and the perfectly inelastic supply curve (that is simply set at the number of available students). The most obvious way in which this improves the efficiency of the allocation of students to faculty is that it maximizes faculty surplus — those who are most willing to pay are those who get the rights to match. Under the previous system, the chair could assign students in any way — and almost certainly did not assign them to those who valued the students most (as expressed through marginal willingness to pay). A slightly more subtle efficiency argument for the system is that it utilizes not only the information (about marginal willingness to pay) that faculty have but also the information that students have (since students negotiate matches with faculty who earned the right to match.) Thus, students try to find matches that work for them — and, assuming they know more than the chair, therefore have more information that is now used in the allocation process. Finally, an even more subtle efficiency argument is that the system causes students to market themselves to faculty — and faculty to market themselves to students, in the process clarifying what needs are most pressing for both the faculty and the students.
24.11 Business Application: Monopoly and Price Discrimination: In Chapter 23, we discussed first, second and third degree price discrimination by a monopolist. Such pricing decisions are strategic choices that can be modeled using game theory — which we proceed to do here. Assume throughout that the monopolist can keep consumers who buy at low prices from selling to those who are offered high prices.

A: Suppose a monopolist faces two types of consumers — a high demand consumer and a low demand consumer. Suppose further that the monopolist can tell which consumer has low demand and which has high demand; i.e. the consumer types are observable to the monopolist.

(a) Can you model the pricing decisions by the monopolist as a set of sequential games with different consumer types?

Answer: In this case, the monopolist essentially plays a different game with each consumer type (since he can tell the consumer type before the interaction begins). Thus, for each consumer type, we have a sequential game in which the monopolist makes a price/quantity offer to the consumer and the consumer decides to accept or reject it.

(b) Suppose the monopolist can construct any set of two-part tariffs — i.e. a per-unit price plus fixed fee for different packages. What is the subgame perfect equilibrium of your games?

Answer: In each game with a consumer type, the monopolist will make an offer that maximizes his surplus — which results in a quantity where per-unit price is equal to marginal cost and the fixed fee is equal to the consumer surplus that the consumer type would get if we were not to charge him a fixed fee. Each consumer type will then accept an offer if it results in at least zero consumer surplus and reject offers that result in negative consumer surplus. The result is that the monopolist engages in first degree price discrimination that results in consumers getting zero consumer surplus.

(c) True or False: First degree price discrimination emerges in the subgame perfect equilibrium but not in other Nash equilibria of the game.

Answer: This is true. We already demonstrated the first part of the statement in the previous exercise. The second part arises from the possibility of consumers engaging in non-credible threats that are taken seriously by monopolists in the first stage of the game. Such a threat might, for instance, involve a strategy by a consumer that goes as follows: “I will accept any offer that gets me at least $x$ consumer surplus, and I will reject all other offers.” Given that the consumer plays this strategy, the monopolist’s best response is to offer a package with consumer surplus $x$ — and given that this is the monopolist’s strategy, the consumer’s strategy is a best response. But it involves the non-credible threat that packages with positive consumer surplus will be rejected.

(d) How is this analysis similar to the game in exercise 24.5?

Answer: It is similar in the sense that the monopolist gets to make an offer of how the overall surplus is to be split between the consumer and the firm — and subgame perfection suggests that this structure of the game allows the monopolist to capture the entire surplus (just as the person who splits the $100 in exercise 24.5 offers the second player zero in expectation that the offer will be accepted.)

(e) Next, suppose that the monopolist cannot charge a fixed fee but only a per-unit price — but he can set different per-unit prices for different consumer types. What is the subgame perfect equilibrium of your games now?

Answer: The subgame perfect equilibrium now is what we called third degree price discrimination in Chapter 23.

B: Next, suppose that the monopolist is unable to observe the consumer type but knows that a fraction $\rho$ in the population are low demand types and a fraction $(1 - \rho)$ are high demand types. Assume that firms can offer any set of price/quantity combinations.

(a) Can you model the price setting decision by the monopolist as a game of incomplete information?

Answer: Yes. In the first stage, Nature moves and assigns low demand type to the consumer with probability $\rho$ and high demand type with probability $(1 - \rho)$. The firm does not know what nature did — i.e. it begins at an information set that contains two nodes, one for each of the two possible moves by Nature. From this information set, the firm has to determine what price/quantity combinations to offer. Then, in the third stage, consumers (who know their type) choose one of the offers or no offer at all.
(b) What is the (subgame) perfect Bayesian equilibrium of this game in the context of concepts discussed in Chapter 23? Explain.

Answer: The equilibrium outcome is second-degree price discrimination in which the monopolist structures the two price/quantity offers in such a way as to get the consumer types to self-identify; and he will reduce the attractiveness of the low demand package sufficiently in order to maximize the overall profit he gets from the two consumer types.
24.12 Business Application: Carrots and Sticks: Efficiency Wages and the Threat of Firing Workers. In our treatment of labor demand earlier in the text, we assumed that firms could observe the marginal revenue product of workers — and thus would hire until wage is equal to marginal revenue product. But suppose a firm cannot observe a worker’s productivity perfectly, and suppose further that the worker himself has some control over his productivity through his choice of whether to exert effort or “shirk” on the job. In part A of the exercise we will consider the subgame perfect equilibrium of a game that models this, and in part B we will see how an extension of this game results in the prediction that firms might combine “above market” wages with the threat to fire the worker if he is not productive. Such wages — known as efficiency wages — essentially have firms employing a “carrot-and-stick” approach to workers: Offer them high wages (the carrot), thus making the threat of firing more potent. (Note: While part B of the exercise conceptually builds on part A, no material from Section B of the chapter is required for any of it.)

A: Suppose the firm begins the game by offering the worker a wage \( w \). Once the worker observes the firm’s offer, he decides to accept or decline the offer. If the worker rejects the offer, the game ends and the worker is employed elsewhere at his market wage \( w^* \).

(a) Suppose the worker’s marginal revenue product is \( MRP = w^* \). What is the subgame perfect equilibrium for this game when marginal revenue product is not a function of effort?

Answer: The worker knows he can get \( w^* \) in the market — so he will reject any wage below \( w^* \). The firm will then offer \( w^* \) and the worker accepts it.

(b) Next, suppose the game is a bit more complicated in that the worker’s effort is correlated with the worker’s marginal revenue product. Assuming he accepted the firm’s wage offer, the worker can decide to exert effort \( e \) if hired by the firm. Since \( e \) is a cost for the worker, how must \( w^* \) be related to \( \gamma \) and \( x \) in order for it to be efficient for the worker not to be employed by the firm if the worker shirks?

Answer: We just concluded that the firm’s expected payoff from hiring a shirking worker is \( \gamma x - w \). The worker’s market wage is \( w^* \) — which is his marginal revenue product in the market. If \( w^* \geq \gamma x \), then the worker’s expected contribution to the firm would be less than his marginal revenue product in the market — implying it would be inefficient to hire the worker (and unprofitable for the firm).

(c) How must \( w^* \) be related to \( \gamma \) and \( x \) in order for it to be efficient for the worker not to be employed by the firm if the worker shirks?

Answer: If the worker shirks, the firm gets \( x - w \) with probability \( \gamma \) and \( -w \) with probability \( (1 - \gamma) \). For the worker, the payoff is \( (w - e) \) if the worker exerts effort and \( w \) if he does not. What is the firm’s expected payoff if the worker shirks?

Answer: If the worker shirks, the firm gets \( x - w \) with probability \( \gamma \) and \( -w \) with probability \( (1 - \gamma) \) — i.e. it’s expected payoff is

\[
\gamma(x - w) + (1 - \gamma)(-w) = \gamma x - w. \tag{24.23}
\]

(d) Suppose the worker exerts effort \( e \) if hired by the firm. Since \( e \) is a cost for the worker, how must \( w^* \) be related to \( x - e \) in order for it to be efficient for non-shirking workers to be hired by the firm?

