What is calculus? Calculus is the study of change. Calculus is important because it helps us to analyze the world of change that we live in. It is used to study gravity, weaponry, flight, planetary motion, heat, light, sound, electricity, magnetism, population growth, resource allocation, and cost minimization. Many of the scientific and technological developments from the late 1700s to modern times are connected with calculus.

Many people consider calculus to be interesting. Some think that it’s beautiful. Sure, they’re mostly mathematicians, scientists, and engineers. But few consider algebra to be beautiful—algebra is more utilitarian than beautiful. Your mathematics background is in algebra, so you might well think that math is utilitarian, not interesting or beautiful.

The beauty of calculus is in its ideas. You might never be in a position in which


WHAT WE WILL DO IN THIS CHAPTER

• We’ll look at the development of algebra and analytic geometry.
• We’ll investigate the four problems that motivated the invention of calculus.
• We’ll look at how Isaac Newton, motivated by those four problems, combined earlier studies of motion and area with the then-new algebra and analytic geometry, and invented calculus.
• We’ll discuss some of Isaac Newton’s key ideas about motion and area.
• While calculations are unavoidable (this is math, after all), we will focus on concepts, not on calculations.
you need calculus. But you are becoming an educated person, and as such, you are in a position to explore its ideas and perhaps become interested in them or even appreciate some of its beauty.

Isaac Newton and Gottfried Wilhelm Leibniz invented calculus, each working independently of the other. Their ideas were built on earlier studies of motion and area. Those earlier efforts were sorely limited by the lack of algebra and analytic geometry.

Algebra was invented in the ninth century by Arabs, but it didn’t reach Europe until the end of the Dark Ages, and it didn’t develop into the algebra that you’ve studied until the 1600s. Analytic geometry (which includes drawing graphs on the x- and y-axes) got its start in Greece in the third century, but it wasn’t until the 1600s that it developed into the graphing system that you’ve studied.

This chapter draws on a diverse set of algebraic and geometric topics. Because calculus is a study of how things change and the rate at which they change, we will review rates and two closely related topics: ratios and similar triangles. One of the questions that originally motivated the invention of calculus involves parabolas, so we will also review parabolas. Finally, because the central concept of calculus involves secant lines and their slopes and uses functional notation, we will review these topics.

13.0 Review of Ratios, Parabolas, and Functions

OBJECTIVES
• Learn about rates and ratios
• Use ratios to solve similar triangle problems
• Be able to graph a parabola and find its vertex and axis of symmetry
• Learn about secant lines
• Become proficient with functional notation

Ratios, Proportions, and Rates

A ratio is a comparison of two quantities, usually expressed as a fraction. In fact, a fraction is frequently called a “rational number,” because one meaning of the word rational is “having to do with ratios.” A college might have a ratio of two men to three women, that is, of (2 men)/(3 women). If 3,000 women are enrolled at the school, then there should be 2,000 men, because (2,000 men)/(3,000 women) = (2 men)/(3 women). This equation is an example of a proportion, which is an equality of two ratios.

A rate is a ratio that is used as a form of measurement. If someone drives 45 miles in one hour, then the ratio of distance to time is distance/time = (45 miles)/(1 hour) = 45 miles per hour. This is the driver’s average speed, or
average rate. If a $2,000 deposit earns $100 interest in 1 year, then the ratio of interest to principal is interest/principal = $100/$2000 = 1/20 = 0.05 = 5%. This is called the interest rate.

EXAMPLE 1

RATES AND PROPORTIONS A car’s odometer read 57,362.5 at 2 P.M. and 57,483.8 at 4 P.M.

a. Find the car’s average rate.

b. Use proportions to find the distance that would be traveled in 3 more hours at the same average rate.

SOLUTION

a. The average rate is the ratio of the change in distance to the change in time:

\[
\text{average rate} = \frac{\text{change in distance}}{\text{change in time}} = \frac{121.3 \text{ mi}}{2 \text{ hr}} = 60.65 \text{ mi/hr}
\]

b. To find the distance traveled in three more hours, we solve the proportion

\[
\frac{60.65 \text{ mi/hr}}{1 \text{ hr}} = \frac{x}{3 \text{ hr}}
\]

\[x = 3 \text{ hr} \cdot 60.65 \text{ mi/hr} = 181.95 \text{ mi}
\]

using dimensional analysis to cancel “hours,” as discussed in Appendix E

The average speed, or average rate, of a car is the ratio of distance to time; speed = distance/time. If we multiply each side of this equation by time, we get distance = speed · time (or distance = rate · time). In part (b) of Example 1, we used a proportion to find a distance. Instead, we could use this formula:

\[\text{distance} = \text{rate} \cdot \text{time} = 60.65 \text{ mi/hr} \cdot 3 \text{ hr} = 181.95 \text{ mi}\]

Delta Notation

Delta notation is frequently used to describe changes in quantities. Delta notation consists of the symbol \(\Delta\) (the Greek letter delta), which means “change in,” followed by a letter that refers to the quantity that changes. Thus, in Example 1, the change in distance could be written \(\Delta d\), and the change in time could be written \(\Delta t\) (read these as “delta \(d\)” and “delta \(t\)”). Expressed in delta notation, the car’s average rate would be written as

\[
\text{Average rate} = \frac{\Delta d}{\Delta t} = \frac{121.3 \text{ mi}}{2 \text{ hr}} = 60.65 \text{ miles per hour}
\]

This average rate is the ratio of the change in distance (\(\Delta d = 121.3 \text{ mi}\)) to the change in time (\(\Delta t = 2 \text{ hr}\)). Thus, it is a ratio of two changes, or a rate of change.

