## CHAPTER 2
Limits and Their Properties

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CHAPTER 2
Limits and Their Properties

Section 2.1 A Preview of Calculus

1. Precalculus: \((20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}\)

3. Calculus required: Slope of the tangent line at \(x = 2\) is the rate of change, and equals about 0.16.

5. (a) Precalculus: Area = \(\frac{1}{2}bh = \frac{1}{2}(5)(4) = 10\) sq. units
   (b) Calculus required: Area = \(bh\)
   \[= 2(2.5)\]
   \[= 5\text{ sq. units}\]

7. \(f(x) = 6x - x^2\)

9. (a) Area \(\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417\)
   Area \(\approx \frac{1}{2}(5 + \frac{5}{15} + \frac{5}{2} + \frac{5}{3} + \frac{5}{3} + \frac{5}{4} + \frac{5}{4}) \approx 9.145\)
   (b) You could improve the approximation by using more rectangles.

11. (a) \(D_1 = \sqrt{(5 - 1)^2 + (1 - 5)^2} = \sqrt{16 + 16} \approx 5.66\)
    (b) \(D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4}\right)^2}\)
    \[\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11\]
    (c) Increase the number of line segments.

Section 2.2 Finding Limits Graphically and Numerically

1. \[
x &| 3.9 & 3.99 & 3.999 & 4.001 & 4.01 & 4.1 \\
f(x) & 0.2041 & 0.2004 & 0.2000 & 0.2000 & 0.1996 & 0.1961
\]
\[
\lim_{x \to 4} \frac{x - 4}{x^2 - 3x - 4} \approx 0.2000 \quad \left(\text{Actual limit is } \frac{1}{5}\right)
\]
3. \[
\lim_{x \to 3} \frac{1/(x + 1) - (1/4)}{x - 3} \approx -0.0625 \quad \text{(Actual limit is } -\frac{1}{16} \text{)}
\]

5. \[
\lim_{x \to 0} \frac{\sin x}{x} \approx 1.0000 \quad \text{(Actual limit is 1.) (Make sure you use radian mode.)}
\]

7. \[
\lim_{x \to 0} \frac{e^x - 1}{x} \approx 1.0000 \quad \text{(Actual limit is 1.)}
\]

9. \[
\lim_{x \to 0} \frac{\ln(x + 1)}{x} \approx 1.0000 \quad \text{(Actual limit is 1.)}
\]

11. \[
\lim_{x \to 1} \frac{x - 2}{x^2 + x - 6} = 0.2500 \quad \text{(Actual limit is } \frac{1}{4} \text{)}
\]

13. \[
\lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1} \approx 0.6666 \quad \text{(Actual limit is } \frac{2}{3} \text{)}
\]

15. \[
\lim_{x \to 0} \frac{\sin 2x}{x} \approx 2.0000 \quad \text{(Actual limit is 2.) (Make sure you use radian mode.)}
\]

17. \[
\lim_{x \to 3} (4 - x) = 1
\]

19. \[
\lim_{x \to 2} \frac{|x - 2|}{x - 2} \text{ does not exist.}
\]
For values of \(x\) to the left of 2, \(\frac{|x - 2|}{x - 2} = -1\), whereas for values of \(x\) to the right of 2, \(\frac{|x - 2|}{x - 2} = 1\).
27. (a) \( f(1) \) exists. The black dot at \((1, 2)\) indicates that 
\[ f(1) = 2. \]

(b) \( \lim_{{x \to 1}} f(x) \) does not exist. As \( x \) approaches 1 from the left, 
\( f(x) \) approaches 3.5, whereas as \( x \) approaches 1 from the right, 
\( f(x) \) approaches 1.

(c) \( f(4) \) does not exist. The hollow circle at 
\((4, 2)\) indicates that \( f \) is not defined at 4.

(d) \( \lim_{{x \to 4}} f(x) \) exists. As \( x \) approaches 4, 
\( f(x) \) approaches 2: 
\[ \lim_{{x \to 4}} f(x) = 2. \]

29. \( \lim_{{x \to c}} f(x) \) exists for all \( c \neq -3. \)

31. 
\[ \lim_{{x \to c}} f(x) \] exists for all \( c \neq 4. \)

33. One possible answer is:

35. \( C(t) = 9.99 - 0.79\sqrt{-(t - 1)} \)

(a) 

(b) 
<table>
<thead>
<tr>
<th>( t )</th>
<th>3</th>
<th>3.3</th>
<th>3.4</th>
<th>3.5</th>
<th>3.6</th>
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<tr>
<td>( C )</td>
<td>11.57</td>
<td>12.36</td>
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\[ \lim_{{t \to 3.5}} C(t) = 12.36 \]

(c) 
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<tr>
<th>( t )</th>
<th>2</th>
<th>2.5</th>
<th>2.9</th>
<th>3</th>
<th>3.1</th>
<th>3.5</th>
<th>4</th>
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<td>( C )</td>
<td>10.78</td>
<td>11.57</td>
<td>11.57</td>
<td>11.57</td>
<td>12.36</td>
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\[ \lim_{{t \to 3}} C(t) \] does not exist because the values of \( C \) approach different values as \( t \) approaches 3 from both sides.

37. You need \( |f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4. \) So, take \( \delta = 0.4. \) If \( 0 < |x - 2| < 0.4, \) then
\[ |x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4, \] as desired.
39. You need to find $\delta$ such that $0 < |x - 1| < \delta$ implies

\[ \left| f(x) - 1 \right| = \left| \frac{1}{x} - 1 \right| < 0.1. \]

That is,

\[ -0.1 < \frac{1}{x} - 1 < 0.1 \]

\[ 1 - 0.1 < \frac{1}{x} < 1 + 0.1 \]

\[ \frac{9}{10} < \frac{1}{x} < \frac{11}{10} \]

\[ \frac{10}{9} > x > \frac{10}{11} \]

\[ \frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1 \]

\[ \frac{1}{9} > x - 1 > -\frac{1}{11}. \]

So take $\delta = \frac{1}{11}$. Then $0 < |x - 1| < \delta$ implies

\[ -\frac{1}{11} < x - 1 < \frac{1}{11} \]

\[ -\frac{1}{11} < x - 1 < \frac{1}{9}. \]

Using the first series of equivalent inequalities, you obtain

\[ \left| f(x) - 1 \right| = \left| \frac{1}{x} - 1 \right| < 0.1. \]

41. $\lim_{x \to 2}(3x + 2) = 8 = L$

\[ \left| (3x + 2) - 8 \right| < 0.01 \]

\[ \left| 3x - 6 \right| < 0.01 \]

\[ 3\left| x - 2 \right| < 0.01 \]

\[ 0 < \left| x - 2 \right| < \frac{0.01}{3} \approx 0.0033 = \delta \]

So, if $0 < \left| x - 2 \right| < \delta = \frac{0.01}{3}$, you have

\[ 3\left| x - 2 \right| < 0.01 \]

\[ \left| 3x - 6 \right| < 0.01 \]

\[ \left| (3x + 2) - 8 \right| < 0.01 \]

\[ \left| f(x) - L \right| < 0.01. \]

43. $\lim_{x \to 2}(x^2 - 3) = 1 = L$

\[ \left| (x^2 - 3) - 1 \right| < 0.01 \]

\[ \left| x^2 - 4 \right| < 0.01 \]

\[ \left| (x + 2)(x - 2) \right| < 0.01 \]

\[ \left| x + 2 \right| \left| x - 2 \right| < 0.01 \]

\[ \left| x - 2 \right| < \frac{0.01}{\left| x + 2 \right|} \]

If you assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$. So, if $0 < \left| x - 2 \right| < \delta = 0.002$, you have

\[ \left| x - 2 \right| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{\left| x + 2 \right|}(0.01) \]

\[ \left| x + 2 \right| \left| x - 2 \right| < 0.01 \]

\[ \left| x^2 - 4 \right| < 0.01 \]

\[ \left| (x^2 - 3) - 1 \right| < 0.01 \]

\[ \left| f(x) - L \right| < 0.01. \]

45. $\lim_{x \to 2}(x + 3) = 5$

Given $\varepsilon > 0$:

\[ \left| (x + 3) - 5 \right| < \varepsilon \]

\[ \left| x - 2 \right| < \varepsilon \]

So, let $\delta = \varepsilon$.

So, if $0 < \left| x - 2 \right| < \delta = \varepsilon$, you have

\[ \left| x - 2 \right| < \varepsilon \]

\[ \left| (x + 3) - 5 \right| < \varepsilon \]

\[ \left| f(x) - L \right| < \varepsilon. \]

47. $\lim_{x \to -4}\left(\frac{x}{2} - 1\right) = \frac{-1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

\[ \left| \left(\frac{x}{2} - 1\right) - (-3) \right| < \varepsilon \]

\[ \frac{1}{2}\left| x + 2 \right| < \varepsilon \]

\[ \frac{1}{2}\left| x + 2 \right| < \varepsilon \]

\[ \left| x - (-4) \right| < \varepsilon \]

\[ \left| f(x) - L \right| < 2\varepsilon. \]

So, let $\delta = 2\varepsilon$.

So, if $0 < \left| x - (-4) \right| < \delta = 2\varepsilon$, you have

\[ \left| x - (-4) \right| < 2\varepsilon \]

\[ \frac{1}{2}\left| x + 2 \right| < \varepsilon \]

\[ \left| \left(\frac{x}{2} - 1\right) + 3 \right| < \varepsilon \]

\[ \left| f(x) - L \right| < \varepsilon. \]
49. \( \lim_{x \to 6} 3 = 3 \)

Given \( \varepsilon > 0 \):

\[
|3 - 3| < \varepsilon \\
0 < \varepsilon
\]

So, any \( \delta > 0 \) will work.