Answer: It is not efficient for the worker to be hired if the overall cost is greater than the benefit to the firm. The cost of taking the worker out of the market (where his marginal revenue product is \( w^* \)) is \( w^* \) — and the effort cost to the worker is \( e \). Thus, the overall net (societal) benefit of hiring the worker into the firm is \( x - w^* - e \). So long as this is positive, it is efficient to hire the non-shirking worker into the firm — i.e. it is efficient so long as \( x - e \geq w^* \).

(e) Suppose \( w^* \) is related to \( \gamma \), \( x \) and \( e \) such that it is efficient for workers to be hired by the firm only if they don’t shirk — i.e. if the conditions you derived in (c) and (d) hold. What is the subgame perfect equilibrium? Will the firm be able to hire workers? Is the equilibrium efficient?

Answer: Combining the conditions from the previous two parts, we now assume that

\[
x - e \geq w^* \geq \gamma x. \tag{24.24}
\]

Consider the worker first: Suppose the worker has accepted the wage offer from the firm. There is nothing at this point to keep the worker from shirking — since he gets the wage
regardless of what payoff is experienced by the firm. Thus, the worker will shirk and exert no effort. Given that he will shirk, the worker will accept any wage that is at least as high as what he gets in the market — i.e. he will accept wage offers that are at least \( w^* \). The firm will know this — and know also that the worker will shirk. Since \( w^* \geq \gamma \), the firm's expected profit from hiring a shirking worker at a wage of \( w^* \) or higher is negative — which implies the firm will not offer a wage of \( w^* \) or higher. Thus, the wage offer will be \( w < w^* \) — which implies it will be rejected and the worker will instead work in the market. The firm therefore hires no workers.

(f) The subgame perfect equilibrium you just derived is inefficient. Why? What is the underlying reason for this inefficiency?

**Answer:** This outcome is inefficient since we have made assumptions that make it efficient for the firm to hire the worker if the worker does not shirk — and the worker would do better off by not shirking and accepting a high wage offer from the firm than he does in the market. Put differently, if the worker could commit to not shirking, the firm would be willing to offer a wage such that the firm makes a positive payoff and the worker does better than he would in the market. But the problem is that there is no way for the worker to commit to not shirking, which keeps the firm from offering the wage that would in fact attract the worker.

B: The problem in the game defined in part A is that we are not adequately capturing the fact that firms and workers do not typically interact just once if a worker is hired by a firm. Suppose, then, that we instead think of the relationship between worker and firm as one that can potentially be repeated infinitely. Each day the firm begins by offering a wage \( w \) to the worker; the worker accepts or rejects the offer — walking away with a market wage \( w^* \) (and ending the relationship) if he rejects. If he accepts, the worker either exerts effort \( e \) or shirks — and the firm observes whether it ends the day with a payoff of \( (x - w) \) (which it gets for sure if the worker exerts effort but only with probability \( \gamma < 1 \) if the worker shirks) or \( (-w) \) (which can happen only if the worker shirks). Everyone goes home at the end of the day and meets again the next day (knowing how all the previous days turned out).

(a) Consider the following strategy for the firm: Offer \( w = \bar{w} > w^* \) on the first day; then offer \( w = \bar{w} \) again every day so long as all previous days have yielded a payoff of \( (\bar{w} - x) \); otherwise offer \( w = 0 \). Is this an example of a trigger strategy?

**Answer:** Yes, this is a trigger strategy: It begins by offering a high wage, in essence giving the worker the benefit of the doubt in anticipation of no shirking. If the worker is caught shirking, however, the firm responds by never again offering a wage above zero. Thus, one instance of getting caught is the “trigger” for a “punishment” that continues forever.

(b) Consider the following strategy for the worker: Accept any offer \( w \) so long as \( w \geq w^* \); reject offers otherwise. Furthermore, exert effort \( e \) upon accepting an offer so long as all previous offers (including the current one) have been at least \( \bar{w} \); otherwise shirk. Is this another example of a trigger strategy?

**Answer:** Yes, this too is an example of a trigger strategy. The worker always accepts offers above his market wage \( w^* \) — but any wage below \( \bar{w} \) “triggers” an end to effort.

(c) Suppose everyone values a dollar next period at \( \delta < 1 \) this period. Suppose further that \( P_e \) is the present discounted value of all payoffs for the worker assuming that firms always offer \( w = \bar{w} \) and the worker always accepts and exerts effort. Explain why the following must then hold: \( P_e = (\bar{w} - e) + \delta P_e \).

**Answer:** If firms always offer \( \bar{w} \) and workers always accept and exert effort, the payoff to the worker in the current period is \( (\bar{w} - e) \) — the wage minus the effort cost. Tomorrow the game will be exactly as it is now — with the present discounted value of payoffs exactly as it is today — i.e. equal to \( P_e \). Viewed from today, however, \( P_e \) tomorrow is worth \( \delta P_e \) today. So the value of the game to the worker is \( (\bar{w} - e) \) that will be earned today plus \( \delta P_e \) — the value of the game starting tomorrow as viewed from today. Adding these together, we get that the present discounted value of the game \( P_e \) is equal to \( (\bar{w} - e) + \delta P_e \).

(d) Use this to determine the present discounted value \( P_e \) of the game (as a function of \( \bar{w} \), \( e \) and \( \delta \)) for the worker assuming it is optimal for the worker to exert effort when working for the firm.
Answer: Solving the equation $P_e = (\overline{w} - e) + \delta P_e$ for $P_e$, we get

$$P_e = \frac{\overline{w} - e}{1 - \delta}.$$  \hspace{1cm} (24.25)

(e) Suppose the firm offers $w = \overline{w}$. Notice that the only way the firm can ever know that the worker shirked is if its payoff on a given day is $(-\overline{w})$ rather than $(x - \overline{w})$ — and we have assumed that this happens with probability $1 - \gamma$ when the worker exerts no effort. Thus, a worker might decide to take a chance and shirk — hoping that the firm will still get payoff of $(x - \overline{w})$ (which happens with probability $\gamma$). What is the worker’s immediate payoff (today) from doing this?

Answer: The worker’s immediate payoff is then $\overline{w}$ — because he collects the wage without incurring any effort cost.

(f) Suppose that the worker gets unlucky and is caught shirking the first time — and that he requires in order to not shirk? How does this premium change with the cost of effort $s$? How much of a premium above the market wage $w^*$ does this imply the worker requires in order to not shirk? How does this premium change with the cost of effort $s$? How does it change with the probability of getting caught shirking? Does this make sense?

Answer: Substituting $P_e$ (from part (d)) and $P_s$ (from part (g)) into the inequality $P_e \geq P_s$, we get

$$P_e \leq \frac{\overline{w} - e}{1 - \delta} = \frac{1 - \delta}{1 - \delta} \frac{(1 - \gamma)w^*}{(1 - \gamma)(1 - \delta)} = P_s.$$  \hspace{1cm} (24.30)
Multiplying both sides by \((1 - \gamma)\), adding \(e\) to both sides and collecting the \(w\) terms on the left hand side, we get

\[
\frac{w}{1 - \delta \gamma} \geq \frac{\delta (1 - \gamma) w^*}{1 - \delta \gamma} + e
\]  
(24.31)

or

\[
\frac{(1 - \delta \gamma) - (1 - \delta)}{1 - \delta \gamma} \geq \frac{\delta (1 - \gamma) w^* + (1 - \delta \gamma) e}{1 - \delta \gamma}
\]  
(24.32)

Multiplying both sides by \((1 - \gamma \delta)\), simplifying the numerator of the left hand side to \(\delta (1 - \gamma)\) and dividing both sides by \(\delta (1 - \gamma)\), we then get

\[
w \geq w^* + \frac{(1 - \delta \gamma) e}{\delta (1 - \gamma)}
\]  
(24.33)

The wage premium necessary to convince workers to work at this firm and put in effort is therefore \((w - w^*)\) or

\[
\text{Wage Premium} = w - w^* = \frac{(1 - \delta \gamma) e}{\delta (1 - \gamma)} \left[ 1 + \frac{1 - \delta}{\delta (1 - \gamma)} \right] e.
\]  
(24.34)

As the effort cost \(e\) increases, the premium increases as well — which makes sense since \(e\) is a cost to workers that they will need to be compensated for. The probability of getting caught shirking is \((1 - \gamma)\) — as this increases, the premium decreases. This too makes sense — the more likely it is that I can actually know that you are shirking, the less of a premium I have to pay you to bribe you to not shirk since you know I am more likely to catch you if you do.