EXAMPLE 2

RATIOS AND PROPORTIONS A karat is a unit for measuring the fineness of gold. Pure gold is 24 karats fine; gold that is less than 24 karats is an alloy rather than pure gold. The number of karats is the numerator of a ratio between the amount of pure gold and the total amount of material; the denominator of that ratio is 24.

a. If a ring is 18-karat gold, find what portion of the ring is pure gold.

b. If the ring weighs 7 grams, find the amount of pure gold in the ring.
SOLUTION

a. \[
\frac{\text{amount of pure gold}}{\text{total amount of material}} = \frac{18 \text{ units of pure gold}}{24 \text{ units of material}}
= \frac{3 \text{ units of pure gold}}{4 \text{ units of material}}
\]

Therefore, three-fourths of the ring is pure gold.

b. To find the amount of pure gold, we solve the following proportion:

\[
\frac{3 \text{ units of pure gold}}{4 \text{ units of material}} = \frac{x \text{ grams pure gold}}{7 \text{ grams of material}}
\]

\[
x = 7 \cdot \frac{3}{4} = \frac{21}{4} = 5.25 \text{ grams}
\]

The ring has 5.25 grams of pure gold in it.

Similar Triangles

If a triangle is magnified, the sides will enlarge, but the angles do not enlarge. For example, if we have a triangle with sides of length 1, 1, and \(\sqrt{2}\), then its angles would be 45°, 45°, and 90°. If we magnified that triangle by a factor of 2, the lengths of the sides of the new triangle would be 2, 2, and \(2\sqrt{2}\), but the angles would still be 45°, 45°, and 90°. These two triangles are called similar triangles because they have the same angles. Similar triangles have the same shape, the same angles, but sides of different sizes, as shown in Figure 13.1.

\[
\text{FIGURE 13.1} \quad \text{Two similar triangles.}
\]

Notice that the ratios of corresponding sides are always equal to 2. In particular,

\begin{itemize}
  \item The ratio of the bottom side of the larger triangle to the bottom side of the smaller triangle is \(2/1 = 2\).
  \item The ratio of the right side of the larger triangle to the right side of the smaller triangle is \(2/1 = 2\).
  \item The ratio of the top side of the larger triangle to the top side of the smaller triangle is \(2\sqrt{2}/\sqrt{2} = 2\).
\end{itemize}

SIMILAR TRIANGLES

If two triangles are similar, the ratios of their corresponding sides are always equal.

\[
\frac{a}{c} = \frac{b}{d}
\]
In the illustrations on the previous page, the two triangles are similar because they have the same angles. In particular, the single slash mark in the left angles of the two triangles means that those two angles are equal. Similarly, the double slash mark in the top angles means that those two angles are equal, and the triple slash mark in the third angles means that those two angles are equal.

**Similar Triangles and the Great Pyramid**

The fact that the ratios of corresponding sides of similar triangles are always equal was first demonstrated by the Greek mathematician Thales around 600 B.C. He used this fact to measure the height of the Great Pyramid of Cheops. He stuck a pole into the ground right at the tip of the shadow of the pyramid. This created two similar triangles, one formed by the pyramid and its shadow, the other formed by the pole and its shadow. Suppose we call the height of the pyramid \( h \), its base \( b \), the length of its shadow \( S \), the height of the pole \( p \), and the length of its shadow \( s \), as shown in Figure 13.2. Then

\[
\frac{h}{p} = \frac{\frac{1}{2}b + S}{s}
\]

\[
h = \frac{\frac{1}{2}b + S}{s} \cdot p
\]

The lengths of the pole, the two shadows, and the base of the pyramid were easily measured, so the height of the pyramid could be calculated.

**EXAMPLE 3**

**USING SIMILAR TRIANGLES** The two triangles in Figure 13.3 are similar because they have the same angles, as indicated by the marks inside their angles. Find the lengths of the unknown sides.
The two triangles are similar, so the ratios of their corresponding sides are equal:

\[
\frac{b}{4} = \frac{12}{3} \quad \text{\textit{b corresponds to 4, and 12 corresponds to 3}}
\]

\[
4 \cdot \frac{b}{4} = 4 \cdot \frac{12}{3} \quad \text{solving for } b
\]

\[
b = 4 \cdot 4 = 16
\]

We can find \(c\) in a similar manner:

\[
\frac{c}{5} = \frac{12}{3} \quad \text{\textit{c corresponds to 5, and 12 corresponds to 3}}
\]

\[
5 \cdot \frac{c}{5} = 5 \cdot \frac{12}{3} \quad \text{solving for } c
\]

\[
c = 5 \cdot 4 = 20
\]

A parabola always has this shape:

A parabola can be fatter or thinner than shown above, or it can be upside down or sideways, as shown in Figure 13.4 on the next page, but it will always have the same basic shape.

One important aspect of a parabola’s shape is that it is \textit{symmetrical}; that is, there is a line along which a mirror could be placed so that one side of the line is the mirror image of the other. That line is called the \textit{line of symmetry}. The point at which the parabola intersects the line of symmetry is called the \textit{vertex}. See Figure 13.5.

The equation of a parabola always involves a quadratic expression. Typically, it is either of the form \(y = ax^2 + bx + c\) or of the form \(x = ay^2 + by + c\). We can graph a parabola by plotting points, as long as we find enough points that the vertex is included.
13.0 Review of Ratios, Parabolas, and Functions

### Figure 13.4
Parabolas.

### Figure 13.5
The line of symmetry.

#### Example 4

**Graphing a Parabola**

Graph the parabola \( y = x^2 - 6x + 5 \) by plotting points until the vertex is found. Find the vertex and the equation of the line of symmetry.

**Solution**

Substituting for \( x \) yields the following points:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>0</td>
<td>-3</td>
<td>-4</td>
<td>-3</td>
<td>0</td>
<td>5</td>
</tr>
</tbody>
</table>

As shown above and in Figure 13-6, the points \((2, -3)\) and \((4, -3)\) have the same \( y \)-values, as do the points \((1, 0)\) and \((5, 0)\), and the points \((0, 5)\) and \((6, 5)\). If we place a mirror halfway between each of these pairs of points, then the points would reflect onto each other. The line of symmetry is the vertical line where we would place the mirror, as shown in Figure 13.6. This line goes through \((3, -4)\) and other points with an \( x \)-coordinate of 3, such as \((3, 0)\) and \((3, 1)\), so the equation of the line of symmetry is \( x = 3 \). The vertex is where the line of symmetry intersects the parabola, so it is the point \((3, -4)\). The parabola resulting from these points is shown in Figure 13.6.