So, for any \( \delta > 0 \), you have

\[
|3 - 3| < \varepsilon \\
|f(x) - L| < \varepsilon.
\]

51. \( \lim_{x \to 0} \sqrt{x} = 0 \)

Given \( \varepsilon > 0 \):

\[
|\sqrt{x} - 0| < \varepsilon \\
|\sqrt{x}| < \varepsilon \\
|x| < \varepsilon^3
\]

So, let \( \delta = \varepsilon^3 \).

So, for \( 0 < |x - 0| < \delta = \varepsilon^3 \), you have

\[
|x| < \varepsilon^3 \\
|\sqrt{x}| < \varepsilon \\
|\sqrt{x} - 0| < \varepsilon \\
|f(x) - L| < \varepsilon.
\]

53. \( \lim_{x \to 2} |x - 2| = |(-2) - 2| = 4 \)

Given \( \varepsilon > 0 \):

\[
| |x - 2| - 4| < \varepsilon \\
|- (x - 2) - 4| < \varepsilon \quad (x - 2 < 0)
\]

\[
|x - 2| = |x + 2| = |x - (-2)| < \varepsilon
\]

So, let \( \delta = \varepsilon \).

So, for \( 0 < |x - (-2)| < \delta = \varepsilon \), you have

\[
|x + 2| < \varepsilon \\
|- (x + 2)| < \varepsilon \\
- (x - 2) - 4| < \varepsilon \\
| |x - 2| - 4| < \varepsilon \quad \text{(because } x - 2 < 0) \\
|f(x) - L| < \varepsilon.
\]

55. \( \lim_{x \to 4} \left(x^2 + 1\right) = 2 \)

Given \( \varepsilon > 0 \):

\[
\left|\left(x^2 + 1\right) - 2\right| < \varepsilon \\
\left|x^2 - 1\right| < \varepsilon \\
\left|(x + 1)(x - 1)\right| < \varepsilon \\
\left|x - 1\right| < \sqrt{\varepsilon + 1} \\
\]

If you assume \( 0 < x < 2 \), then \( \delta = \varepsilon/3 \).

So, for \( 0 < |x - 1| < \delta = \varepsilon/3 \), you have

\[
|x - 1| < \frac{1}{3} \varepsilon < \frac{1}{|x + 1|} \varepsilon \\
\left|x^2 - 1\right| < \varepsilon \\
\left|(x^2 + 1) - 2\right| < \varepsilon \\
\left|f(x) - 2\right| < \varepsilon.
\]

57. \( \lim_{x \to \pi} f(x) = \lim_{x \to \pi} \pi = 4 \)

59. \( f(x) = \frac{\sqrt{x + 5} - 3}{x - 4} \)

\[
\lim_{x \to 4} f(x) = \frac{1}{6}
\]

The domain is \([-5, 4) \cup (4, \infty) \). The graphing utility does not show the hole at \( \left(4, \frac{1}{6}\right) \).

61. \( f(x) = \frac{x - 9}{\sqrt{x} - 3} \)

\[
\lim_{x \to 9} f(x) = 6
\]

The domain is all \( x \geq 0 \) except \( x = 9 \). The graphing utility does not show the hole at \( (9, 6) \).
63. \( \lim_{{x \to 8}} f(x) = 25 \) means that the values of \( f \) approach 25 as \( x \) gets closer and closer to 8.

65. (i) The values of \( f \) approach different numbers as \( x \) approaches \( c \) from different sides of \( c \):

(ii) The values of \( f \) increase or decrease without bound as \( x \) approaches \( c \):

(iii) The values of \( f \) oscillate between two fixed numbers as \( x \) approaches \( c \):

67. (a) \( C = 2\pi r \)

\[ r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm} \]

(b) When \( C = 5.5 \):

\[ r = \frac{5.5}{2\pi} \approx 0.87535 \text{ cm} \]

(c) \( \lim_{{x \to 3/\pi}} (2\pi r) = 6; e = 0.5; \delta \approx 0.0796 \)

69. \( f(x) = (1 + x)^{1/4} \)

\( \lim_{{x \to 0}} (1 + x)^{1/4} = e \approx 2.71828 \)

71. Using the zoom and trace feature, \( \delta = 0.001 \). So, \( (2 - \delta, 2 + \delta) = (1.999, 2.001) \).

Note: \( \frac{x^2 - 4}{x - 2} = x + 2 \) for \( x \neq 2 \).

73. False. The existence or nonexistence of \( f(x) \) at \( x = c \) has no bearing on the existence of the limit of \( f(x) \) as \( x \to c \).

75. False. Let

\[ f(x) = \begin{cases} 
  x - 4, & x \neq 2 \\
  0, & x = 2 
\end{cases} \]

\( f(2) = 0 \)

\( \lim_{{x \to 2}} f(x) = \lim_{{x \to 2}} (x - 4) = 2 \neq 0 \)

77. \( f(x) = \sqrt{x} \)

\( \lim_{{x \to 0.25}} \sqrt{x} = 0.5 \) is true.

As \( x \) approaches 0.25 = \( \frac{1}{4} \) from either side,

\( f(x) = \sqrt{x} \) approaches \( \frac{1}{2} = 0.5 \).
79. Using a graphing utility, you see that
\[
\lim_{x \to 0} \sin x = 1
\]
\[
\lim_{x \to 0} \sin 2x = 2, \text{ etc.}
\]
So, \( \lim_{x \to 0} \sin nx = n \).

81. If \( \lim_{x \to c} f(x) = L_1 \) and \( \lim_{x \to c} f(x) = L_2 \), then for every \( \varepsilon > 0 \), there exists \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that
\[
|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon \quad \text{and} \quad |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon. \]
Let \( \delta \) equal the smaller of \( \delta_1 \) and \( \delta_2 \). Then for \( |x - c| < \delta \), you have
\[
|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon.
\]
So, \( |L_1 - L_2| < 2\varepsilon \). Because \( \varepsilon > 0 \) is arbitrary, it follows that \( L_1 = L_2 \).

83. \( \lim_{x \to c} [f(x) - L] = 0 \) means that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if
\[
0 < |x - c| < \delta,
\]
then
\[
|f(x) - L - 0| < \varepsilon.
\]
This means the same as \( |f(x) - L| < \varepsilon \) when
\[
0 < |x - c| < \delta.
\]
So, \( \lim_{x \to c} f(x) = L \).

85. Answers will vary.

87. The radius \( OP \) has a length equal to the altitude \( z \) of the triangle plus \( \frac{h}{2} \). So,
\[
z = 1 - \frac{h}{2}.
\]
Area triangle = \( \frac{1}{2}bh \left( 1 - \frac{h}{2} \right) \)
Area rectangle = \( bh \)
Because these are equal, \( \frac{1}{2}bh \left( 1 - \frac{h}{2} \right) = bh \)
\[1 - \frac{h}{2} = 2h\]
\[\frac{5}{2}h = 1\]
\[h = \frac{2}{5}\]

Section 2.3 Evaluating Limits Analytically

1. \[ h(x) = -x^2 + 4x \]
   (a) \( \lim_{x \to 4} h(x) = 0 \)
   (b) \( \lim_{x \to 1} h(x) = -5 \)

3. \[ f(x) = x \cos x \]
   (a) \( \lim_{x \to \pi} f(x) = 0 \)
   (b) \( \lim_{x \to \pi/3} f(x) \approx 0.524 \)
   \( \left( = \frac{\pi}{6} \right) \)
5. \( \lim_{x \to 2} x^3 = 2^3 = 8 \)

7. \( \lim_{x \to 0} (2x - 1) = 2(0) - 1 = -1 \)

9. \( \lim_{x \to -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7 \)

11. \( \lim_{x \to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2 \)

13. \( \lim_{x \to 2} \frac{1}{x} = \frac{1}{2} \)

15. \( \lim_{x \to 1} \frac{x}{x^2 + 4} = \frac{1}{1^2 + 4} = \frac{1}{5} \)

17. \( \lim_{x \to 3} \frac{3x}{\sqrt{x} + 2} = \frac{3(7)}{\sqrt{7} + 2} = \frac{21}{3} = 7 \)

19. \( \lim_{x \to \pi/2} \sin x = \sin \frac{\pi}{2} = 1 \)

21. \( \lim_{x \to 1} \cos \frac{\pi x}{3} = \cos \frac{\pi}{3} = \frac{1}{2} \)

37. (a) \( \lim_{x \to 2} 5g(x) = 5 \lim_{x \to 2} g(x) = 5(2) = 10 \)

(b) \( \lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 3 + 2 = 5 \)

(c) \( \lim_{x \to 2} f(x)g(x) = \left[ \lim_{x \to 2} f(x) \right] \left[ \lim_{x \to 2} g(x) \right] = (3)(2) = 6 \)

(d) \( \lim_{x \to 3} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 3} f(x)}{\lim_{x \to 3} g(x)} = \frac{3}{2} \)

39. (a) \( \lim_{x \to \infty} [f(x)]^3 = \left[ \lim_{x \to \infty} f(x) \right]^3 = (4)^3 = 64 \)

(b) \( \lim_{x \to \infty} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x \to \infty} f(x)} = \sqrt[3]{4} = 2 \)

(c) \( \lim_{x \to \infty} [3f(x)] = 3 \lim_{x \to \infty} f(x) = 3(4) = 12 \)

(d) \( \lim_{x \to \infty} [f(x)]^{3/2} = \left[ \lim_{x \to \infty} f(x) \right]^{3/2} = (4)^{3/2} = 8 \)

41. \( f(x) = x - 1 \) and \( g(x) = \frac{x^2 - x}{x} \) agree except at \( x = 0 \).

(a) \( \lim_{x \to 0} g(x) = \lim_{x \to 0} f(x) = 0 - 1 = -1 \)

(b) \( \lim_{x \to -1} g(x) = \lim_{x \to -1} f(x) = -1 - 1 = -2 \)

43. \( f(x) = x(x + 1) \) and \( g(x) = \frac{x^3 - x}{x - 1} \) agree except at \( x = 1 \).