(i) What is the highest that \(w\) can get in order for the firm to best respond to workers (who play the strategy in (b)) by playing the strategy in (a)? Combining this with your answer to (h), how must \((x - e)\) be related to \(w^*, \delta, \gamma\) and \(e\) in order for the strategies in (a) and (b) to constitute a Nash equilibrium? Given your answer to A(d), will it always be the case that firms hire non-shirking workers whenever it is efficient?

\text{Answer:} \quad \text{Firms must get a positive payoff from hiring workers than don't shirk at wage } \overline{w} — \text{ i.e. } x - \overline{w} \geq 0 \text{ or } x \geq \overline{w}. \text{ Combining this with equation (24.33), we get that}

\[
x \geq \overline{w} \geq w^* + \frac{(1 - \delta \gamma) e}{\delta (1 - \gamma)}
\]  
(24.35)

which can also be written as

\[
x \geq w^* + \left[ 1 + \frac{1 - \delta}{\delta (1 - \gamma)} \right] e
\]  
(24.36)

or

\[
(x - e) \geq w^* + \left[ 1 + \frac{1 - \delta}{\delta (1 - \gamma)} \right] e.
\]  
(24.37)

We concluded in A(d) that it is efficient for non-shirking workers to be hired by the firm if \((x - e) \geq w^* \geq \gamma x\). Since there exist cases when \((x - e) \geq w^* \geq \gamma x\) and equation (24.37) is not satisfied, we can then conclude that there will be cases when it would be efficient for non-shirking workers to be hired by the firm, but even with efficiency wages this will not happen. However, there are also cases when \((x - e) \geq w^* \geq \gamma x\) and equation (24.37) are both satisfied — and thus efficiency wages (combined with the threat of firing shirking workers) result in efficiency enhancements — i.e. in the hiring of non-shirking workers by firms when this is efficient.
24.13 Policy Application: Negotiating with Pirates, Terrorists (and Children). While we often think of pirates as a thing of the past, piracy in international waters has been on the rise. Typically, pirates seize a commercial vessel and then demand a monetary ransom to let go of the ship. This is similar to some forms of terrorism where, for instance, terrorists kidnap citizens of a country with which the terrorists have a grievance — and then demand some action by the country in exchange for the hostages.

A: Oftentimes, countries have an explicit policy that “we do not negotiate with terrorists” — but still we often discover after the fact that a country (or a company that owns a shipping vessel) paid a ransom or took some other action demanded by terrorists in order to resolve the crisis.

(a) Suppose the ships of many countries are targeted by pirates. In every instance of piracy, a country faces the decision of whether or not to negotiate, and the more likely it is that pirates find victims amenable to negotiating a settlement, the more likely it is that they will commit more acts of piracy. Can you use the logic of the Prisoners’ Dilemma to explain why so many countries negotiate even though they say they don’t? (Assume Pirates cannot tell who owns a ship before they board it.)

Answer: If no country negotiated, pirates and terrorists would have the lowest possible incentives to continue. But each country faces Prisoners’ Dilemma incentives in the following sense: If the country negotiates, it reaps the benefits but the costs get spread over all countries. Thus, regardless of what other countries do, it may be a dominant strategy for each country to itself choose negotiation when faced with a pirate or terrorist threat. (There are, of course, other things at play, like large countries attempting to exercise leadership that makes it easier for others to follow, politicians facing political consequences if negotiations become public, etc.).

(b) Suppose that only a single country is targeted by terrorists. Does the Prisoner’s Dilemma still apply?

Answer: No, it does not apply. The single country will face all the costs and benefits of engaging (or not engaging) with terrorists.

(c) If you had to guess, do you think small countries or large countries are more likely to negotiate with pirates and terrorists?

Answer: Small countries would be more likely to negotiate with terrorists. The benefit of such negotiations is that the country will be able to get something in return for negotiating: the cost is that terrorism is more likely to pay off — which implies that the cost of the small country negotiating will be borne by all countries (with only a small fraction borne by the small country). A larger country faces more of this cost.

(d) Children can be like terrorists — screaming insanely to get their way and implicitly suggesting that they will stop screaming if parents give in. In each instance, it is tempting to just give them what they want, but parents know that this will teach children that they can get their way by screaming, thus leading to an increased frequency of outbursts by the little terrors. If a child lives with a single parent, is there a Prisoners’ Dilemma?

Answer: Just as there is no Prisoner’s Dilemma in part (a), there is none here — the single parent faces all the costs and benefits of giving into the screaming child.

(e) What if the child lives in a two-parent household? What if the child is raised in a commune where everyone takes care of everyone’s children?

Answer: The Prisoner’s Dilemma problem is introduced when there are two parents — and gets worse as the number of adults in charge of the children increases (for the same reasons as in (b)).

(f) All else being equal, where would you expect the most screaming per child: in a single-parent household, a two-parent household or in a commune?

Answer: For reasons articulated above, we would expect the most screaming in the commune.
Everyday Business and Policy Application: Education as a Signal

In Chapter 22, we briefly discussed the signaling role of education — i.e., the fact that part of the reason many people get more education is not to learn more but rather to signal high productivity to potential employers (in hopes of getting a higher wage offer). We return to this in part B of this exercise in the context of an incomplete information game (built on concepts from Section B of the chapter) but first consider the lack of a role for signaling in a complete information game. Throughout, suppose that there are two types of workers — type 1 workers with low productivity and type 2 workers with high productivity, with a fraction $\delta$ of all workers being type 2 and a fraction $1-\delta$ being type 1. Both types can earn education by expending effort, but it costs type 1 workers $e$ to get education level $e > 0$ while it costs type 2 workers only $e/2$. An employer gets profit $(2-w)$ if she hires a type 2 worker at wage $w$ and $(1-w)$ if she hires a type 1 worker at wage $w$. (Employers get zero profit if they do not hire a worker). We then assume that the worker decides in stage 1 how much education to get; then, in stage 2, he approaches two competing employers who decide simultaneously how much of a wage $w$ to offer; and finally, in stage 3, he decides which wage offer to accept.

A: Suppose first that worker productivity is directly observable by employers; i.e., firms can tell who is a type 1 and who is a type 2 worker by just looking at them.

(a) Solving this game backwards, what strategy will the worker employ in stage 3 when choosing between wage offers?

Answer: The worker will choose the higher of the offers. If the offers are the same, his strategy can be to accept firm 1’s offer or firm 2’s offer or flip a coin.

(b) Given that firms know what will happen in stage 3, what wage will they offer to each of the two types in the simultaneous move game of stage 2 (assuming that they best respond to one another)? (Hint: Ask yourself if the two employers could offer two different wages to the same worker type, and — if not — how competition between them impacts the wage that they will offer in equilibrium.)