Notice that the following shorter list of points would have been sufficient for graphing our parabola.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>0</td>
<td>-3</td>
<td>-4</td>
<td>-3</td>
</tr>
</tbody>
</table>

The locations of the line of symmetry and the vertex are apparent as soon as we see that if \( x \) is either 2 or 4, the \( y \)-value is the same. We don’t have to list any more points, because symmetry tells us that the \( y \)-value at \( x = 5 \) (2 units to the right of the vertex) will be the same as the \( y \)-value at \( x = 1 \) (2 units to the left of the vertex). When you graph a parabola, list points until you can see the symmetry.
A line is secant to a curve if the line intersects the curve in at least two points.

**EXAMPLE 5**

**DRAWING A SECANT LINE**

**a.** Sketch the curve \( y = x^2 \).

**b.** Sketch the line secant to the curve through \( x = 2 \) and \( x = 3 \).

**c.** Find the slope of the secant line.

**SOLUTION**

**a.** The equation is of the form \( y = ax^2 + bx + c \) (with \( a = 1, b = 0, \) and \( c = 0 \)), so the curve is a parabola. Substituting for \( x \) yields the following points:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

The symmetry is not yet apparent, since no two points have the same \( y \)-value. We can continue to substitute larger values for \( x \), or we can try some negative values.
The graph of \( y = x^2 \).

**FIGURE 13.8**
The graph of \( y = x^2 \).

The graph, with a secant line.

**FIGURE 13.9**
The graph, with a secant line.

b. The secant line intersects the curve at the points \((2, 4)\) and \((3, 9)\). See Figure 13.9.

c. The slope of the secant line is

\[
m = \frac{\Delta y}{\Delta x} = \frac{9 - 4}{3 - 2} = \frac{5}{1} = 5
\]

### Functions

A **function** from \( x \) to \( y \) is a rule that associates a single value of \( y \) to each value of \( x \). The equation

\[
y = 3x - 5
\]

is a function, because it associates a single value of \( y \) to each value of \( x \). For example, to the \( x \)-value 0 it associates the *single* \( y \)-value \( 3 \cdot 0 - 5 = -5 \), and to the \( x \)-value 12 it associates the *single* \( y \)-value \( 3 \cdot 12 - 5 = 31 \).

When an equation is a function, it can be rewritten with functional notation.

**Functional notation** involves giving the function a name. Usually, the name is the letter \( f \) (which stands for “function”) or some other letter near \( f \) in the alphabet. If we named the above function \( f \), then we would write

\[ f(x) = 3x - 5 \]

instead of

\[ y = 3x - 5 \]

The notation \( f(x) \) looks like multiplication, but it is not. It is read “\( f \) of \( x \)” rather than “\( f \) times \( x \).”

The instructions “find \( f(0) \)” mean “substitute 0 for \( x \) in the function named \( f \).”

Thus, for the function \( f(x) = 3x - 5 \),

\[
f(0) = 3 \cdot 0 - 5 = -5
\]

### EXAMPLE 6

**USING FUNCTIONAL NOTATION**

Given \( g(x) = 5x + 2 \) and \( h(x) = x^2 - 1 \), find the following:

**a.** \( g(7) \)  
**b.** \( h(4) \)  
**c.** \( g(3x - 1) \)  
**d.** \( h(x + \Delta x) \)

**SOLUTION**

**a.** “Find \( g(7) \)” means “substitute 7 for \( x \) in the function named \( g \).”

\[
g(x) = 5x + 2
\]

\[
g(7) = 5 \cdot 7 + 2 = 37
\]

**b.** “Find \( h(4) \)” means “substitute 4 for \( x \) in the function named \( h \).”

\[
h(x) = x^2 - 1
\]

\[
h(4) = 4^2 - 1 = 15
\]

**c.** “Find \( g(3x - 1) \)” means “substitute 3\( x \) - 1 for \( x \) in the function named \( g \).”

\[
g(x) = 5x + 2
\]

\[
g(3x - 1) = 5 \cdot (3x - 1) + 2 = 15x - 5 + 2 = 15x - 3
\]

same y-value

The graph is sketched as shown in Figure 13.8.
d. “Find $h(x + \Delta x)$” means “substitute $x + \Delta x$ for $x$ in the function named $h$."

\[
h(x) = x^2 - 1
\]

\[
h(x + \Delta x) = (x + \Delta x)^2 - 1 = x^2 + 2x\Delta x + \Delta x^2 - 1
\]

## 13.0 Exercises

In Exercises 1–8, use ratios to find the unknown lengths.

1. 

2. 

3. 

4. 

5. 

6. 

7. 

8. 

9. The University of Utopia has a student/teacher ratio of 15 to 1.
   a. Write this ratio as a fraction.
   b. How large is the University of Utopia faculty if the university has 5,430 students?

10. Sylvia Silver received $2,330 in interest on an investment of $29,125.
    a. Find the ratio of interest to principal.
    b. Express this ratio as an interest rate.