(a) \( \lim_{x \to 1} g(x) = \lim_{x \to 1} f(x) = 2 \)

(b) \( \lim_{x \to 1} g(x) = \lim_{x \to 1} f(x) = 0 \)

45. \( f(x) = \frac{x^2 - 1}{x + 1} \) and \( g(x) = x - 1 \) agree except at \( x = -1 \).

\( \lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = -2 \)
47. \( f(x) = \frac{x^3 - 8}{x - 2} \) and \( g(x) = x^2 + 2x + 4 \) agree except at \( x = 2 \).

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = 12
\]

51. \( \lim_{x \to 0} \frac{x}{x^2 - x} = \lim_{x \to 0} \frac{x}{x(x - 1)} = \lim_{x \to 0} \frac{1}{x - 1} = -1 \)

53. \( \lim_{x \to -4} \frac{x - 4}{x^2 - 16} = \lim_{x \to -4} \frac{x - 4}{(x + 4)(x - 4)} = \lim_{x \to -4} \frac{1}{x + 4} = \frac{1}{8} \)

55. \( \lim_{x \to 3} \frac{x^2 + x - 6}{x^2 - 9} = \lim_{x \to 3} \frac{(x + 3)(x - 2)}{(x + 3)(x - 3)} = \lim_{x \to 3} \frac{x - 2}{x - 3} = -\frac{5}{6} \)

57. \( \lim_{x \to 4} \sqrt{x + 5} - 3 = \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4} \cdot \frac{\sqrt{x + 5} + 3}{\sqrt{x + 5} + 3} = \lim_{x \to 4} \frac{(x + 5) - 9}{\sqrt{x + 5} + 3} = \lim_{x \to 4} \frac{1}{\sqrt{x + 5} + 3} = \frac{1}{6} \)

59. \( \lim_{x \to 0} \frac{x + 5 - \sqrt{5}}{x} = \lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x} \cdot \frac{\sqrt{x + 5} + \sqrt{5}}{\sqrt{x + 5} + \sqrt{5}} = \lim_{x \to 0} \frac{(x + 5) - 5}{\sqrt{x + 5} + \sqrt{5}} = \lim_{x \to 0} \frac{1}{\sqrt{x + 5} + \sqrt{5}} = \frac{\sqrt{5}}{10} \)

61. \( \lim_{x \to 0} \frac{3 + x - 1}{x} = \lim_{x \to 0} \frac{3 - (3 + x)}{3 + x} = \lim_{x \to 0} \frac{-x}{3 + x} = \lim_{x \to 0} \frac{-1}{3 + x} = -\frac{1}{9} \)

63. \( \lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2 \)

65. \( \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2 \)

67. \( \lim_{x \to 0} \frac{\sin x}{5x} = \lim_{x \to 0} \left[ \frac{\sin x}{x} \cdot \frac{1}{5} \right] = (1) \left( \frac{1}{5} \right) = \frac{1}{5} \)

69. \( \lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^2} = \lim_{x \to 0} \left[ \sin x \cdot \frac{1 - \cos x}{x} \cdot \frac{1}{x} \right] = (1)(0) = 0 \)

71. \( \lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} \left[ \sin x \cdot \frac{\sin x}{x} \right] = (1) \sin 0 = 0 \)

73. \( \lim_{h \to 0} \frac{(1 - \cos h)^2}{h} = \lim_{h \to 0} \left[ \frac{1 - \cos h}{h} \cdot \frac{1 - \cos h}{h} \right] = (0)(0) = 0 \)

75. \( \lim_{x \to \sqrt{2}} \frac{\cos x}{\sqrt{2}} = \lim_{x \to \sqrt{2}} \sin x = 1 \)

77. \( \lim_{x \to 0} \frac{1 - e^{-x}}{e^x - 1} = \lim_{x \to 0} \frac{1 - e^{-x}}{e^x - 1} \cdot \frac{e^x}{e^x} = \lim_{x \to 0} \frac{(1 - e^{-x})e^x}{e^x} = \lim_{x \to 0} e^x = 1 \)

79. \( \lim_{t \to 0} \frac{\sin 3t}{3t} = \lim_{t \to 0} \left( \frac{\sin 3t}{3t} \cdot \frac{3}{3} \right) = (1) \left( \frac{3}{2} \right) = \frac{3}{2} \)
81. \( f(x) = \frac{\sqrt{x + 2} - \sqrt{2}}{x} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.358</td>
<td>0.354</td>
<td>0.354</td>
<td>?</td>
<td>0.354</td>
<td>0.353</td>
<td>0.349</td>
</tr>
</tbody>
</table>

It appears that the limit is 0.354.

The graph has a hole at \( x = 0 \).

Analytically,
\[
\lim_{x \to 0} \frac{\sqrt{x + 2} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{\frac{2}{\sqrt{x + 2} + \sqrt{2}}}{1} = \lim_{x \to 0} \frac{1}{\frac{2}{\sqrt{x + 2} + \sqrt{2}}} = \frac{\sqrt{2}}{4} \approx 0.354.
\]

83. \( f(x) = \frac{1}{2 + x} - \frac{1}{2} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-0.263</td>
<td>-0.251</td>
<td>-0.250</td>
<td>?</td>
<td>-0.250</td>
<td>-0.249</td>
<td>-0.238</td>
</tr>
</tbody>
</table>

It appears that the limit is -0.250.

The graph has a hole at \( x = 0 \).

Analytically,
\[
\lim_{x \to 0} \frac{1}{2 + x} - \frac{1}{2} = \lim_{x \to 0} \frac{2 - (2 + x)}{2(2 + x)} = \lim_{x \to 0} \frac{-x}{2(2 + x)} = -\frac{1}{4}.
\]

85. \( f(t) = \frac{\sin 3t}{t} \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>2.96</td>
<td>2.9996</td>
<td>3</td>
<td>?</td>
<td>3</td>
<td>2.9996</td>
<td>2.96</td>
</tr>
</tbody>
</table>

It appears that the limit is 3.

The graph has a hole at \( t = 0 \).

Analytically,
\[
\lim_{t \to 0} \frac{\sin 3t}{t} = \lim_{t \to 0} \left( \frac{\sin 3t}{3t} \right) = 3(1) = 3.
\]
87. \[ f(x) = \frac{\sin x^2}{x} \]

<table>
<thead>
<tr>
<th>x</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-0.099998</td>
<td>-0.01</td>
<td>-0.001</td>
<td>?</td>
<td>0.001</td>
<td>0.01</td>
<td>0.099998</td>
</tr>
</tbody>
</table>

It appears that the limit is 0.

The graph has a hole at \( x = 0 \).

Analytically, \( \lim_{x \to 0} \frac{\sin x^2}{x} = \lim_{x \to 0} \left( \frac{\sin x^2}{x} \right) = 0(1) = 0 \).

89. \[ f(x) = \frac{\ln x}{x - 1} \]

<table>
<thead>
<tr>
<th>x</th>
<th>0.5</th>
<th>0.9</th>
<th>0.99</th>
<th>1.01</th>
<th>1.1</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1.3863</td>
<td>1.0536</td>
<td>1.0050</td>
<td>0.950</td>
<td>0.9531</td>
<td>0.8109</td>
</tr>
</tbody>
</table>

It appears that the limit is 1.

Analytically, \( \lim_{x \to 1} \frac{\ln x}{x - 1} = 1 \).

91. \[ \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x + 30x - 2 - 3x + 2}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3 \]

93. \[ \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{x + \Delta x + 3} - \frac{1}{x + 3} = \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \frac{1}{\Delta x} \]

\[ = \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)} = \lim_{\Delta x \to 0} \frac{-1}{(x + 3)^2} = \frac{-1}{(x + 3)^2} \]

95. \[ \lim_{x \to 0^+} (4 - x^2) \leq \lim_{x \to 0^+} f(x) \leq \lim_{x \to 0^+} (4 + x^2) \]

\[ 4 \leq \lim_{x \to 0^+} f(x) \leq 4 \]

So, \( \lim_{x \to 0^+} f(x) = 4 \).

97. \[ f(x) = x \cos x \]

\[ \lim_{\Delta x \to 0} \left( \frac{x \cos x}{x} \right) = 0 \]

99. \[ f(x) = |x| \sin x \]

\[ \lim_{x \to 0} |x| \sin x = 0 \]
101. \( f(x) = x \sin \frac{1}{x} \)

\[
\lim_{x \to 0} x \sin \frac{1}{x} = 0
\]

103. You say that two functions \( f \) and \( g \) agree at all but one point (on an open interval) if \( f(x) = g(x) \) for all \( x \) in the interval except for \( x = c \), where \( c \) is in the interval.

105. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as \( \frac{0}{0} \). That is, \( \lim_{x \to c} f(x) \) for which \( \lim_{x \to c} f(x) \) does not exist. However, \( \lim_{x \to c} f(x) + g(x) \) and \( \lim_{x \to c} f(x) - g(x) \) do exist.

107. \( f(x) = x, g(x) = \sin x, h(x) = \frac{\sin x}{x} \)

When the \( x \)-values are "close to" 0 the magnitude of \( f \) is approximately equal to the magnitude of \( g \). So, \( \left| \frac{f}{g} \right| \approx 1 \) when \( x \) is "close to" 0.

109. \( s(t) = -16t^2 + 500 \)

\[
\lim_{t \to 2} \frac{s(2) - s(t)}{2 - t} = \lim_{t \to 2} \frac{-16(2)^2 + 500 - (-16t^2 + 500)}{2 - t} = \lim_{t \to 2} \frac{436 + 16t^2 - 500}{2 - t} = \lim_{t \to 2} \frac{16(t^2 - 4)}{2 - t} = \lim_{t \to 2} \frac{16(t - 2)(t + 2)}{2 - t} = \lim_{t \to 2} -16(t + 2) = -64 \text{ ft/sec}
\]

The wrench is falling at about 64 feet/second.