Answer: In equilibrium, it must be that the two firms offer the same wage $w_1$ to type 1 workers and the same wage $w_2$ to type 2 workers. Suppose this were not the case. Then one firm is offering a higher wage than another to some type of worker — which means that the firm that is offering the higher wage is essentially not competing for the worker and settles for a payoff of zero. So long as the firm that is offering the lower wage is making profit by hiring the worker, the other firm could be doing better by bidding slightly above what the first firm bid and thus getting the worker. This continues to be true until the wage for each worker type gives the firm that hires the worker zero profit. Thus, in equilibrium, $w_2 = 2$ and $w_1 = 1$. (This is an example of what we will call “Bertrand competition” in the next chapter.)

(c) Note that we have assumed that worker productivity is not influenced by the level of education $e$ chosen by a worker in stage 1. Is there any way that the level of $e$ can then have any impact on the wage offers that a worker gets in equilibrium?

Answer: No, not when firms can already identify the productivity of job applicants without knowing anything about education level.

(d) Would the wages offered by the two employers be any different if the employers moved in sequence — with employer 2 being able to observe the wage offer from employer 1 before the worker chooses an offer?

Answer: No, the same logic applies: The best response of employer 2 is to offer a wage slightly higher than that offered by employer 1 — as long as profit from doing so is non-negative — and to offer a zero-profit wage otherwise. Employer 1 knows this and therefore sets a zero-profit wage — causing employer 2 to best respond by matching the wage offered by employer 1.

(e) What level of $e$ will the two worker types then get in any subgame perfect equilibrium?

Answer: The workers will get no education (since there is no value to the education) — at least assuming that they don’t get some intrinsic pleasure that we have not modeled) from education.

(f) True or False: If education does not contribute to worker productivity and firms can directly observe the productivity level of job applicants, workers will not expend effort to get education, at least not for the purpose of getting good wage offer.

Answer: This is true as we have shown above.
B: Now suppose that employers cannot tell the productivity level of workers directly — all they know is the fraction $\delta$ of workers that have high productivity and the education level $e$ of job applicants.

(a) Will workers behave any differently in stage 3 than they did in part A of the exercise?

Answer: No, they will not. In stage 3, workers simply have to decide between competing offers, and they will still optimize by choosing the higher of the two (or flipping a coin if the offers are the same).

(b) Suppose that there is a separating equilibrium in which type 2 workers get education $\bar{e}$ that differs from the education level type 1 workers get — and thus firms can identify the productivity level of job applicants by observing their education level. What level of education must type 1 workers be getting in such a separating equilibrium?

Answer: If the equilibrium is a separating equilibrium in which firms can tell infer from the education signals which worker is type 1 and which is type 2, then type 1 workers have nothing to gain from getting any education. Thus, $e = 0$ for type 1 workers.

(c) What wages will the competing firms offer to the two types of workers? State their complete strategies and the beliefs that support these.

Answer: For reasons analogous to those in part A of the exercise, the competition between employers will cause them to offer $w_2 = 2$ to workers identified as type 2 and $w_1 = 1$ to workers they believe are type 1 workers. Their strategy is therefore to offer $w = 2$ if an applicant has education level $\bar{e}$ and $w = 1$ otherwise. In order for this to be Bayesian equilibrium behavior, firms must believe that a signal $e$ identifies type 2 workers — and that any other signal identifies a worker as a type 1 worker. Even though low productivity workers will not get any education in equilibrium and high productivity workers will all get $e$ in equilibrium, it is still necessary to specify what firms believe a signal other than the two equilibrium signals means — and inferring from any signal other than $e$ that the worker is low productivity does not violate any Bayesian updating rules since no such behavior actually occurs in equilibrium.

(d) Given your answers so far, what values could $e$ take in this separating equilibrium? Assuming $\bar{e}$ falls in this range, specify the separating perfect Bayesian Nash equilibrium — including the strategies used by workers and employers as well as the full beliefs necessary to support the equilibrium.

Answer: It must be that $\bar{e}$ is such that type 1 workers do not find it worthwhile to pretend to be type 2 workers and type 2 workers get enough from paying the cost of getting the education to justify doing so. First, consider type 1 workers: Their payoff from getting $\bar{e}$ and thus being offered $w_2 = 2$ is $(2 - \bar{e})$ while their payoff from getting $e = 0$ is $w_1 = 1$. Thus, it must be that $(2 - \bar{e}) \leq 1$, or $\bar{e} \geq 1$. Next, consider type 2 workers: They get $w_2 = 2$ if they have the education $\bar{e}$ which costs them $\bar{e}/2$ — thus giving a payoff of $(2 - \bar{e}/2) = (4 - \bar{e})/2$. If they get $e = 0$ instead, their payoff is $w_1 = 1$. Thus, in order for high productivity workers to get the education level $\bar{e}$, it must be that $(4 - \bar{e})/2 \geq 1$ or $\bar{e} \leq 2$. Combining these two conclusions, we get that

$$1 \leq \bar{e} \leq 2.$$  \hspace{1cm} (24.38)

Assuming $\bar{e}$ falls in this range, the following then constitutes a perfect Bayesian separating equilibrium: Workers choose $e = \bar{e}$ if they are type 2 and $e = 0$ if they are type 1 in stage 1 and accept the highest wage offer in stage 3 (or randomly pick between the wage offers if they are the same). Firms offer $w = 2$ if they observe $\bar{e}$ and $w = 1$ if they observe any other education level; and they believe $\bar{e}$ reveals a type 2 worker while any other education signal reveals a type 1 worker.

(e) Next, suppose instead that the equilibrium is a pooling equilibrium — i.e. an equilibrium in which all workers get the same level of education $\bar{e}$ and firms therefore cannot infer anything about the productivity of a job applicant. Will the strategy in stage 3 be any different than it has been?

Answer: No, workers will still choose the higher wage offer in the third stage, and can randomize between offers if the wage offers are the same.
(f) Assuming that every job applicant is type 2 with probability $\delta$ and type 1 with probability $(1-\delta)$, what wage offers will firms make in stage 2?

**Answer:** For the same reasons as explored in the other scenarios, firms will again make zero-profit wage offers. When facing a job applicant, an employer knows he has a high productivity worker with probability $\delta$ — and thus will make profit $(2-w)$ with probability $\delta$; and the employer knows he has a low productivity worker with probability $(1-\delta)$ — and thus will make a profit of $(1-w)$ with probability $(1-\delta)$. Thus, the expected profit from offering a wage $w$ is

$$\delta(2-w)+(1-\delta)(1-w) = 1+\delta-w.$$  \hfill (24.39)

The zero (expected) profit wage offer is therefore $w = (1+\delta)$.

(g) What levels of education $e$ could in fact occur in such a perfect Bayesian pooling equilibrium? Assuming $e$ falls in this range, specify the pooling perfect Bayesian Nash equilibrium — including the strategies used by workers and employers as well as the full beliefs necessary to support the equilibrium.

**Answer:** We are at liberty to specify any out-of-equilibrium beliefs for the firm — and in particular can specify that firms will believe workers who do not have education level $e$ to be type 1 workers. In such a case, firms would offer $w = 1$ if they had a job applicant who does not have education level $\bar{\mathcal{E}}$ (which they will never face in equilibrium). Type 2 workers then get a payoff of $(1+\delta-\bar{\mathcal{E}}/2)$ if they get education level $\bar{\mathcal{E}}$ and at most 1 if they do not. Thus, it must be that $(1+\delta-\bar{\mathcal{E}}/2) \geq 1$ or $\bar{\mathcal{E}} \leq 2\delta$. Type 1 workers will get payoff $(1+\delta-\bar{\mathcal{E}})$ if they get the education and at most 1 otherwise — so it must be that $(1+\delta-\bar{\mathcal{E}}) \geq 1$ or $\bar{\mathcal{E}} \leq \delta$. The second condition is strictly more binding than the first — so we can conclude that, in any pooling equilibrium,

$$\bar{\mathcal{E}} \leq \delta.$$ \hfill (24.40)

For $\bar{\mathcal{E}} \leq \delta$, a pooling perfect Bayesian Nash equilibrium is then as follows: Regardless of what type, workers will get education $\bar{\mathcal{E}}$ and will then choose the highest wage offer or randomize between wage offers if the two offers are the same. Firms will offer the wage $w = 1 + \delta$ if the the applicant has education level $\bar{\mathcal{E}}$ and $w = 1$ otherwise, and firms believe that every job applicant with education level $\bar{\mathcal{E}}$ has high productivity with probability $\delta$ and low productivity with probability $(1-\delta)$ — and any job applicant with a different level of education is believed to be a low productivity worker.