Selected exercises available online at www.webassign.net/brookscole
11. In one year, Gordon Gotrox received $1,575 in interest on an investment of $17,500.
   a. Find the ratio of interest to principal.
   b. Express this ratio as an interest rate.
12. A car’s odometer read 33,482.4 at 9 A.M. and 33,812.9 at 4 P.M.
   a. Find the car’s average rate.
   b. Use proportions to find the distance that would be traveled in four more hours at the same average rate.
   c. Use a formula to find the distance that would be traveled in four more hours at the same average rate.
13. A car’s odometer read 101,569.3 at 10 A.M. and 101,633.5 at 12 noon.
   a. Find the car’s average rate.
   b. Use proportions to find the distance that would be traveled in four more hours at the same average rate.
   c. Use a formula to find the distance that would be traveled in four more hours at the same average rate.
14. A car’s odometer read 101,569.3 at 10:00 A.M. and 101,633.5 at 11:30 A.M.
   a. Find the car’s average rate.
   b. Use proportions to find the distance that would be traveled in four more hours at the same average rate.
   c. Use a formula to find the distance that would be traveled in four more hours at the same average rate.
15. A car’s odometer read 72,938.0 at 9:00 A.M. and 73,130.3 at 11:30 A.M.
   a. Find the car’s average rate.
   b. Use proportions to find the distance that would be traveled in four more hours at the same average rate.
   c. Use a formula to find the distance that would be traveled in four more hours at the same average rate.
16. Phyllis Peterson has a new Toyonda Suppord. When she filled the tank with gas, the odometer read 37.6 miles. Later, she added 7.3 gallons, and the odometer read 274.1 miles.
   a. Find the ratio of miles traveled to gallons.
   b. Use a proportion to predict the number of miles she could travel on a full tank if her tank holds 14 gallons.
   c. Use a formula to predict the number of miles she could travel on a full tank if her tank holds 14 gallons.
17. Rick Mixter has a new Nissota Suppette. When he filled the tank with gas, the odometer read 5.4 miles. Later, he added 13.3 gallons, and the odometer read 332.5 miles.
   a. Find the ratio of miles traveled to gallons consumed.
   b. Use a proportion to predict the number of miles he could travel on a full tank if his tank holds 15 gallons.
   c. Use a formula to predict the number of miles he could travel on a full tank if his tank holds 15 gallons.
18. If a ring is 18-karat gold and weighs 4 grams, find the amount of pure gold in the ring.
19. If a ring is 12-karat gold and weighs 8 grams, find the amount of pure gold in the ring.
20. If a ring is 18-karat gold and weighs 8 grams, find the amount of pure gold in the ring.
21. When Thales determined the height of the Great Pyramid of Cheops, he found that the base of the pyramid was 756 feet long, the pyramid’s shadow was 342 feet long, the pole was 6 feet tall, and the pole’s shadow was 9 feet long. (Naturally, the unit of measurement was not feet.) Find the height of the Great Pyramid.
22. Draw the two similar triangles that are a part of Figure 13.2. List the two pairs of corresponding sides.

In Exercises 23–32, (a) graph the parabola, (b) give the equation of its line of symmetry, and (c) find its vertex.
23. $y = 3x^2 + 4$
24. $y = 2 - 4x$
25. $y = -x^2 - 4x + 3$
26. $y = x^2 + 2x + 1$
27. $y = 2x + x^2 + 3$
28. $y = x^2 - 8x + 10$
29. $y = 8x - x^2 - 14$
30. $y = 4x - x^2$
31. $y = 12x - 2x^2$
32. $y = 2x^2 - 8x + 4$

In Exercises 33–36, (a) find the slope of the secant line that intersects the given parabola at the given points and (b) graph the parabola and the secant line.
33. $y = 2 - 3x^2$, $x = 1$, $x = 3$
34. $y = x^2 - 2x + 2$, $x = -2$, $x = 0$
35. $y = x^2 - 6x + 11$, $x = 1$, $x = 5$
36. $y = 3 - 4x^2$, $x = 1$, $x = 2$

In Exercises 37–48, use the following functions:

$$f(x) = 8x - 11$$
$$g(x) = 3x^2$$
$$h(x) = x^2 - 3$$
$$k(x) = x^2 + 2x - 4$$
37. Find $f(4)$.
38. Find $g(-2)$.
39. Find $h(-3)$.
40. Find $k(-5)$.
41. Find $f(x + 3)$.
42. Find $g(x - 2)$.
43. Find $h(x - 7)$.
44. Find $k(x + 1)$.
45. Find $f(x + \Delta x)$.
46. Find $g(x + \Delta x)$.
47. Find $h(x + \Delta x)$.
48. Find $k(x + \Delta x)$. 

13.1 The Antecedents of Calculus

Objectives

- Learn about Apollonius and his studies of parabolas
- Discover how al-Khowarizmi invented algebra and how Viète and Descartes later improved it
- Learn of Oresme’s early work in analytic geometry and how Descartes and Fermat later improved it

Calculus was invented because there was a need for it and because some mathematical tools necessary to calculus had already been invented. We’ll discuss the need for calculus in Section 13.2. We’ll discuss the mathematical tools necessary to calculus in this section.

Greek Mathematics—Geometry

There were many Greek mathematicians who made important discoveries in geometry. We will look at one particular Greek mathematician, whose work contributed to the invention of calculus in an indirect but important way.

Around 220 B.C., the Greek mathematician Apollonius of Perga investigated the curves obtained by taking cross sections of a double cone. These curves—the circle, ellipse, parabola, and hyperbola—are called **conic sections**. (See Figure 13.10.) Apollonius’s work was the standard reference on conic sections for almost 2,000 years, even though several of its eight volumes were lost.

Algebra had not been invented yet, so Apollonius studied parabolas by superimposing line segments and squares over the curves. Apollonius arbitrarily selected a point on a parabola. As shown in Figure 13.11, he then drew a square with one side on the parabola’s line of symmetry and a second side connecting the line of symmetry and the selected point. He also drew a line segment along the line of symmetry, connecting the vertex and the square.

Apollonius then selected a second point on the parabola and drew a second square and a second line segment, just as he had done with the first point. See...
Figure 13.12. He found that the ratio of the lengths of the line segments is always equal to the ratio of the areas of the squares. In other words,

\[
\frac{\text{length of line segment 1}}{\text{length of line segment 2}} = \frac{\text{area of square 1}}{\text{area of square 2}}
\]

If Apollonius had had the more modern idea of \( x \)- and \( y \)-coordinates, had made the \( y \)-axis the line of symmetry and the origin the vertex, and had labeled the two points on the parabola \((x_1, y_1)\) and \((x_2, y_2)\), he might have realized that the lengths of the line segments are \( y_1 \) and \( y_2 \) and the bases of the squares are \( x_1 \) and \( x_2 \). See Figure 13.13.

Thus, the areas of the squares are \( x_1^2 \) and \( x_2^2 \). The modern version of Apollonius’s conjecture that

\[
\frac{\text{length of line segment 1}}{\text{length of line segment 2}} = \frac{\text{area of square 1}}{\text{area of square 2}}
\]

would be

\[
\frac{y_1}{y_2} = \frac{x_1^2}{x_2^2}
\]

**EXAMPLE 1**

**USING APOLLONIUS’S APPROACH TO PARABOLAS** The parabola \( y = 2x^2 \) goes through the points \((1, 2)\) and \((2, 8)\).

a. At each of these two points, compute the length of the line segment and the area of the square described by Apollonius.

b. Compute \((\text{length of line segment 1})/\text{(length of line segment 2)}\) and \((\text{area of square 1})/\text{(area of square 2)}\).

a. The parabola is graphed in Figure 13.14.