111. \( s(t) = -4.9t^2 + 200 \)

\[
\lim_{t \to 3} s(3) - s(t) = \lim_{t \to 3} \frac{-4.9(3)^2 + 200 - (-4.9t^2 + 200)}{3 - t} = \lim_{t \to 3} \frac{4.9(t^2 - 9)}{3 - t} = \lim_{t \to 3} \frac{4.9(t - 3)(t + 3)}{3 - t} = \lim_{t \to 3} \frac{-4.9(t + 3)}{3 - t} = -29.4 \text{ m/sec}
\]

The object is falling about 29.4 m/sec.

113. Let \( f(x) = \frac{1}{x} \) and \( g(x) = \frac{-1}{x} \). \( \lim_{x \to 0} f(x) \) and \( \lim_{x \to 0} g(x) \) do not exist. However,

\[
\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \frac{1}{x} + \frac{-1}{x} = \lim_{x \to 0} 0 = 0
\]

and therefore does exist.

115. Given \( f(x) = b \), show that for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |f(x) - b| < \varepsilon \) whenever \( |x - c| < \delta \). Because \( |f(x) - b| = |b - b| = 0 < \varepsilon \) for every \( \varepsilon > 0 \), any value of \( \delta > 0 \) will work.

117. If \( b = 0 \), the property is true because both sides are equal to 0. If \( b \neq 0 \), let \( \varepsilon > 0 \) be given. Because \( \lim_{x \to c} f(x) = L \), there exists \( \delta > 0 \) such that \( |f(x) - L| < \varepsilon/|b| \) whenever \( 0 < |x - c| < \delta \). So, whenever \( 0 < |x - c| < \delta \), we have

\[
|b| |f(x) - L| < \varepsilon \quad \text{or} \quad |bf(x) - bL| < \varepsilon
\]

which implies that \( \lim_{x \to c} bf(x) = bL \).
119. \[ M|f(x)| \leq f(x)g(x) \leq M|f(x)| \]
\[
\lim_{x \to c^-}(-M|f(x)|) \leq \lim_{x \to c^-}f(x)g(x) \leq \lim_{x \to c^-}(M|f(x)|)
\]
\[ -M(0) \leq \lim_{x \to c^-}f(x)g(x) \leq M(0) \]
\[ 0 \leq \lim_{x \to c^-}f(x)g(x) \leq 0 \]
So, \( \lim_{x \to c^-}f(x)g(x) = 0 \).

121. Let \( f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases} \)
\[
\lim_{x \to 0^+}f(x) = \lim_{x \to 0^-}4 = 4.
\]
\[
\lim_{x \to 0^-}f(x) \text{ does not exist because for } x < 0, f(x) = -4 \text{ and for } x \geq 0, f(x) = 4.
\]

123. False. The limit does not exist because the function approaches \( 1 \) from the right side of \( 0 \) and approaches \( -1 \) from the left side of \( 0 \).

125. True

127. False. The limit does not exist because \( f(x) \) approaches \( 3 \) from the left side of \( 2 \) and approaches \( 0 \) from the right side of \( 2 \).

129. \[
\lim_{x \to 0^-} \frac{1 - \cos x}{x} = \lim_{x \to 0^+} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}
\]
\[
= \lim_{x \to 0^-} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \to 0^+} \frac{\sin^2 x}{x(1 + \cos x)}
\]
\[
= \lim_{x \to 0^-} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} = \left[ \lim_{x \to 0^-} \frac{\sin x}{x} \right] \left[ \lim_{x \to 0^+} \frac{\sin x}{1 + \cos x} \right] = (1)(0) = 0
\]

131. \( f(x) = \frac{\sec x - 1}{x^2} \)

(a) The domain of \( f \) is all \( x \neq 0, \pi/2 + n\pi \).

(b) The domain is not obvious. The hole at \( x = 0 \) is not apparent.

(c) \( \lim_{x \to 0^-} f(x) = \frac{1}{2} \)

(d) \[
\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2} \cdot \frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}
\]
\[
= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x} \left( \frac{\sin^2 x}{x^2} \right) \frac{1}{\sec x + 1}
\]
So, \( \lim_{x \to 0^-} \frac{\sec x - 1}{x^2} = \lim_{x \to 0^+} \frac{\sec^2 x - 1}{x^2(\sec x + 1)} = \lim_{x \to 0} \frac{1}{\sec x + 1} = 0 \)
\[
= \frac{1}{2}
\]

133. The graphing utility was set in degree mode, instead of radian mode.

Section 2.4 Continuity and One-Sided Limits

1. (a) \( \lim_{x \to 4^-} f(x) = 3 \)
   (b) \( \lim_{x \to 4^-} f(x) = 3 \)
   (c) \( \lim_{x \to 4^+} f(x) = 3 \)
   The function is continuous at \( x = 4 \) and is continuous on \((\infty, \infty)\).

3. (a) \( \lim_{x \to 3^+} f(x) = 0 \)
   (b) \( \lim_{x \to 3^-} f(x) = 0 \)
   (c) \( \lim_{x \to 3} f(x) = 0 \)
   The function is NOT continuous at \( x = 3 \).
5. (a) \( \lim_{x \to 2^+} f(x) = -3 \)
(b) \( \lim_{x \to 2^-} f(x) = 3 \)
(c) \( \lim_{x \to 2} f(x) \) does not exist.
The function is NOT continuous at \( x = 2 \).

7. \( \lim_{x \to 3} \frac{1}{x^2 + 8} \) = \( \frac{1}{8 + 8} = \frac{1}{16} \)

15. \( \lim_{\Delta x \to 0^-} \frac{1}{x + \Delta x} - \frac{1}{x} = \lim_{\Delta x \to 0^-} \frac{x - (x + \Delta x)}{x(x + \Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0^-} -\frac{\Delta x}{x(x + \Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0^-} -\frac{1}{x(x + \Delta x)} = \frac{1}{x(x + 0)} = \frac{1}{x^2} \)

19. \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 2 \)
\( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 + 1) = 2 \)
\( \lim_{x \to 1} f(x) = 2 \)

21. \( \lim_{x \to \pi} \cot x \) does not exist because \( \lim_{x \to \pi^-} \cot x \) and \( \lim_{x \to \pi^+} \cot x \) do not exist.

23. \( \lim_{x \to 4} (5[x] - 7) = 5(3) - 7 = 8 \)
\( [x] = 3 \) for \( 3 \leq x < 4 \)

25. \( \lim_{x \to 3} (2 - [x]) \) does not exist because \( \lim_{x \to 3^-} (2 - [x]) = 2 - (-3) = 5 \)
and \( \lim_{x \to 3^+} (2 - [x]) = 2 - (-4) = 6. \)

27. \( \lim_{x \to 3^+} \ln(x - 3) = \ln 0 \) does not exist.

29. \( \lim_{x \to 2^-} x^2(3 - x) = \ln[4(1)] = \ln 4 \)

31. \( f(x) = \frac{1}{x^2 - 4} \)
has discontinuities at \( x = -2 \) and \( x = 2 \) because \( f(-2) \) and \( f(2) \) are not defined.

9. \( \lim_{x \to 5} \frac{x - 5}{x^2 - 25} = \lim_{x \to 5} \frac{1}{x + 5} = \frac{1}{10} \)

11. \( \lim_{x \to -3} \frac{x}{\sqrt{x^2 - 9}} \) does not exist because \( \frac{x}{\sqrt{x^2 - 9}} \) decreases without bound as \( x \to -3^- \).

13. \( \lim_{x \to 0^-} x = \lim_{x \to 0^-} \frac{-x}{x} = -1 \)

33. \( f(x) = \frac{x}{2} + x \) has discontinuities at each integer \( k \)
because \( \lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x) \).

35. \( g(x) = \sqrt{49 - x^2} \) is continuous on \([-7, 7]\).

37. \( \lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x) \) is continuous on \([-1, 4]\).

39. \( f(x) = \frac{6}{x} \) has a nonremovable discontinuity at \( x = 0 \).

41. \( f(x) = x^2 - 9 \) is continuous for all real \( x \).

43. \( f(x) = \frac{1}{4 - x^2} = \frac{1}{(2 - x)(2 + x)} \) has nonremovable discontinuities at \( x = \pm 2 \) because \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \) do not exist.

45. \( f(x) = 3x - \cos x \) is continuous for all real \( x \).

47. \( f(x) = \frac{x}{x^2 - x} = \frac{1}{x - 1} \) for \( x \neq 0, x = 0 \) is a removable discontinuity, whereas \( x = 1 \) is a nonremovable discontinuity.

49. \( f(x) = \frac{x}{x^2 + 1} \) is continuous for all real \( x \).
51. \( f(x) = \frac{x + 2}{(x + 2)(x - 5)} \) has a nonremovable discontinuity at \( x = 5 \) because \( \lim_{x \to 5} f(x) \) does not exist, and has a removable discontinuity at \( x = -2 \) because
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{2}{x + 2} = \frac{1}{7}.
\]
53. \( f(x) = \frac{x + 2}{x + 2} \) has a nonremovable discontinuity at \( x = -2 \) because \( \lim_{x \to -2} f(x) \) does not exist.

55. \( f(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases} \) has a possible discontinuity at \( x = 1 \).

1. \( f(1) = 1 \)
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1 \\
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1
\]
\[\implies \lim_{x \to 1} f(x) = 1\]

2. \( f(1) = \lim_{x \to 1} f(x) \)

3. \( f(1) = \lim_{x \to 1^-} f(x) \)

\( f \) is continuous at \( x = 1 \), so \( f \) is continuous for all real \( x \).