(h) Could there be an education level $\bar{\mathcal{E}}$ that high productivity workers get in a separating equilibrium and that all workers get in a pooling equilibrium?

**Answer:** No, this is not possible. The highest $\bar{\mathcal{E}}$ that can exist in a pooling equilibrium is $\delta$, and the lowest possible $\bar{\mathcal{E}}$ that can exist in a separating equilibrium is 1. Since $\delta$ is a probability, the highest it can possible get is 1 — but if it is 1, there is no information asymmetry to begin with and firms automatically know that every job applicant is a high productivity worker (because there are no low productivity workers in the economy). In that case, they would simply offer a wage of $w = 2$ to all workers, and there would be no need for anyone the get an education. For any $\delta < 1$, $\bar{\mathcal{E}}$ is strictly lower in a pooling equilibrium than in a separating equilibrium.

(i) What happens to the pooling wage relative to the highest possible wage in a separating equilibrium as $\delta$ approaches 1? Does this make sense?

**Answer:** As $\delta$ approaches 1, the pooling equilibrium wage $w = (1+\delta)$ approaches 2 which is the highest wage (offered to only high productivity workers) in the separating equilibrium. This should make sense — as the likelihood of encountering a low productivity worker diminishes (when $\delta$ approaches 1), firms have increasingly little reason not to offer a wage close to the high productivity (zero profit) wage.
24.15 Everyday Business and Policy Application: To Fight or not to Fight. In many situations, we are confronted with the decision of whether to challenge someone who is currently engaged in a particular activity. In personal relationships, for instance, we decide whether it is worthwhile to push our own agenda over that of a partner; in business, potential new firms have to decide whether to challenge an incumbent firm (as discussed in one of the examples in the text); and in elections, politicians have to decide whether to challenge incumbents in higher level electoral competitions.

A: Consider the following game that tries to model the decisions confronting both challenger and incumbent: The potential challenger moves first — choosing between staying out of the challenge, preparing for the challenge and engaging in it, or entering the challenge without much preparation. We will call these three actions O (for "out"), P (for "prepared entry") and U (for "unprepared entry"). The incumbent then has to decide whether to fight the challenge (F) or give into the challenge (G) if the challenge takes place; otherwise the game simply ends with the decision of the challenger to play O.

(a) Suppose that the payoffs are as follows for the five potential combinations of actions, with the first payoff indicating the payoff to the challenger and the second payoff indicating the payoff to the incumbent: (P, G) leads to (3,3); (P, F) leads to (1,1); (U, G) leads to (4,3); (U, F) leads to (0,2); and O leads to (2,4). Graph the full sequential game tree with actions and payoffs.

Answer: This is illustrated in Graph 24.4.

Graph 24.4: The Challenge to an Incumbent

(b) Illustrate the game using a payoff matrix (and be careful to account for all strategies).

Answer: This is illustrated in Table 24.15. The strategy pairs for the incumbent are such that the first action is the one planned if the challenger plays P and the second is the action planned if the challenger plays U. For instance, the pair (G, F) indicates a strategy that responds to a prepared challenge with G and an unprepared challenge with F.

<table>
<thead>
<tr>
<th>Challenger</th>
<th>Incumbent</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(G, G) 3,3</td>
</tr>
<tr>
<td>U</td>
<td>(G, G) 4,3</td>
</tr>
<tr>
<td>O</td>
<td>(G, G) 2,4</td>
</tr>
</tbody>
</table>

Table 24.15: The Challenge to an Incumbent

(c) Identify the pure strategy Nash equilibria of the game and indicate which of these is subgame perfect.
Answer: If the challenger plays $P$, the incumbent best responds by playing either $(G, G)$ or $(G, F)$. If the incumbent plays $(G, G)$, however, the challenger would best respond by playing $U$ but if the incumbent plays $(G, F)$, the challenger best responds by playing $P$. Thus, $P$ is a best response to $(G, F)$ and $(G, F)$ is a best response to $P$ — implying that $(P, (G, F))$ is a Nash equilibrium. If the challenger plays $U$, the incumbent best responds by playing $(G, G)$ or $(F, G)$ — and $U$ by the challenger is a best response to both of these. Thus, both $(U, (G, G))$ and $(U, (F, G))$ are Nash equilibria. Finally, if the challenger plays $O$, all four of the incumbent’s strategies are best responses, but $O$ is a best response by the challenger only to $(F, F)$. Thus, $(O, (F, F))$ is a Nash equilibrium. We therefore have identified four pure strategy Nash equilibria. But of these, only $(U, (G, G))$ is subgame perfect. You can see this by solving the game tree in part (a) backwards: If the incumbent encounters a prepared challenger, his best response is to play $G$ (thus giving him payoff of 3 rather than 1), and if he encounters an unprepared challenger, his best response is also to play $G$ (thus getting 3 rather than 2). Knowing that the incumbent will respond by playing $G$ to any challenge, the challenger knows he will get payoff of 3 if he plays $P$, 4 if he plays $U$ and 2 if he plays $O$ — and he will therefore choose $U$.

(d) Next, suppose that the incumbent only observes whether or not the challenger is engaging in the challenge (or staying out) but does not observe whether the challenger is prepared or not. Can you use the logic of subgame perfection to predict what the equilibrium will be?

Answer: Yes. The incumbent’s best response to a challenge is the same — i.e. play $G$ — regardless of whether or not the challenger prepared. Thus, the challenger knows that the incumbent will play $G$ if challenged, and thus he knows he can get his highest payoff of 4 by playing $U$. The strategies in this case would then be for the challenger to play $U$ and the incumbent to play $G$ when facing a challenge.

(e) Next, suppose that the payoffs for $(P, G)$ changed to $(3, 2)$, the payoffs for $(U, G)$ changed to $(4, 2)$ and the payoffs for $(U, F)$ changed to $(0, 3)$ (with the other two payoff pairs remaining the same). Assuming again that the incumbent fully observes whether or not he is being challenged and whether the challenger is prepared, what is the subgame perfect equilibrium?

Answer: Solving the game tree backwards again, we can consider what the incumbent’s best response to $P$ and $U$ are. If he encounters a prepared challenger, he will do best by playing $G$ (thus getting a payoff of 2 rather than 1), but if he encounters an unprepared challenger, he will do best playing $F$ (and getting a payoff of 3 rather than 2). Thus, the subgame perfect strategy by the incumbent — i.e. the strategy that employs non-credible threats — is to play $(G, F)$. The challenger knows this — and thus expects a payoff of 3 from $P$, a payoff of 0 from $U$ and a payoff of 2 from $O$. He will therefore play $P$ — leading to the subgame perfect equilibrium $(P, (G, F))$ where the challenger issues a prepared challenge and the incumbent gives in.

(f) Can you still use the logic of subgame perfection to arrive at a prediction of what the equilibrium will be if the incumbent cannot tell whether the challenger is prepared or not as you did in part (d)?