At the point \((1, 2)\), the line segment has length \( y_1 = 2 \). The base of the square has length \( x_1 = 1 \), so the area of the square is \( 1^2 = 1 \).

At the point \((2, 8)\), the line segment has length \( y_2 = 8 \). The base of the square has length \( x_2 = 2 \), so the area of the square is \( 2^2 = 4 \).
As Apollonius observed, the ratio of the length of the line segments is equal to the ratio of the areas of the squares—both equal 1/4.

**Arabic Mathematics—Algebra**

The Arab mathematician Mohammed ibn Musa al-Khowarizmi wrote two important books around A.D. 830, each of which was translated into Latin in the twelfth century. Much of the mathematical knowledge of medieval Europe was derived from the Latin translations of al-Khowarizmi’s two works.

Al-Khowarizmi’s first book, on arithmetic, was titled *Algorithmi de numero Indorum* (or *al-Khowarizmi on Indian Numbers*). The Latin translation of this book introduced Arabic numbers (1, 2, 3 etc.) to Europe along with the procedures for multiplication and long division that we now use. Before this book, multiplying and dividing had to be done with Roman numerals (I, II, III, etc.) and were extremely difficult. The book’s title is the origin of the word *algorithm*, which means a procedure for solving a certain type of problem, such as the procedure for long division.

Al-Khowarizmi’s second book, *Al-Jabr w’al Muqabalah*, is considered the first algebra book ever written. It discussed linear and quadratic equations. In fact, the word *algebra* comes from the title of this second book. This title, which translates literally as *Restoration and Opposition*, refers to the solving of an equation by adding the same thing to each side of the equation (which “restores the balance” of the equation) and simplifying the result by canceling opposite
terms (which is the title’s “opposition”). For example (using modern symbolic algebra),

\[ 6x = 5x + 11 \]
\[ 6x + -5x = 5x + 11 + -5x \]
\[ x = 11 \]

"Al-jabr" or restoration of balance

"Al-muqabalah" or opposition

The quote below, from a translation of *Al-Jabr w'al Muqabalah,* demonstrates several important features of al-Khowarizmi’s algebra. First, it is entirely verbal, as was the algebra of Apollonius—there is no symbolic algebra at all. Second, this algebra differs from that of Apollonius in that it is not based on proportions. Third, the terminology betrays the algebra’s connections with geometry. When al-Khowarizmi refers to “a square,” he is actually referring to the area of a square; when he refers to “a root,” he is actually referring to the length of one side of the square (hence the modern phrase “square root”). Modern symbolic algebra uses the notations \( x^2 \) and \( x \) in place of “a square” and “a root.”

The quote from *Al-Jabr w'al Muqabalah* is on the left; a modern version of the same instructions is on the right. You might recognize this modern version from intermediate algebra, where it is called “completing the square.”

The following is an example of squares and roots equal to numbers: a square and 10 roots are equal to 39 units. The question therefore in this type of equation is about as follows: what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken then the square root of this which is 8, subtract from it the half of the roots, 5, leaving 3. The number three therefore represents one root of this square, which itself, of course, is 9. Nine therefore gives that square.

Al-Khowarizmi did not recognize negative numbers. Modern algebra recognizes negative numbers and gives two solutions to the above equation:

\[ (x + 5)^2 = 64 \]
\[ x + 5 = 8 \quad \text{or} \quad x + 5 = -8 \]
\[ x = 3 \quad \text{or} \quad x = -13 \]

Al-Khowarizmi justifies his verbal algebraic solution with a geometrical demonstration: “Now, however, it is necessary that we should demonstrate geometrical the truth of the same problems which we have explained in numbers. Therefore our first proposition is this, that a square and 10 roots equal 39 units. The proof is that we construct a square of unknown sides and let this square figure represent the square which together with its root you wish to find.” (He is describing a square with sides of length $x$.)

He next instructs the reader to construct a rectangle with one side of length 10 and the other of length equal to that of the square. If the square’s side is of length $x$, then the area of the square is $x^2$ and the area of the rectangle is $10x$, so the total area is $x^2 + 10x$. We are given that “a square and 10 roots equals 39 units,” so we know that $x^2 + 10x = 39$, and thus the combined areas of the square and the rectangle must be 39, as shown in Figure 13.15.
The rectangle can be cut into four strips, each with one side of length $x$ and the other of length $\frac{10}{4} = \frac{5}{2}$, as shown in Figure 13.16. Each strip can be glued to the edge of the square, resulting in a figure whose area is still $x^2 + 4 \cdot \frac{5}{2}x = x^2 + 10x = 39$, as shown in Figure 13.17.

If we filled in the corners of this figure, the result would be a new square that is larger than the original square. (This step explains the name of the modern version of al-Khowarizmi’s method—“completing the square.”) The corners are each squares of area $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$, so the area of the four corners is $4 \cdot \frac{25}{4} = 25$. The area of the figure with its corners filled in, shown in Figure 13.18, is $39 + 25 = 64$. Because the area of the large square is 64, it must have sides of length 8. To find the length of a side of the original square and to solve the problem, we would have to subtract $2 \cdot \frac{5}{2} = 5$ from 8 and get $8 - 5 = 3$. Thus, the area of the smaller, original square is $3 \cdot 3 = 9$.

### Mathematics during the Middle Ages—Analytic Geometry

As trade expanded, Arab and Greek knowledge was transmitted throughout Europe. European merchants began to visit the East to acquire more scientific information. Many of the famous European universities, including Oxford, Cambridge, Paris, and Bologna, were built.

During the Middle Ages, the Greek philosopher Aristotle’s concepts of change and motion were favorite topics at the universities, especially at Oxford and Paris. Around 1360, the Parisian scholar Nicole Oresme drew a picture to describe the speed of a moving object, which he described as the “intensity of motion.” He drew a horizontal line, along which he marked points representing different points in time. At each of these points, he drew a vertical line, the length of which represented the speed of the object at that point in time, as shown in Figure 13.19. Oresme called his horizontal and vertical lines *longitudo* and *latitudo*, which indicate the concept’s origins in mapmaking. If the speed of the object increased in a uniform way, the endpoints of these vertical lines would themselves lie in a straight line.