57. \( f(x) = \begin{cases} \frac{x}{2} + 1, & x \leq 2 \\ 3 - x, & x > 2 \end{cases} \)

has a possible discontinuity at \( x = 2 \).

1. \( f(2) = \frac{2}{2} + 1 = 2 \)
\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left( \frac{x}{2} + 1 \right) = 2
\]
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 1
\]

\( \lim_{x \to 2^-} f(x) \) does not exist.

So, \( f \) has a nonremovable discontinuity at \( x = 2 \).

59. \( f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases} \)

has possible discontinuities at \( x = -1, x = 1 \).

1. \( f(-1) = -1 \quad f(1) = 1 \)
\[
\lim_{x \to -1} f(x) = -1 \\
\lim_{x \to 1} f(x) = 1
\]

3. \( f(-1) = \lim_{x \to -1} f(x) \\
\quad f(1) = \lim_{x \to 1} f(x) \\
\]

\( f \) is continuous at \( x = \pm 1 \), so \( f \) is continuous for all real \( x \).

61. \( f(x) = \begin{cases} \ln(x + 1), & x \geq 0 \\ 1 - x^2, & x < 0 \end{cases} \)

has a possible discontinuity at \( x = 0 \).

1. \( f(0) = \ln(0 + 1) = \ln 1 = 0 \)
\[
\lim_{x \to 0^+} f(x) = 1 - 0 = 1 \\
\lim_{x \to 0^-} f(x) = 0
\]

\( \lim_{x \to 0^-} f(x) \) does not exist.

So, \( f \) has a nonremovable discontinuity at \( x = 0 \).

63. \( f(x) = \csc 2x \) has nonremovable discontinuities at integer multiples of \( \pi/2 \).

65. \( f(x) = \begin{cases} x - 8, & \text{has nonremovable discontinuities at each integer } k. \end{cases} \)

67. \( \lim_{x \to 0^+} f(x) = 0 \\
\lim_{x \to 0^-} f(x) = 0 \\
\)

\( f \) is not continuous at \( x = -2 \).

69. \( f(l) = 3 \\
\)

\( \text{Find } a \text{ so that } \lim_{x \to -4} (ax - 4) = 3 \)
\[
a(1) - 4 = 3 \\
a = 7.
\]

71. \( f(2) = 8 \\
\)

\( \text{Find } a \text{ so that } \lim_{x \to 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2. \)
73. Find \(a\) and \(b\) such that \(\lim_{{x \to -1^+}} (ax + b) = -a + b = 2\) and \(\lim_{{x \to 3^-}} (ax + b) = 3a + b = -2\).

\[
\begin{align*}
  a - b &= -2 \\
  (3a + b) &= -2 \\
  4a &= -4 \\
  a &= -1 \\
  b &= 2 + (-1) = 1
\end{align*}
\]

89. \(f(x) = \frac{\sin x}{x}\)

The graph appears to be continuous on the interval \([-4, 4]\). Because \(f(0)\) is not defined, you know that \(f\) has a discontinuity at \(x = 0\). This discontinuity is removable so it does not show up on the graph.

91. \(f(x) = \frac{\ln(x^2 + 1)}{x}\)

The graph appears to be continuous on the interval \([-4, 4]\). Because \(f(0)\) is not defined, you know that \(f\) has a discontinuity at \(x = 0\). This discontinuity is removable so it does not show up on the graph.
97. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.
$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the root feature, you find that $x \approx 0.6823$.

99. $g(t) = 2\cos t - 3t$

$g$ is continuous on $[0, 1]$.
$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the root feature, you find that $t \approx 0.5636$.

101. $f(x) = x + e^x - 3$

$f$ is continuous on $[0, 1]$.
$f(0) = e^0 - 3 = -2 < 0$ and
$f(1) = 1 + e - 3 = e - 2 > 0$.

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of $c$ between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.79$. Using the root feature, you find that $x \approx 0.7921$.

103. $f(x) = x^2 + x - 1$

$f$ is continuous on $[0, 5]$.
$f(0) = -1$ and $f(5) = 29$
$-1 < 11 < 29$

The Intermediate Value Theorem applies.
$x^2 + x - 1 = 11$
$x^2 + x - 12 = 0$
$(x + 4)(x - 3) = 0$
$x = -4$ or $x = 3$
$c = 3$ (x = -4 is not in the interval.)

So, $f(3) = 11$.

105. $f(x) = x^3 - x^2 + x - 2$

$f$ is continuous on $[0, 3]$.
$f(0) = -2$ and $f(3) = 19$
$-2 < 4 < 19$

The Intermediate Value Theorem applies.
$x^3 - x^2 + x - 2 = 4$
$x^3 - x^2 + x - 6 = 0$
$(x - 2)(x^2 + x + 3) = 0$
$x = 2$

$x^2 + x + 3$ has no real solution.
$c = 2$

So, $f(2) = 4$.

107. (a) The limit does not exist at $x = c$.
(b) The function is not defined at $x = c$.
(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.
(d) The limit does not exist at $x = c$.

109. If $f$ and $g$ are continuous for all real $x$, then so is $f + g$ (Theorem 2.11, part 2). However, $f/g$ might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then $f$ and $g$ are continuous for all real $x$, but $f/g$ is not continuous at $x = \pm 1$.

111. True

1. $f(c) = L$ is defined.
2. $\lim_{x \to c} f(x) = L$ exists.
3. $f(c) = \lim_{x \to c} f(x)$

All of the conditions for continuity are met.

113. False. A rational function can be written as $P(x)/Q(x)$ where $P$ and $Q$ are polynomials of degree $m$ and $n$, respectively. It can have, at most, $n$ discontinuities.

115. $\lim_{t \to -4^-} f(t) \approx 28$

$\lim_{t \to -4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 oz. At the beginning of day 4, more chlorine was added, and the amount is now about 56 oz.
117. \( C(t) = \begin{cases} 
0.40, & 0 < t \leq 10 \\
0.40 + 0.05(t - 9), & t > 10, \ t \text{ not an integer} \\
0.40 + 0.05(t - 10), & t > 10, \ t \text{ an integer} 
\end{cases} \)

There is a nonremovable discontinuity at each integer greater than or equal to 10.

Note: You could also express \( C \) as
\[
C(t) = \begin{cases} 
0.40, & 0 < t \leq 10 \\
0.40 - 0.05[10 - t], & t > 10 
\end{cases}
\]

119. Let \( s(t) \) be the position function for the run up to the campsite. \( s(0) = 0 \) (\( t = 0 \) corresponds to 8:00 A.M., \( s(20) = k \) (distance to campsite)). Let \( r(t) \) be the position function for the run back down the mountain: \( r(0) = k, \ r(10) = 0 \). Let \( f(t) = s(t) - r(t) \).

When \( t = 0 \) (8:00 A.M.),
\[
f(0) = s(0) - r(0) = 0 - k < 0.
\]

When \( t = 10 \) (8:10 A.M.), \( f(10) = s(10) - r(10) > 0 \).

Because \( f(0) < 0 \) and \( f(10) > 0 \), there must be a value \( t \) in the interval \([0, 10]\) such that \( f(t) = 0 \).

If \( f(t) = 0 \), then \( s(t) - r(t) = 0 \), which gives us \( s(t) = r(t) \). So, at some time \( t \), where \( 0 \leq t \leq 10 \), the position functions for the run up and the run down are equal.

121. Suppose there exists \( x_1 \) in \([a, b]\) such that \( f(x_1) > 0 \) and there exists \( x_2 \) in \([a, b]\) such that \( f(x_2) < 0 \). Then by the Intermediate Value Theorem, \( f(x) \) must equal zero for some value of \( x \) in \([x_1, x_2]\) or \([x_2, x_1]\) if \( x_2 < x_1 \). So, \( f \) would have a zero in \([a, b]\) which is a contradiction. Therefore, \( f(x) > 0 \) for all \( x \) in \([a, b]\) or \( f(x) < 0 \) for all \( x \) in \([a, b]\).

123. If \( x = 0 \), then \( f(0) = 0 \) and \( \lim_{x \to 0} f(x) = 0 \). So, \( f \) is continuous at \( x = 0 \).

If \( x \neq 0 \), then \( \lim_{x \to x} f(t) = 0 \) for \( x \) rational, whereas
\[
\lim_{x \to x} f(t) = \lim_{x \to x} kx \neq 0 \text{ for } x \text{ irrational. So, } f \text{ is not continuous for all } x \neq 0.
\]

125. (a) \( f(x) = \begin{cases} 
0.40, & 0 < t \leq 10 \\
0.40 - 0.05[10 - t], & t > 10 
\end{cases} \)

(b) There appears to be a limiting speed and a possible cause is air resistance.

127. \( f(x) = \begin{cases} 
1 - x^2, & x \leq c \\
x, & x > c 
\end{cases} \)

\( f \) is continuous for \( x < c \) and for \( x > c \). At \( x = c \), you need \( 1 - c^2 = c \). Solving \( c^2 + c - 1 = 0 \), you obtain
\[
c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.
\]

131. \( h(x) = x[x] \)

\( h \) has nonremovable discontinuities at \( x = \pm 1, \pm 2, \pm 3, \ldots \).
133. \( f(x) = \frac{4}{1 + 2^{4/5}} \)

(a) Domain: all \( x \neq 0 \)

(b) 

(c) \( \lim_{x \to 0^+} f(x) = 4, \quad \lim_{x \to 0^-} f(x) = 0 \)

(d) For \( x \) near 0 and negative, \( \frac{4}{1 + 2^{4/5}} \approx \frac{4}{1 + 0} \approx 4 \).

For \( x \) near 0 and positive, \( \frac{4}{1 + 2^{4/5}} \approx 0 \).