Answer: No, it is no longer as clear. The reason it was clear in (d) derived from the fact that the challenger knew that the incumbent would always give in — regardless of whether he believed the challenger to be prepared or not. But now the best response by the incumbent depends on whether the incumbent believes the challenger to be prepared or not. Without knowing what beliefs the incumbent operates with as he interprets a challenge, we cannot apply the logic of subgame perfection. (For this reason, we need to move to the material of section B of the chapter where beliefs play a role in the formation of an equilibrium.)

B: Consider the game you ended with in part A(f).

(a) Suppose that the incumbent believes that a challenger who issues a challenge is prepared with probability $\delta$ and not prepared with probability $(1 - \delta)$. What is the incumbent’s expected payoff from playing $G$? What is his expected payoff from playing $F$?

Answer: If he plays $G$, he will get a payoff of 2 with probability $\delta$ when the challenger is prepared and a payoff of 2 with probability $(1 - \delta)$ when the challenger is unprepared. Thus, his expected payoff from $G$ is 2. If he plays $F$, he gets a payoff of 1 with probability $\delta$ and a payoff of 3 with probability $(1 - \delta)$. Thus, the expected payoff from $F$ is $\delta + 3(1 - \delta) = 3 - 2\delta$. 


(b) For what range of $\delta$ is it a best response for the incumbent to play $G$? For what range is it a best response to play $F$?
Answer: The incumbent best responds by playing $G$ if $2 \geq (3 - \delta)$ — i.e. if $\delta \geq 1/2$. He best responds by playing $F$ if $\delta \leq 1/2$.

(c) What combinations of strategies and (incumbent) beliefs constitute a pure strategy subgame perfect equilibrium? (Be careful: In equilibrium, it should not be the case that the incumbent’s beliefs are inconsistent with the strategy played by the challenger!)
Answer: If $\delta \leq 1/2$, the incumbent will play $F$ when encountering a challenge — implying that the challenger will get payoff of 1 by playing $P$, 0 by playing $U$ and 2 by playing $O$. Thus, the challenger plays $O$. Given that the challenger never challenges, any beliefs about whether the challenger prepared or not when he issues a challenge are consistent with the strategies played — including the beliefs $\delta \leq 1/2$. The following is then a set of pure strategy Bayesian Nash equilibria: players play the strategies $(O, F)$ and incumbents believe challengers who challenge are prepared with probability $\delta \leq 1/2$. The incumbent will fight when challenged but never is challenged in equilibrium.

If the incumbent has beliefs $\delta > 1/2$, on the other hand, he will play $G$ in response to a challenge. The challenger then expects a payoff of 3 from playing $P$, 4 from playing $U$ and 2 from playing $O$. He will therefore choose $U$. This might suggest equilibrium $(U, G)$ with beliefs $\delta \geq 1/2$. The only problem is that the beliefs are now not consistent with the strategies that are played — because the challenge plays $U$ with probability 1 while the incumbent believes the challenger to have played $U$ with only probability $(1 - \delta) \leq 1/2$ when a challenge is observed. Thus, we do not have pure strategy equilibria with beliefs $\delta < 1/2$.

(d) Next, suppose that the payoffs for $(P, G)$ changed to $(4, 2)$ and the payoffs for $(U, G)$ changed to $(3, 2)$ (with the remaining payoff pairs remaining as they were in (f)). Do you get the same pure strategy subgame perfect equilibrium?
Answer: Yes, the same equilibria still hold. Nothing has changed in the payoffs for the incumbent — who will therefore still play $F$ if $\delta \leq 1/2$. If the incumbent fights any challenge, the challenger is still better off playing $O$ — and given he does not challenge, the beliefs of $\delta \leq 1/2$ are not inconsistent with the strategies that are being played.

(e) In which equilibrium — the one in part (c) or the one in part (d) — do the equilibrium beliefs of the incumbent seem more plausible?
Answer: Consider first the game in (c): If the challenger challenges here, he is better off preparing if he expects $P$ and not preparing if he expects $G$. Next, consider the game in (d): In this version, if the challenger challenges, he is better off preparing regardless of whether the incumbent gives in or fights. In the latter case, it would seem reasonable that the incumbent then expects that any challenge is one by a prepared challenger — but the equilibrium beliefs we have derived require the incumbent to believe that a challenger is prepared with only probability $\delta \leq 1/2$. That seems unreasonable in a way that the same beliefs are not unreasonable in the game in (c) where there is no clear dominant choice between $P$ and $U$ for the challenger.
24.16 Everyday and Policy Application: Reporting a Crime. Most of us would like to live in a world where crimes are reported and dealt with, but we'd sure prefer to have others bear the burden of reporting a crime. Suppose a crime is witnessed by N people, and suppose the cost of picking up the phone and reporting a crime is $c > 0$.

As Begin by assuming that everyone places a value $x > c$ on the crime being reported, and if the crime goes unreported, everyone's payoff is 0. (Thus, they payoff to me if you report a crime is $x$, and the payoff to me if I report a crime is $(x - c)$.)

(a) Each person then has to simultaneously decide whether or not to pick up the phone to report the crime. Is there a pure strategy Nash equilibrium in which no one reports the crime?

**Answer:** No, there cannot be such an equilibrium. If no one reports the crime, everyone's payoff is 0. But, given that no one else is reporting the crime, each person could do better by reporting the crime and getting $(x - c) > 0$. Thus, when everyone is playing the strategy of not reporting the crime, no one is best responding to what everyone else is playing.

(b) Is there a pure strategy Nash equilibrium in which more than one person reports the crime?

**Answer:** No, there is no such equilibrium either. Suppose you and I both report the crime. Then both of us get a payoff of $(x - c)$. But it only takes one person to report the crime — so, given you are reporting, I would do better not reporting and thus get $x$ rather than $(x - c)$ (and given that I am reporting, you could do better by not reporting.) Thus, we are not best responding to each other's strategies if we both report the crime. The same logic holds if more than two people are reporting the crime.

(c) There are many pure strategy Nash equilibria in this game. What do all of them have in common?

**Answer:** We have concluded that no pure strategy Nash equilibrium could have crime go unreported, and no pure strategy Nash equilibrium could have more than one person report the crime. Thus, we are only left with the possibility of 1 person reporting the crime. Suppose if I am the only one to report the crime. Then I get payoff of $(x - c)$ while everyone else gets the maximum payoff of $x$. Clearly everyone else is best responding. But I am best responding as well — given that no one else is reporting the crime. If I were to not report it, then the crime would go unreported — and I would get 0 rather than $(x - c)$ > 0. Thus, every pure strategy Nash equilibrium has the feature that one person is reporting the crime — but it could be any one of the $N$ different people who witness the crime. Thus, there are $N$ different such Nash equilibria.

(d) Next, suppose each person calls with probability $\delta < 1$. In order for this to be a mixed strategy equilibrium, what has to be the relationship between the expected payoff from not calling and the expected payoff from calling for each of the players?

**Answer:** Suppose everyone is calling with probability $\delta$. Then the only way I am best responding by also calling with probability $\delta$ is if I am indifferent between not calling and calling; i.e. in order for this to be a mixed strategy equilibrium, it has to be that my expected payoff from not calling is equal to my expected payoff from calling given everyone else calls with probability $\delta$. Since we are all the same, the same then also has to hold for everyone else.

(e) What is the payoff from calling when everyone calls with probability $\delta < 1$?

**Answer:** If I end up calling, then I pay $c$ and get $x$ for sure — i.e. my payoff is $(x - c)$ if I call.

(f) What is the expected payoff from not calling when everyone calls with probability $\delta$? (Hint: The probability that one person does not call is $(1 - \delta)$ — and the probability that $(N - 1)$ people don’t call is $(1 - \delta)^{(N-1)}$.)