Using modern terminology, Oresme’s horizontal line was the $x$-axis, the points along the line were $x$-coordinates, the length of the vertical lines were $y$-coordinates, and the diagonal line was the graph of the speed. Oresme’s work is considered one of the earliest appearances of the idea of $x$- and $y$-coordinates and of the graphing of a variable quantity; for this reason, his work is a cornerstone of analytic geometry.
Mathematics during the Renaissance—Better Science and Algebra

In the 1400s, Europe was undergoing a massive transformation. The Black Death, a plague that killed as much as three-quarters of the population of Europe, was over, and Europe was becoming revitalized. In 1453, the Turks conquered Constantinople, the last remaining center of Greek culture. Many Eastern scholars moved from Constantinople to Europe, bringing Greek knowledge and manuscripts with them. Around the same time, Gutenberg invented the movable-type printing press, which greatly increased the availability of scientific information in the form of both new works and translations of ancient works. Translations of Apollonius’s work on geometry and al-Khowarizmi’s work on algebra were printed.

The ready availability of Greek and Arab knowledge, together with information about the Greek way of life, created a pointed contrast in the medieval European mind. The Greeks attempted to understand the physical world, whereas the Church suppressed any theory that was opposed to established doctrines. A general spirit of skepticism replaced the acceptance of old ideas, and knowledge expanded in all directions. Many libraries and universities were established, and scholars, poets, artists, and craftsmen received encouragement and material support from wealthy benefactors.

Technology grew quickly. The introduction of gunpowder in the thirteenth century had revolutionized warfare. Europe was engulfed in an arms race; a nation’s fate hinged on its advances in the design and use of cannons, muskets, and fortifications. Some of Europe’s best minds—including those of Galileo, Leonardo da Vinci, Michelangelo, and Albrecht Dürer—were devoted to various aspects of the problem. It was necessary to learn about the trajectories of projectiles, their range, the heights they could reach, and the effect of muzzle velocity. In fact, it became necessary to investigate the principles of motion itself.

Algebra was still a set of specific techniques that could be used to solve specific equations. There was little generalization, and there was no way to write an equation to represent an entire class of equations, as we would now write $ax^2 + bx + c = 0$ to represent all quadratic equations. There were only ways to write specific equations such as $3x^2 + 5x + 7 = 0$. Thus, it was impossible to write a formula such as the quadratic formula [if $ax^2 + bx + c = 0$, then $x = (-b ± \sqrt{b^2 - 4ac})/2a$]. It was possible only to give an example, such as al-Khowarizmi’s example of completing the square.
In the late sixteenth century, algebra matured into a much more powerful tool. It became more symbolic. Exponents were introduced; what had been written as “cubus,” “A cubus” or “AAA” could now be written as “A^3.” The symbols +, −, and = were also introduced.

FRANÇOIS VIÉTE AND ALGEBRA

François Viète, a French lawyer who studied mathematics as a hobby, began using vowels to represent variables and consonants to represent constants. (The more modern convention of using letters from the end of the alphabet, such as x and y, to represent variables and letters from the beginning of the alphabet, such as a and b, to represent constants was introduced later by Descartes.) This allowed mathematicians to represent the entire class of quadratic equations by writing “A^2 + BA = C” (where the vowel A is the variable and the consonants B and C are the constants) and made it possible to discuss general techniques that could be used to solve classes of equations. These notational changes were slow to gain acceptance. No one mathematician adopted all the new notations. Viète’s algebra was quite verbal; he did not even adopt the symbol ÷ until late in his life.

RENÉ DESCARTES, ANALYTIC GEOMETRY, AND ALGEBRA

In 1637, the famous French philosopher and mathematician René Descartes published *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Discourse on the Method of Reasoning Well and Seeking Truth in the Sciences). Descartes’s *Discours* contained an appendix, *La géométrie*, that explored the relationship between algebra and geometry in a way unforeseen by Apollonius, al-Khowarizmi, and Oresme. In it, Descartes showed that algebra could be applied to geometric problems. This combined algebra and geometry in new and unique ways. However, it did not especially resemble our modern analytic geometry, which consists of ordered pairs, x- and y-axes, and a correspondence between algebraic equations and their graphs. Descartes used an x-axis, but he did not have a y-axis. Descartes did not use what we now call Cartesian coordinates in his honor. Although he knew that an equation in two unknowns determines a curve, he had very little interest in sketching curves; he never plotted a new curve directly from its equation.

Descartes’s algebra, on the other hand, was more modern than that of any of his predecessors or contemporaries. Algebra had been advancing steadily since the Renaissance, and it found its culmination in Descartes’s *La géométrie*. Here we see a symbolic, equation-based algebra rather than a verbal, proportion-based algebra. With Descartes, algebra had finally evolved enough that calculus could be invented.

PIERRE DE FERMAT AND MODERN ANALYTIC GEOMETRY

In 1629, eight years before Descartes’s *La géométrie*, the French lawyer and amateur mathematician Pierre de Fermat attempted to recreate one of the lost works of Apollonius on conic sections using references to that work made by other Greek mathematicians. Fermat applied Viète’s algebra to Apollonius’s analytic geometry and created an analytic geometry much more similar to the modern one than was Descartes’s. Fermat emphasized the sketching of graphs of equations. He showed a parallelism between certain types of equations and certain types of graphs; for example, he showed that the graph of a quadratic function is always a parabola.

Analytic geometry is usually considered to be an invention of Descartes, because Descartes published before Fermat. However, Fermat had written his work well before Descartes wrote his. Unfortunately, Fermat never published his work; it was released only after his death, almost 50 years after it was written. It would be most correct to say that analytic geometry was invented by Apollonius, al-Khowarizmi, Oresme, Descartes, and Fermat.
RENÉ DESCARTES, 1596–1650

René Descartes’s family was rather well off, and he inherited enough money to be able to afford a life of study and travel. At the age of eight, he was sent away to a Jesuit school where, at first because of health problems, he was allowed to stay in bed all morning. He maintained this habit all his life and felt that his morning meditative hours were his most productive.