**Section 2.5 Infinite Limits**

1. \( f(x) = \frac{1}{x - 4} \)

As \( x \) approaches 4 from the left, \( x - 4 \) is a small negative number. So,

\[ \lim_{x \to 4^-} f(x) = -\infty \]

As \( x \) approaches 4 from the right, \( x - 4 \) is a small positive number. So,

\[ \lim_{x \to 4^+} f(x) = \infty \]

3. \( f(x) = \frac{1}{(x - 4)^2} \)

As \( x \) approaches 4 from the left or right, \( (x - 4)^2 \) is a small positive number. So,

\[ \lim_{x \to 4} f(x) = \lim_{x \to 4} \left(\frac{1}{(x - 4)^2}\right) = \infty \]

5. \( \lim_{x \to -2^+} \frac{x}{x^2 - 4} = \infty \)

\( \lim_{x \to -2^-} \frac{x}{x^2 - 4} = \infty \)

7. \( \lim_{x \to -1^+} \frac{\pi x}{4} = -\infty \)

\( \lim_{x \to -1^-} \frac{\pi x}{4} = \infty \)

9. \( f(x) = \frac{1}{x^3 - 9} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -3.5 )</th>
<th>( -3.1 )</th>
<th>( -3.01 )</th>
<th>( -3.001 )</th>
<th>( -2.999 )</th>
<th>( -2.99 )</th>
<th>( -2.9 )</th>
<th>( -2.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.308</td>
<td>1.639</td>
<td>16.64</td>
<td>166.6</td>
<td>-166.7</td>
<td>-16.69</td>
<td>-1.695</td>
<td>-0.364</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to -3^{-}} f(x) = \infty \]

\[ \lim_{x \to -3^{+}} f(x) = -\infty \]
11. \( f(x) = \frac{x^2}{x^2 - 9} \)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & -3.5 & -3.1 & -3.01 & -2.999 & -2.99 & -2.9 \\
\hline
f(x) & 3.769 & 15.75 & 150.8 & 1501 & -149.3 & -14.25 \\
\hline
\end{array}
\]

\( \lim_{x \to -3} f(x) = \infty \)
\( \lim_{x \to -3} f(x) = -\infty \)

So, \( x = 0 \) is a vertical asymptote.

13. \( \lim_{x \to 0^+} \frac{1}{x} = \infty = \lim_{x \to 0^-} \frac{1}{x} \)

So, \( x = 0 \) is a vertical asymptote.

15. \( \lim_{x \to -2} \frac{x^2}{x^2 - 4} = \infty \) and \( \lim_{x \to -2^+} \frac{x^2}{x^2 - 4} = -\infty \)

So, \( x = -2 \) is a vertical asymptote.

\( \lim_{x \to -2^-} \frac{x^2}{x^2 - 4} = -\infty \) and \( \lim_{x \to -2^+} \frac{x^2}{x^2 - 4} = \infty \)

So, \( x = 2 \) is a vertical asymptote.

17. No vertical asymptote because the denominator is never zero.

19. \( \lim_{x \to 2^-} \frac{x^2 - 2}{x - 2(x + 1)} = \infty \)

\( \lim_{x \to 2^-} \frac{x^2 - 2}{x - 2(x + 1)} = -\infty \)

So, \( x = 2 \) is a vertical asymptote.

\( \lim_{x \to -1^+} \frac{x^2 - 2}{x - 2(x + 1)} = \infty \)

\( \lim_{x \to -1^-} \frac{x^2 - 2}{x - 2(x + 1)} = -\infty \)

So, \( x = -1 \) is a vertical asymptote.

21. \( \lim_{t \to 0^+} \left( 1 - \frac{4}{t^2} \right) = -\infty = \lim_{t \to 0^-} \left( 1 - \frac{4}{t^2} \right) \)

So, \( t = 0 \) is a vertical asymptote.

23. \( f(x) = \frac{3}{x^2 + x - 2} = \frac{3}{(x + 2)(x - 1)} \)

Vertical asymptotes at \( x = -2 \) and \( x = 1 \).

25. \( f(x) = \frac{x^3 + 1}{x + 1} = \frac{(x + 1)(x^2 - x + 1)}{x + 1} \)

No vertical asymptote because

\( \lim_{x \to -1} f(x) = \lim_{x \to -1} (x^2 - x + 1) = 3. \)

The graph has a hole at \( x = -1 \).

27. \( f(x) = \frac{e^{2x}}{x - 1} \)

\( x = 1 \) is a vertical asymptote.

29. \( h(t) = \frac{\ln(t^2 + 1)}{t + 2} \)

\( t = -2 \) is a vertical asymptote.

31. \( f(x) = \frac{1}{e^t - 1} \)

\( x = 0 \) is a vertical asymptote.

33. \( f(x) = \tan \pi x = \frac{\sin \pi x}{\cos \pi x} \) has vertical asymptotes at \( x = \frac{2n + 1}{2} \), \( n \) any integer.

35. \( s(t) = \frac{t}{\sin t} \) has vertical asymptotes at \( t = n\pi, n \) a nonzero integer. There is no vertical asymptote at \( t = 0 \) because

\( \lim_{t \to 0} \frac{t}{\sin t} = 1 \).
37. \( \lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} (x - 1) = -2 \)

Removable discontinuity at \( x = -1 \)

39. \( \lim_{x \to -1^-} \frac{x^2 + 1}{x + 1} = \infty \)
\( \lim_{x \to -1^+} \frac{x^2 + 1}{x + 1} = -\infty \)

Vertical asymptote at \( x = -1 \)

41. \( f(x) = \frac{e^{2(x+1)} - 1}{e^{x+1} - 1} = \frac{(e^{x+1} - 1)(e^{x+1} + 1)}{e^{x+1} - 1} = e^{x+1}, \ x \neq -1 \)

Removable discontinuity at \( x = -1 \)

43. \( \lim_{x \to -1^+} \frac{1}{x + 1} = \infty \)

45. \( \lim_{x \to -2^+} \frac{x}{x^2} = \infty \)

47. \( \lim_{x \to -1} \frac{x^2}{(x - 2)^2} = \infty \)

49. \( \lim_{x \to -3} \frac{x + 3}{x^2 + x - 6} = \lim_{x \to -3} \frac{x + 3}{(x + 3)(x - 2)} = \lim_{x \to -3} \frac{1}{x - 2} = \frac{1}{5} \)

51. \( \lim_{x \to 1^-} \frac{x - 1}{(x^2 + 1)(x - 1)} = \lim_{x \to 1^-} \frac{1}{x^2 + 1} = \frac{1}{2} \)

53. \( \lim_{x \to 0^-} (1 + \frac{1}{x}) = -\infty \)

55. \( \lim_{x \to 0^+} \frac{2}{\sin x} = \infty \)

57. \( \lim_{x \to 8^-} \frac{e^{x^2}}{(x - 8)^3} = -\infty \)

59. \( \lim_{x \to (\pi/2)^-} \ln |\cos x| = \lim_{x \to (\pi/2)^-} \ln \left| \frac{\pi}{2} \right| = \ln 0 = -\infty \)

61. \( \lim_{x \to (\pi/2)^-} x \sec(\pi x) = \infty \) and \( \lim_{x \to (\pi/2)^+} x \sec(\pi x) = -\infty \).

So, \( \lim_{x \to (\pi/2)^-} x \sec(\pi x) \) does not exist.

63. \( f(x) = \frac{x^2 + x + 1}{x^3 - 1} = \frac{x^2 + x + 1}{(x - 1)(x^2 + x + 1)} \)
\( \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{1}{x - 1} = \infty \)

65. \( f(x) = \frac{1}{x^2 - 25} \)
\( \lim_{x \to 5^-} f(x) = -\infty \)

67. A limit in which \( f(x) \) increases or decreases without bound as \( x \) approaches \( c \) is called an infinite limit.
\( \infty \) is not a number. Rather, the symbol
\( \lim_{x \to c} f(x) = \infty \)
says how the limit fails to exist.

69. One answer is
\( f(x) = \frac{x - 3}{(x - 6)(x + 2)} = \frac{x - 3}{x^2 - 4x - 12} \)

71. 

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73. \[ m = \frac{m_0}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \]

\[ \lim_{v \to c} m = \lim_{v \to c} \frac{m_0}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} = \infty \]

75. (a) \[ r = 50\pi \sec^2 \frac{\pi}{6} = \frac{200\pi}{3} \text{ ft/sec} \]

(b) \[ r = 50\pi \sec^2 \frac{\pi}{3} = 200\pi \text{ ft/sec} \]

(c) \[ \lim_{\theta \to \pi/2^+} [50\pi \sec^2 \theta] = \infty \]

77. (a) Average speed = \[ \frac{\text{Total distance}}{\text{Total time}} \]

\[ 50 = \frac{2d}{(d/x) + (d/y)} \]

50 = \[ \frac{2xy}{y + x} \]

50y + 50x = 2xy

\[ 50x = 2xy - 50y \]

50x = 2y(x - 25)

\[ \frac{25x}{x - 25} = y \]

Domain: \( x > 25 \)

(b) \[
\begin{array}{|c|c|c|c|c|}
\hline
x & 30 & 40 & 50 & 60 \\
\hline
y & 150 & 66.667 & 50 & 42.857 \\
\hline
\end{array}
\]

(c) \[ \lim_{x \to 25^+} \frac{25x}{\sqrt{x - 25}} = \infty \]

As \( x \) gets close to 25 mi/h, \( y \) becomes larger and larger.