**Answer:** If I don’t call, then I get 0 if no one else calls and $x$ if at least one other person calls. The probability that no one else calls is $(1 - \delta)^{(N-1)}$ — which implies that the probability that at least one other person calls is $(1-(1 - \delta)^{(N-1)})$. The expected payoff from not calling is then

$$0 \left(1 - \delta\right)^{(N-1)} + x \left[1 - (1 - \delta)^{(N-1)}\right] = x \left[1 - (1 - \delta)^{(N-1)}\right].$$
Using your answers to (d) through (f), derive δ as a function of c, x and N such that it is a mixed strategy equilibrium for everyone to call with probability δ. What happens to this probability as N increases?

**Answer:** We concluded in (d) that, in order for everyone calling with probability δ to be a mixed strategy equilibrium, it has to be that the expected payoff from calling (derived in (e)) is equal to the expected payoff from not calling (derived in (f)); i.e. it has to be that

\[(x - c) = x \left(1 - (1 - δ)^{(N-1)}\right)\]  \hspace{1cm} (24.42)

which, after subtracting x from both sides, multiplying by -1 and dividing by x, can also be written as

\[\frac{c}{x} = (1 - δ)^{(N-1)}.\]  \hspace{1cm} (24.43)

Solving for δ, we get

\[δ = 1 - \left(\frac{c}{x}\right)^{1/(N-1)}.\]  \hspace{1cm} (24.44)

As N increases, 1/(N − 1) decreases. Since (c/x) lies between 0 and 1 (because x > c), we can then conclude that (c/x)^{1/(N-1)} increases as N increases. Thus, as N increases, we are subtracting a larger number from 1 — implying that δ decreases as N increases; i.e. as more people witness the crime, the probability of any one of them reporting the crime falls.

(h) What is the probability that a crime will be reported in this mixed strategy equilibrium? (Hint: From your work in part (f), you should be able to conclude that the probability that no one else reports the crime — i.e. \((1 - δ)^{(N-1)}\) — is equal to \(c/x\) in the mixed strategy equilibrium. The probability that no one reports a crime is then equal to this times the probability that the last person also does not report the crime.) How does this change as N increases?

**Answer:** The probability that no one else reports the crime in the mixed strategy equilibrium is therefore independent of N — but the probability that the last person reports the crime is δ which we have just concluded falls as N increases. Thus, the probability that a crime is reported falls as N increases — i.e. crimes are more likely to go unreported the more people witness the crime.

(i) True or False: If the reporting of crimes is governed by such mixed strategy behavior, it is advantageous for few people to observe a crime — whereas if the reporting of crime is governed by pure strategy Nash equilibrium behavior, it does not matter how many people witnessed a crime.

**Answer:** This is true. We just concluded that the crime is more likely to go unreported as N increases in the mixed strategy equilibrium — but we previously concluded that the crime gets reported by exactly one person in all pure strategy Nash equilibria.

(j) If the cost of reporting the crime differed across individuals (but is always less than x), would the set of pure Nash equilibria be any different? Without working it out, can you guess how the mixed strategy equilibrium would be affected?

**Answer:** The logic of parts (a) through (c) still holds — implying that the set of pure strategy equilibria is unchanged. In the mixed strategy equilibrium, however, it could no longer be the case that everyone calls with the same probability — because everyone needs to be indifferent between calling and not calling given what everyone else is doing in the mixed strategy equilibrium. And the payoff from calling is now higher for those whose cost of calling is lower — implying that their probability of calling will be higher than it is for those with higher costs.

B: Suppose from here on out that everyone values the reporting of crime differently, with person n’s value of having a crime reported denoted \(x_n\). Assume that everyone still faces the same cost c of reporting the crime. Everyone knows that c is the same for everyone, and person n discovers \(x_n\) prior to having to decide whether to call. But the only thing each individual knows about the x values for others is that they fall in some interval [0, b], with c falling inside that interval and with the probability that \(x_n\) is less than x given by \(p(x)\) for all individuals.
(a) What is \( P(0) \)? What is \( P(b) \)?

**Answer:** Since 0 is the lowest possible \( x \), the probability that \( x_n \) falls below 0 for some individual \( n \) is zero — i.e. \( P(0) = 0 \). And since \( b \) is the highest possible value \( x \) can take, \( P(b) = 1 \). (Since \( P(x) \) is defined as the probability that \( x_n \) is below \( x \), \( P(b) \) is actually just very close to 1.)

(b) From here on out, suppose that \( P(x) = xb \). Does what you concluded in (a) hold?

**Answer:** Yes, it does — because

\[
P(a) \frac{0}{b} = 0 \quad \text{and} \quad P(b) = \frac{b}{b} = 1.
\]  

(c) Consider now whether there exists a Bayesian Nash equilibrium in which each player \( n \) plays the strategy of reporting the crime if and only if \( x_n \) is greater than or equal to some critical value \( y \). Suppose that everyone other than \( n \) plays this strategy. What is the probability that at least one person other than individual \( n \) reports a crime? (Hint: Given this strategy, the probability that person \( k \) will not report a crime is equal to the probability that \( x_k \) is less than \( y \) — which is equal to \( P(y) \). The probability that \( K \) individuals do NOT report the crime is then \( P(y)^K \).)

**Answer:** The probability that no one else reports a crime is equal to the probability that \( (N-1) \) people do not report the crime. This probability is \( P(y)^{N-1} \). The probability that at least one of the \( (N-1) \) individuals reports the crime is then \( (1 - P(y))^{N-1} \) which is

\[
1 - \left( \frac{y}{b} \right)^{N-1}.
\]

(d) What is the expected payoff of not reporting the crime for individual \( n \) whose value is \( x_n \)? What is the expected payoff of reporting the crime for this individual?

**Answer:** The expected payoff of reporting the crime is \( (x_n - c) \), and the expected payoff of not reporting the crime is equal to 0 times the probability that no one else reported it either plus \( x_n \) times the probability that at least one other individual reported the crime — i.e.

\[
0 \left( \frac{y}{b} \right)^{N-1} + x_n \left(1 - \left( \frac{y}{b} \right)^{N-1}\right) = x_n \left[ 1 - \left( \frac{y}{b} \right)^{N-1}\right].
\]

(e) What is the condition for individual \( n \) to optimally not report the crime if \( x_n < y \)? What is the condition for individual \( n \) to optimally report the crime when \( x_n \geq y \)?

**Answer:** The condition for \( n \) to optimally not report the crime if \( x_n < y \) is that the expected payoff from not reporting is greater than or equal to the expected payoff from reporting. The condition for \( n \) to optimally report the crime if \( x_n \geq y \) is that the expected payoff from reporting is greater than or equal to the expected payoff from not reporting. These two conditions translate to

\[
x_n \left[ 1 - \left( \frac{y}{b} \right)^{N-1}\right] \geq (x_n - c) \quad \text{if} \quad x_n < y
\]

and

\[
(x_n - c) \geq x_n \left[ 1 - \left( \frac{y}{b} \right)^{N-1}\right] \quad \text{if} \quad x_n \geq y.
\]

Subtracting \( x_n \) from both sides of each of these inequalities and multiplying by -1, these can be written as

\[
x_n \left( \frac{y}{b} \right)^{N-1} \leq c \quad \text{if} \quad x_n < y \quad \text{and} \quad x_n \left( \frac{y}{b} \right)^{N-1} \geq c \quad \text{if} \quad x_n \geq y.
\]

(f) For what value of \( y \) have we identified a Bayesian Nash equilibrium?