At the age of sixteen, he left school and went to Paris, where he studied mathematics. Four years later, he became a professional soldier, enlisting in the army of Prince Maurice of Nassau and later in the Bavarian army. He found that a soldier’s life, though busy and dangerous at times, provided him with sufficient leisure time to continue his studies.

After quitting the army, he spent several years traveling in Denmark, Switzerland, Italy, and Holland. He eventually settled in Holland for 20 years, where he studied science, philosophy, and mathematics and spent four years writing a book on the workings of the physical world. Holland allowed much more freedom of thought than did most of Europe at that time. Even so, when he heard of Galileo’s condemnation by the Church and his imprisonment for writing that the earth revolves around the sun, Descartes prudently abandoned his work and instead wrote on his philosophy of science. The resulting work, Discourse on the Method of Rightly Conducting Reason and Seeking Truth in the Sciences, was followed by two more books: Meditations, in which he further explains his philosophical views, and Principia philosophiae (Philosophical Principles), in which he details a theory on the workings of the universe that became the accepted scientific view for about 100 years.

Descartes’s philosophy was based on systematic doubt. His doubt was so systematic that he doubted his own existence. He decided that the only thing he couldn’t doubt was doubt itself. This led to the conclusion “I think, therefore I am.” He hoped that, through this doubting process, he would be able to reach clear and distinct ideas.

Holland’s liberal attitude toward new views was not without exception. At one time, it was forbidden to print or sell any of Descartes’s works. Another time, he was brought before a judge on charges of atheism.

Although he made numerous advances in optics and wrote on physics, physiology, and psychology, Descartes is most famous for his philosophical and mathematical works. He has been called the father of modern philosophy, because he attempted to build a completely new system of thought.

When Queen Christina of Sweden invited Descartes to become her tutor, he accepted. Stockholm was cold and miserable, and the queen forced Descartes to break his habit of staying in bed by requiring him to instruct her daily at 5:00 A.M. After four months in Sweden, Descartes came down with pneumonia and died at the age of fifty-four.

In Exercises 1–4, do the following.

a. Sketch the parabola given by the equation.

b. At $x = 1$ and $x = 2$, compute the length of the line segment and the area of the square described by Apollonius. Sketch these two line segments and two squares in their proper places, superimposed over the parabola.

c. Compute the ratio of the lengths of the two line segments and the ratio of the areas of the two squares in part (b).

d. At $x = 3$ and $x = 4$, compute the length of the line segment and the area of the square described by Apollonius. Sketch these two line segments and two squares in their proper places, superimposed over the parabola.
e. Compute the ratio of the lengths of the two line segments and the ratio of the areas of the two squares in part (d).

1. \( y = \frac{1}{4}x^2 \)  
2. \( y = \frac{1}{2}x^2 \)
3. \( y = 3x^2 \)  
4. \( y = 10x^2 \)

In Exercises 5–8, do the following.

a. Sketch the parabola given by the equation.

b. At \( x = 1 \) and \( x = 2 \), compute the length of the line segment and the area of the square described by Apollonius. Sketch these two line segments and two squares in their proper places, superimposed over the parabola.

c. Compute the ratio of the lengths of the two line segments and the ratio of the areas of the two squares in part (b).

5. \( y = x^2 + 1 \)  
6. \( y = x^2 + 3 \)
7. \( y = x^2 + 2x + 1 \)  
8. \( y = x^2 + 6x + 9 \)

In Exercises 9–14, do the following.

a. Solve the problem using the method of al-Khowarizmi. Use language similar to his, but don’t just copy the example given in this section. Do not use modern terminology and notation.

b. Illustrate your solution to the problem in the manner of al-Khowarizmi.

c. Give a modern version of your solution from part (a).

9. A square and 12 roots are equal to 45 units.

10. A square and 8 roots are equal to 48 units.

11. A square and 6 roots are equal to 7 units.

12. A square and 6 roots are equal to 55 units.

13. A square and 2 roots are equal to 80 units.

14. A square and 4 roots are equal to 96 units.

15. Al-Khowarizmi’s verbal description of the solution of \( x^2 + 10x = 39 \) differs from his geometrical description in one area. In his verbal description, he says to take one-half of the roots in the problem \( (\frac{1}{2} \cdot 10 = 5) \), to multiply the result by itself \( (5 \cdot 5 = 25) \), and to add this to 39 \( (25 + 39 = 64) \). In his geometrical description, he takes one-fourth of the ten roots \( (\frac{1}{4} \cdot 10 = 2) \) and fills in the four corners of a figure with small squares where each square has a side of length \( \frac{1}{2} \). This gives a total area of \( 39 + 4 \cdot (\frac{1}{2})^2 = 39 + 25 = 64 \). By answering the following questions, show that these seemingly different sets of instructions always have the same result.

a. Consider the more general problem \( x^2 + bx = c \) (rather than al-Khowarizmi’s specific example \( x^2 + 10x = 39 \)), and find the result of taking half of the \( b \) roots, squaring the result, and adding this to the given area \( c \).

b. In the more general problem \( x^2 + bx = c \), find the total area that results from filling in the four corners of a figure with small squares, each of length of one-fourth of the \( b \) roots.

16. Apply al-Khowarizmi’s method to the problem \( x^2 + bx = c \) and obtain a formula for the solution of all equations of this type.

\[ \text{HINT: Follow the modern version of the example in the text but with } b \text{ and } c \text{ in place of 10 and 39. You will need to use the } \sqrt{\text{ symbol in the next-to-last step.} \]

17. Apply al-Khowarizmi’s method to the problem \( ax^2 + bx = c \) and obtain a formula for the solution of all equations of this type.

\[ \text{HINT: Divide each side by } a \text{ and get } x^2 + (b/a)x = c/a. \text{ Then pattern your work after what you did in Exercise 16.} \]

\[ \text{Answer the following questions using complete sentences and your own words.} \]

**Concept and History Questions**

18. Al-Khowarizmi described his method of solving quadratic equations with an example; he did not generalize his method into a formula, as shown in Exercise 16. What characteristic of the mathematics of his time limited him to this form of a description? What change in mathematics lifted this limitation? To whom is that change due? Approximately how many years after al-Khowarizmi did this change occur?