79. (a) \[ A = \frac{1}{2} \left( bh - \frac{1}{2}r^2\theta \right) = \frac{1}{2} \left( 10 \tan \theta - \frac{1}{2} \left( \frac{10}{2}\right)^2 \theta \right) = 50 \tan \theta - 50\theta \]

Domain: \( \left( 0, \frac{\pi}{2} \right) \)

(b) \[
\begin{array}{|c|c|c|c|c|}
\hline
\theta & 0.3 & 0.6 & 0.9 & 1.2 & 1.5 \\
\hline
f(\theta) & 0.47 & 4.21 & 18.0 & 68.6 & 630.1 \\
\hline
\end{array}
\]

(c) \[ \lim_{\theta \to \pi/2^-} A = \infty \]

81. False. For instance, let \[ f(x) = \frac{x^2 - 1}{x - 1} \text{ or } g(x) = \frac{x}{x^2 + 1}. \]

83. False. The graphs of \( y = \tan x, y = \cot x, y = \sec x \) and \( y = \csc x \) have vertical asymptotes.

85. Let \( f(x) = \frac{1}{x^2} \) and \( g(x) = \frac{1}{x^3}, \) and \( c = 0. \)

\[ \lim_{x \to 0} \frac{1}{x^2} = \infty \text{ and } \lim_{x \to 0} \frac{1}{x^3} = \infty, \] but

\[ \lim_{x \to 0} \left( \frac{1}{x^2} - \frac{1}{x^3} \right) = \lim_{x \to 0} \left( \frac{x^2 - 1}{x^4} \right) = -\infty \neq 0. \]

87. Given \( \lim_{x \to c} f(x) = \infty, \) let \( g(x) = 1. \) Then

\[ \lim_{x \to c} \frac{g(x)}{f(x)} = 0 \] by Theorem 2.15.
89. \( f(x) = \frac{1}{x-3} \) is defined for all \( x > 3 \). Let \( M > 0 \) be given. You need \( \delta > 0 \) such that \( f(x) = \frac{1}{x-3} > M \) whenever \( 3 < x < 3 + \delta \).

Equivalently, \( x - 3 < \frac{1}{M} \) whenever \( |x - 3| < \delta, x > 3 \).

So take \( \delta = \frac{1}{M} \). Then for \( x > 3 \) and \( |x - 3| < \delta, \frac{1}{x - 3} > \frac{1}{8} = M \) and so \( f(x) > M \).

**Review Exercises for Chapter 2**

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.

![Graph of f(x) = 1/(x-3)]

3. \( f(x) = \frac{4}{x + 2} - 2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-1.0526</td>
<td>-1.0050</td>
<td>-1.0005</td>
<td>-0.9995</td>
<td>-0.9950</td>
<td>-0.9524</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} f(x) \approx -1.0 \]

![Graph of f(x) = 1/(x + 2) - 2]

5. \( f(x) = 4 - x^2 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.8867</td>
<td>0.0988</td>
<td>0.0100</td>
<td>-0.0100</td>
<td>-0.1013</td>
<td>-1.1394</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} f(x) = 0 \]

![Graph of f(x) = 4 - x^2]

7. \( h(x) = \frac{4x - x^2}{x} = \frac{x(4 - x)}{x} = 4 - x, x \neq 0 \)

(a) \( \lim_{x \to 0} h(x) = 4 - 0 = 4 \)

(b) \( \lim_{x \to -1} h(x) = 4 - (-1) = 5 \)

9. \( f(t) = \frac{\ln(t + 2)}{t} \)

(a) \( \lim_{t \to 0} f(t) \) does not exist.

(b) \( \lim_{t \to -1} f(t) = 0 \)
11. \( \lim_{x \to 4}(x + 4) = 1 + 4 = 5 \)

Let \( \varepsilon > 0 \) be given. Choose \( \delta = \varepsilon \). Then for \( 0 < |x - 1| < \delta = \varepsilon \), you have
\[
|x - 1| < \varepsilon \\
|(x + 4) - 5| < \varepsilon \\
f(x) - L < \varepsilon.
\]

13. \( \lim_{x \to 2}(1 - x^2) = 1 - 2^2 = -3 \)

Let \( \varepsilon > 0 \) be given. You need
\[
|1 - x^2 - (-3)| < \varepsilon \Rightarrow |x^2 - 4| = |x - 2||x + 2| < \varepsilon \Rightarrow |x - 2| < \frac{1}{|x + 2|} \varepsilon.
\]

Assuming \( 1 < x < 3 \), you can choose \( \delta = \frac{\varepsilon}{5} \).

So, for \( 0 < |x - 2| < \frac{\varepsilon}{5} \), you have
\[
|x - 2| < \frac{\varepsilon}{5} < \frac{\varepsilon}{|x + 2|} \\
|x - 2||x + 2| < \varepsilon \\
|x^2 - 4| < \varepsilon \\
|4 - x^2| < \varepsilon \\
|(1 - x^2) - (-3)| < \varepsilon \\
f(x) - L < \varepsilon.
\]

23. \( \lim_{x \to 0} \frac{1/(x + 1) - 1}{x} = \lim_{x \to 0} \frac{1 - (x + 1)}{x(x + 1)} = \lim_{x \to 0} \frac{1}{x(x + 1)} = -1 \)

25. \( \lim_{x \to 5} \frac{x^3 + 125}{x + 5} = \lim_{x \to 5} \frac{(x + 5)(x^2 - 5x + 25)}{x + 5} = \lim_{x \to 5} (x^2 - 5x + 25) = 75 \)

27. \( \lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \lim_{x \to 0} \frac{x}{\sin x} \left( \frac{1 - \cos x}{x} \right) = (1)(0) = 0 \)

29. \( \lim_{x \to 1} e^{x-1} \sin \frac{\pi x}{2} = e^0 \sin \frac{\pi}{2} = 1 \)

31. \( \lim_{\Delta x \to 0} \frac{\sin[(\pi/6) + \Delta x] - (1/2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\pi/6)\cos \Delta x + \cos(\pi/6)\sin \Delta x - (1/2)}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{1/2 \cdot \cos \Delta x}{\Delta x} + \lim_{\Delta x \to 0} \frac{\sqrt{3}}{2} \cdot \frac{\sin \Delta x}{\Delta x} \\
= 0 + \frac{\sqrt{3}}{2} \cdot 1 \\
= \frac{\sqrt{3}}{2} \)
33. \( \lim_{x \to -\infty} [f(x) \cdot g(x)] = \left(-\frac{3}{4}\right) \left(\frac{2}{3}\right) = -\frac{1}{2} \)

35. \( \lim_{x \to -\infty} [f(x) + 2g(x)] = -\frac{3}{4} + 2\left(\frac{2}{3}\right) = \frac{7}{12} \)

37. \( f(x) = \frac{\sqrt{2x + 1} - \sqrt{3}}{x - 1} \)

(a)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.1</th>
<th>1.01</th>
<th>1.001</th>
<th>1.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.5680</td>
<td>0.5764</td>
<td>0.5773</td>
<td>0.5773</td>
</tr>
</tbody>
</table>

\( \lim_{x \to 1^+} \frac{\sqrt{2x + 1} - \sqrt{3}}{x - 1} \approx 0.577 \) \((\text{Actual limit is } \sqrt{3}/3)\)

(b) The graph has a hole at \( x = 1 \).

\( \lim_{x \to 1^+} f(x) \approx 0.5774 \)

(c) \( \lim_{x \to 1^+} \frac{\sqrt{2x + 1} - \sqrt{3}}{x - 1} = \lim_{x \to 1^+} \frac{\sqrt{2x + 1} - 3}{x - 1} \cdot \frac{\sqrt{2x + 1} + \sqrt{3}}{\sqrt{2x + 1} + \sqrt{3}} \)

\[ = \lim_{x \to 1^+} \frac{(2x + 1) - 3}{(x - 1)(\sqrt{2x + 1} + \sqrt{3})} \]

\[ = \lim_{x \to 1^+} \frac{2}{\sqrt{2x + 1} + \sqrt{3}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \]

39. \( v = \frac{s(4) - s(t)}{4 - t} \)

\[ = \lim_{t \to 4} \left[-4.9(16^2 + 250) - [-4.9t^2 + 250]\right] \]

\[ = 4.9(t^2 - 16) \]

\[ = \lim_{t \to 4^+} 4.9(t - 4)(t + 4) \]

\[ = 4.9 \cdot 0 = 0 \]

\[ \text{The object is falling at about 39.2 m/sec.} \]

41. \( \lim_{x \to 3^-} \frac{x - 3}{x - 3} = \lim_{x \to 3^-} \frac{(x - 3)}{x - 3} = -1 \)

43. \( \lim_{x \to 2^+} f(x) = 0 \)

45. \( \lim_{t \to 1^+} h(t) \text{ does not exist because } \lim_{t \to 1^+} h(t) = 1 + 1 = 2 \)

and \( \lim_{t \to 1^+} h(t) = \frac{1}{2}(1 + 1) = 1 \).

47. \( f(x) = -3x^2 + 7 \)

Continuous on \((-\infty, \infty)\)

49. \( f(x) = \left[ x + 3 \right] \)

\( \lim_{x \to k^+} \left[ x + 3 \right] = k + 3 \text{ where } k \text{ is an integer.} \)

\( \lim_{x \to k^-} \left[ x + 3 \right] = k + 2 \text{ where } k \text{ is an integer.} \)

Nonremovable discontinuity at each integer \( k \)

Continuous on \((k, k + 1)\) for all integers \( k \)
51. \( f(x) = \frac{3x^2 - x - 2}{x - 1} = \frac{(3x + 2)(x - 1)}{x - 1} \)
\[ \lim_{x \to 1} f(x) = \lim_{x \to 1} (3x + 2) = 5 \]
Removable discontinuity at \( x = 1 \)
Continuous on \((-\infty, 1) \cup (1, \infty)\)

53. \( f(x) = \frac{1}{(x - 2)^2} \)
\[ \lim_{x \to 2} \frac{1}{(x - 2)^2} = \infty \]
Nonremovable discontinuity at \( x = 2 \)
Continuous on \((-\infty, 2) \cup (2, \infty)\)

55. \( f(x) = \frac{3}{x + 1} \)
\[ \lim_{x \to -1^+} f(x) = -\infty \]
\[ \lim_{x \to -1^-} f(x) = \infty \]
Nonremovable discontinuity at \( x = -1 \)
Continuous on \((-\infty, -1) \cup (-1, \infty)\)

57. \( f(x) = \csc \frac{\pi x}{2} \)
Nonremovable discontinuities at each even integer.
Continuous on \((2k, 2k + 2)\) for all integers \( k \).