**Answer:** These conditions imply that

\[
y \left( \frac{y}{b} \right)^{N-1} = c
\]
which we can solve to get

\[ y = \left( \frac{c b^{(N-1)}}{N} \right)^{1/N} = c^{1/N} b^{(N-1)/N}. \]  

(24.52)

(g) *What happens to the equilibrium probability of a crime being reported as \( N \) increases?*

**Answer:** The probability that an individual \( n \) will *not* report a crime is equal to the probability that \( x_n \) is less than \( y \) — i.e. \( P(y) \). Now that we have calculated the equilibrium \( y \), we can write this as

\[ P(y) = c^{1/N} b^{(N-1)/N} = \left( \frac{c}{b} \right)^{1/N}. \]

(24.53)

The equilibrium probability that no one reports the crime is then

\[ (P(y))^N = \left( \frac{c}{b} \right)^{N} = \frac{c}{b} \]

(24.54)

This is the probability that no one will report the crime. The probability that the crime will be reported is therefore

\[ 1 - (P(y))^N = 1 - \frac{c}{b}; \]

(24.55)

i.e. the probability of a crime being reported does not change as the number of witnesses increases.

(h) *How is the probability of a crime being reported (in this equilibrium) affected by \( c \) and \( b \)? Does this make sense?*

**Answer:** Looking at the probability of a crime being reported as derived in equation (24.55), we can see that this probability falls as \( c \) increases and rises as \( b \) increases. This makes sense — as it becomes more costly to report a crime, it is less likely that crimes will be reported; and as people value having crime reported more, i.e. as \( b \) increases, crime is more likely to be reported.
24.17 Policy Application. Some Prisoners’ Dilemmas. We mentioned in this chapter that the incentives of the prisoners’ dilemma appear frequently in real world situations.

At each of the following, explain how these are prisoners’ dilemmas and suggest a potential solution that might address the incentive problems identified in such games.

(a) When I teach the topic of prisoners’ dilemmas in large classes that also meet in smaller sections once a week, I sometimes offer the following extra credit exercise: Every student is given 10 points. Each student then has to decide how many of these points to donate to a “section account” and convey this to me privately. Each student’s payoff is a number of extra credit points equal to the number of points they did not donate to their section plus twice the average contribution to the section account by students registered in their section. For instance, if a student donates 4 points to his section and the average student in the section donated 3 points, then this student’s payoff would be 12 extra credit points — 6 because the student only donated 4 of his 10 points, and 6 because he gets twice the average donated in his section.

Answer: This is a prisoners’ dilemma because — although it is a dominant strategy for everyone to keep all their points (and not contribute to the section account), everyone would be better off if everyone contributed all their points. (In the case where everyone holds onto their points, each student in the section gets 10 extra credit points; in the case where everyone donates all points, the average donation is 10 and thus all students in the section would get 20 extra credit points; but no matter what everyone else in the section does, it is better for an individual student to not contribute his own points so long as there are more than 2 students in the class.) This is the classic prisoners’ dilemma. And students find all sorts of creative ways out of it. One student once designed a program through which students would send their e-mails to me, with the program set so that it would submit all points from each student so long as all students agreed — but would send no points otherwise. This changed the incentives in the game — because now students knew that the would get 10 for sure if they submitted anything other than all their points — but they had a chance at 20 points if they submitted all their points. Other students organized ways of checking each other’s e-mails before they were sent. When given the option of voting for a 100% tax that would transfer all points in the section to the section account, most students voted for this — essentially hiring the government to force them to do what they otherwise knew they would have trouble getting done.

(b) People get in their cars without thinking about the impact they have on other drivers by getting on the road — and at certain predictable times, this results in congestion problems on roads.

Answer: Everyone would be better off if all drivers would take into account the costs they impose on others when getting on the road — but it is in no one's interest to do so. Put differently, everyone would like better an outcome where we cooperated but it is in everyone's incentive not to cooperate no matter what others do — the classic prisoners' dilemma. This is the same for all externalities. One possible solution is to have congestion pricing on roads — i.e. fees that increase during times when there is otherwise much congestions. This changes the payoffs of the prisoners' dilemma, with drivers essentially hiring the government to change those payoffs. Alternatively, private road-owners might charge differential tolls that achieve a similar outcome.

(c) Everyone in your neighborhood would love to see some really great neighborhood fireworks on the next national independence day — but somehow no fireworks ever happen in your neighborhood.

Answer: This is another externality problem that creates a prisoners’ dilemma — we all would be better off if we contributed to such fireworks, but it is in each of our individual interests not to contribute regardless of what everyone else does. We might solve this by hiring a local home owner’s association to pay for the fireworks with mandatory dues paid by all homeowners in the neighborhood; or someone who is a great social entrepreneur might make everyone feel guilty enough to overcome the prisoners’ dilemma incentives.

(d) People like downloading pirated music for free but would like to have artists continue to produce lots of great music.

Answer: In order for lots of great music to be produced, it might be necessary for artists to be able to benefit financially from their work. But pirating music circumvents this —
and everyone has an incentive to do this regardless of whether others are also doing it. But differently, we might all be better off by not having any of us pirate music, but we are individually better off by getting what we can for free regardless of how much everyone else is doing this. We might overcome this through strict enforcement of copyright laws, or there might be public campaigns to make people feel guilty for pirating music — thus changing the payoffs in the prisoners' dilemma.

(e) Small business owners would like to keep their businesses open during "business hours" and not on evenings and weekends. In some countries, they have successfully lobbied the government to force them to close in the evenings and weekends. (Laws that restrict business activities on Sunday are sometimes called blue laws.)

Answer: Each business owner has an incentive to remain open to attract customers — regardless of whether other businesses stay open. (If they don't, then I get all the customers by staying open; if they do, I lose customers because I don't offer hours that are as convenient.) But if all businesses close at the same time, no one loses any customers. So the incentives are precisely those of the prisoners' dilemma — and forcing everyone to close at the same time enforces cooperation that could not otherwise be sustained.

B: In Chapter 21, we introduced the Coase Theorem, and we mentioned in Section 21.A.4.4 the example of beekeeping on apple orchards. Apple trees, it turns out, don't produce much honey (when frequented by bees), but bees are essential for cross-pollination.

(a) In an area with lots of apple orchards, each owner of an orchard has to ensure that there are sufficient numbers of bees to visit the trees and do the necessary cross-pollination. But bees cannot easily be kept to just one orchard — which implies that an orchard owner who maintains a bee hive is also providing some cross-pollination services to neighboring orchards. In what sense to orchard owners face a prisoners' dilemma?

Answer: This is another externality problem resulting in prisoners' dilemma incentives. Each orchard owner can get away with providing fewer bee hives than is optimal because the bees from neighboring orchards will come by — and the fact that individual orchard owners do not take into account the benefits of each bee hive for other orchard owners, too few bee hives will be set up and everyone agrees that everyone would be better off if all were forced to have more bee hives.

(b) How does the Coase Theorem suggest that orchard owners will deal with this problem?

Answer: The Coase Theorem suggests that, when property rights are well specified, orchard owners will find a way to contract with each other to find the efficient solution.

(c) We mentioned in Chapter 21 that some have documented a "custom of the orchards" — an implicit understanding among orchard owners that each will employ the same number of bee hives per acre as the other owners in the area. How might such a custom be an equilibrium outcome in a repeated game with indefinite end?

Answer: Suppose each orchard owner believes that they will be in the same vicinity as other orchard owners in the area with some probability again and again — i.e. there is no definitive end to the relationship. We have shown that there is no dominant strategy in this case. So suppose all orchard owners adopt the "custom of the orchards" so long as all others did so the previous year but otherwise switch to the non-cooperative number of bee hives on their land. If the probability of future interactions is sufficiently large and orchard owners do not discount the future too much, then each orchard owner is best responding to every other orchard owner by playing this strategy — and the custom of the orchards is observed in equilibrium. This is similar to the tit-for-tat strategy leading to cooperation in repeated prisoners' dilemmas that have no clear end — so long as it is sufficiently likely that future interactions occur and so long as the future is not heavily discounted. The increased payoff from future cooperation is larger in expected value terms than the gains one could make by violating the custom of the orchards this year.