59. \( g(x) = 2e^{\sqrt[3]{x}} \) is continuous on all intervals \((n, n + 1)\),
where \( n \) is an integer. \( g \) has nonremovable discontinuities at each \( n \).

61. \( f(2) = 5 \)
Find \( c \) so that \( \lim_{x \to 2^+} (cx + 6) = 5 \).
\[ c(2) + 6 = 5 \]
\[ 2c = -1 \]
\[ c = -\frac{1}{2} \]

63. \( f \) is continuous on \([1, 2] \)
\[ f(1) = -1 < 0 \]
\[ f(2) = 13 > 0 \]
So by the Intermediate Value Theorem, there is at least one value \( c \) in \((1, 2)\) such that
\[ 2c^3 - 3 = 0. \]

65. \( A = 5000(1.06)^{2t} \)
Nonremovable discontinuity every 6 months

67. \( g(x) = 1 + \frac{2}{x} \)
Vertical asymptote at \( x = 0 \)

69. \( f(x) = \frac{8}{(x - 10)^2} \)
Vertical asymptote at \( x = 10 \)

71. \( g(x) = \ln(25 - x^2) \)
Vertical asymptotes at \( x = 5 \) and \( x = -5 \)

73. \( \lim_{x \to -2^-} \frac{2x^2 + x + 1}{x + 2} = -\infty \)

75. \( \lim_{x \to -1^+} \frac{x + 1}{x^3 + 1} = \lim_{x \to -1^+} \frac{1}{x^2 - x + 1} = \frac{1}{3} \)

77. \( \lim_{x \to -1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty \)

79. \( \lim_{x \to 0^+} \frac{\sin 4x}{5x} = \lim_{x \to 0^+} \left[ \frac{4(\sin 4x)}{4x} \right] = \frac{4}{5} \)

81. \( \lim_{x \to 0^+} \frac{\csc 2x}{x} = \lim_{x \to 0^+} \frac{1}{x \sin 2x} = \infty \)

83. \( \lim_{x \to 0^+} \ln(\sin x) = -\infty \)
85. \( f(x) = \frac{\tan 2x}{x} \)

(a) | \( x \) | \(-0.1\) | \(-0.01\) | \(-0.001\) | \(0.001\) | \(0.01\) | \(0.1\) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(2.0271)</td>
<td>(2.0003)</td>
<td>(2.0000)</td>
<td>(2.0000)</td>
<td>(2.0003)</td>
<td>(2.0271)</td>
</tr>
</tbody>
</table>

\[ \lim_{x \to 0} \frac{\tan 2x}{x} = 2 \]

(b) Yes, define \( f(x) = \begin{cases} \frac{\tan 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases} \).

Now \( f(x) \) is continuous at \( x = 0 \).

**Problem Solving for Chapter 2**

1. (a) Perimeter \( \Delta P AO = \sqrt{x^2 + (y - 1)^2} + \sqrt{x^2 + y^2} + 1 \)
   \[ = \sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1 \]

Perimeter \( \Delta P BO = \sqrt{(x - 1)^2 + y^2} + \sqrt{x^2 + y^2} + 1 \)
   \[ = \sqrt{(x - 1)^2 + x^2} + \sqrt{x^2 + x^4} + 1 \]

(b) \( r(x) = \frac{\sqrt{x^2 + (x^2 - 1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x - 1)^2 + x^2} + \sqrt{x^2 + x^4} + 1} \)

\[
\begin{array}{|c|c|c|c|c|}
\hline
x & 4 & 2 & 1 & 0.1 \\
\hline
\text{Perimeter} \Delta P AO & 33.02 & 9.08 & 3.41 & 2.10 \\
\hline
\text{Perimeter} \Delta P BO & 33.77 & 9.60 & 3.41 & 2.00 \\
\hline
r(x) & 0.98 & 0.95 & 1 & 1.05 \\
\hline
\end{array}
\]

(c) \( \lim_{x \to 0^+} r(x) = \frac{1 + 0 + 1}{1 + 0 + 1} = \frac{2}{2} = 1 \)
3. (a) There are 6 triangles, each with a central angle of $60^\circ = \pi/3$. So,

\[
\text{Area hexagon} = 6 \left( \frac{1}{2}bh \right) = 6 \left( \frac{1}{2} \right) \sin \left( \frac{\pi}{3} \right) = \frac{3\sqrt{3}}{2} \approx 2.598.
\]

(b) There are $n$ triangles, each with central angle of $\theta = 2\pi/n$. So,

\[
A_n = \left( \frac{1}{2} \right) h \left( \frac{2\pi}{n} \right) = \frac{n \sin(2\pi/n)}{2}.
\]

(c) 

<table>
<thead>
<tr>
<th>$n$</th>
<th>6</th>
<th>12</th>
<th>24</th>
<th>48</th>
<th>96</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>2.598</td>
<td>3.106</td>
<td>3.133</td>
<td>3.139</td>
<td></td>
</tr>
</tbody>
</table>

(d) As $n$ gets larger and larger, $2\pi/n$ approaches 0. Letting $x = 2\pi/n$, $A_n = \sin(2\pi/n)/(2\pi/n) = \sin x/x$, which approaches $(\pi)=\pi$.

5. (a) Slope $= -\frac{12}{5}$

(b) Slope of tangent line is $\frac{5}{12}$.

\[
y + 12 = \frac{5}{12}(x - 5)
\]

\[
y = \frac{5}{12}x - \frac{169}{12}.
\]

(c) 

\[
Q = (x, y) = \left( x, -\sqrt{169 - x^2} \right)
\]

\[
m_s = \frac{-\sqrt{169 - x^2} + 12}{x - 5}
\]

(d) \[
\lim_{x \to 5} m_s = \lim_{x \to 5} \frac{12 - \sqrt{169 - x^2}}{x - 5} \cdot \frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}}
\]

\[
= \lim_{x \to 5} \frac{144 - (169 - x^2)}{(x - 5)(12 + \sqrt{169 - x^2})}
\]

\[
= \lim_{x \to 5} \frac{x^2 - 25}{(x - 5)(12 + \sqrt{169 - x^2})}
\]

\[
= \lim_{x \to 5} \frac{10}{12 + \sqrt{169 - x^2}} = \frac{10}{12 + 12} = \frac{5}{12}
\]

This is the same slope as part (b).
7. (a) \(3 + x^{\frac{1}{3}} \geq 0\)
\[x^{\frac{1}{3}} \geq -3\]
\[x \geq -27\]

Domain: \(x \geq -27, x \neq 1\) or \([-27, 1) \cup (1, \infty)\)

(b) \[
\begin{align*}
\text{Graph}
\end{align*}
\]

(c) \[
\lim_{x \to -27^+} f(x) = \frac{\sqrt[3]{3 + (-27)^{\frac{1}{3}}} - 2}{-27 - 1} = \frac{-2}{-28} = \frac{1}{14} \approx 0.0714
\]

(d) \[
\lim_{x \to a^+} f(x) = \frac{\sqrt[3]{3 + x^{\frac{1}{3}} - 2}}{x - 1} - \frac{\sqrt[3]{3 + x^{\frac{1}{3}} + 2}}{\sqrt[3]{3 + x^{\frac{1}{3}} + 2}}
\]
\[
= \lim_{x \to a^+} \frac{3 + x^{\frac{1}{3}} - 4}{(x - 1)(\sqrt[3]{3 + x^{\frac{1}{3}}} + 2)}
\]
\[
= \lim_{x \to a^+} \frac{x^{\frac{1}{3}} - 1}{(x^{\frac{1}{3}} - 1)(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(\sqrt[3]{3 + x^{\frac{1}{3}}} + 2)}
\]
\[
= \lim_{x \to a^+} \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}
\]

9. (a) \[
\lim_{x \to 2^+} f(x) = 3: g_1, g_4
\]

(b) \(f\) continuous at 2: \(g_1\)

(c) \[
\lim_{x \to 2^-} f(x) = 3: g_1, g_4
\]

11.

(a) \[
\begin{align*}
f(1) &= [1] + [-1] = 1 + (-1) = 0 \\
f(0) &= 0 \\
f\left(\frac{1}{2}\right) &= 0 + (-1) = -1 \\
f(-2.7) &= -3 + 2 = -1
\end{align*}
\]

(b) \[
\begin{align*}
\lim_{x \to -1^-} f(x) &= -1 \\
\lim_{x \to -1^+} f(x) &= -1 \\
\lim_{x \to -1^+} f(x) &= -1
\end{align*}
\]

(c) \(f\) is continuous for all real numbers except \(x = 0, \pm 1, \pm 2, \pm 3, \ldots\)

13. (a) \[
\begin{align*}
\text{Graph}
\end{align*}
\]

(b) (i) \[
\lim_{x \to a^+} P_{a,b}(x) = 1
\]

(ii) \[
\lim_{x \to a^-} P_{a,b}(x) = 0
\]

(iii) \[
\lim_{x \to b^+} P_{a,b}(x) = 0
\]

(iv) \[
\lim_{x \to b^-} P_{a,b}(x) = 1
\]

(c) \(P_{a,b}\) is continuous for all positive real numbers except \(x = a, b\).

(d) The area under the graph of \(U\), and above the \(x\)-axis, is 1.