19. What type of mathematics are the Greeks known for?

20. What are the conic sections?

21. Which Greek mathematician is known for his work on conic sections?

22. Why is our modern number system called the Arabic number system? What is important about this system?

23. The mathematics and science of the Greeks could well have been lost if it were not for a certain culture. Which culture saved this Greek knowledge, expanded it, and reintroduced it to Europe?

24. What were the subjects of al-Khowarizmi’s two books?

25. What is analytic geometry?

26. What did Apollonius contribute to analytic geometry?

27. What did Oresme contribute to analytic geometry?

28. What did Descartes contribute to analytic geometry?

29. What did Fermat contribute to analytic geometry?

30. What did Viète contribute to algebra?

31. What did Descartes contribute to algebra?

32. Describe Descartes’s philosophy.

33. What habit did Descartes maintain all his life?

34. Why did Descartes abandon his book on the workings of the physical world? What did he write about instead?

**Projects**

35. Write several paragraphs tracing the evolution of algebra as well as the evolution of analytic geometry, as described in this section. Use this text as well as other sources.
36. Write an essay summarizing the contributions of Apollonius, Oresme, Descartes, and Fermat to analytic geometry. Use this text as well as other sources.

37. Algebra underwent several qualitative changes before calculus was invented. Write an essay in which you compare and contrast the algebras of Apollonius, al-Khowarizmi, and Descartes. Use this text as well as other sources.

38. Pierre de Fermat once wrote in the margin of a book a brief mathematical statement involving the Pythagorean theorem, along with a claim that he had found a proof of that statement. He never actually wrote the proof. For 350 years, mathematicians were unsuccessful at proving what became known as Fermat’s Last Theorem, until a Princeton mathematician is thought to have succeeded in 1993. Write a research paper on Fermat’s Last Theorem and the reaction to its recent proof.

39. Write a research paper on a historical topic referred to in this section, or a related topic. Following is a partial list of topics from this section. (The topics in italics were not mentioned in the section.)

- al-Khowarizmi and his arithmetic and algebra
- Apollonius of Perga and his geometry
- Constantinople’s fall and its effect on Europe
- Pierre de Fermat and his analytic geometry
- Hindu and Arabic numerals
- Omar Khayyam, Persian poet and mathematician, author of The Rubaiyat
- The Moors
- The Islamic Empire
- Islamic mathematicians
- Nicole Oresme and his analytic geometry
- François Viète and his algebra

Some useful links for these web projects are listed on the text companion web site. Go to www.cengage.com/math/johnson to access the web site.

13.2 Four Problems

Objective

- Investigate four problems that motivated the invention of calculus

Four problems that were instrumental in the invention of calculus came together in the 1600s. The earliest of these problems is to find the area of any shape. This is one of the most ancient of all mathematical problems. The Ahmes Papyrus, written in Egypt in 1650 B.C., contains methods for computing some areas. The Greeks had a great deal of success with finding areas; they developed methods for finding areas of circles, rectangles, triangles, and other shapes. However, they were not able to find the area of any shape.

The second problem is to find the line tangent to a given curve at any point on that curve. This problem is almost as old as the area problem; it, too, was a subject of Greek geometry. The Greeks attempted to find methods of constructing a line tangent to a curve at a point using a compass and a straightedge. If you’ve had a class in geometry, you’ve constructed a line tangent to a circle. The construction involves drawing a radius and then using a compass to construct a line perpendicular to the radius at the point where the radius touches the circle. This line is tangent to the circle. See Figure 13.20.

What does it mean to say that a line is tangent to a curve? The line tangent to a circle is frequently described as a line that intersects the circle once. However, if the curve is not a circle, the tangent line might hit the curve more than once. The tangent line is the straight line to which the curve is most similar in orientation at the point of tangency. If you take that part of the curve that is near the point of tangency and magnify it, as shown in Figure 13.21 on the next page, the curve will look as if it’s almost a straight line. That line is the tangent line.
The third problem has to do with gravity. In the early 1600s, the Italian scientist Galileo Galilei was attempting to describe the motion of a falling object. In particular, he was attempting to find the distance traveled by a falling object, as well as its speed. He did pursue this matter in the Italian town of Pisa, but he did not actually drop balls from the Leaning Tower of Pisa, as the story claims.

The fourth problem is related to arms and war. In the 1500s and 1600s, Europe was the scene of almost constant warfare. Ever since the introduction of gunpowder in the thirteenth century, Europe was immersed in an arms race; scientists in different countries were trying to find how to use cannons better. Aiming a cannon so that the ball would hit the target was difficult. If one simply aimed the cannon directly at the target, the ball would fall short. If one pointed the cannon too high, the ball would either never reach the target or pass completely over it. Aiming a cannon so that the ball would go over a fortification wall and into the enemy’s stronghold was even more difficult. It became necessary to find the path (or trajectory) of a cannonball and determine how to control that trajectory. In fact, it became necessary to investigate the principles of motion itself. The first country that found how to do this would have a significant advantage over other countries. Some of Europe’s best minds—including Galileo, Leonardo da Vinci, Michelangelo, and Albrecht Dürer—were devoted to various aspects of the problem. Calculus was, to some extent, the result of this arms race.
These and other problems motivated the invention of calculus. In this section, we discuss the partial solutions to these problems that preceded calculus. In an attempt to keep things in their proper historical perspective, we discuss these partial solutions in the order of their solution, rather than in the order of their inception, given at the beginning of this section.

**Problem 1: Find the Distance Traveled by a Falling Object—Galileo and Oresme**

In about 1360, Parisian Scholar Nicole Oresme drew the graph of the speed of a moving object (Figure 13.22), which he thought of as the “intensity” of the object’s motion. In this graph, point \( A \) represents the time at which the motion started, and points \( B \) and \( D \) represent later times. The length of a horizontal line segment, such as \( AB \), represents the amount of time from the beginning of motion until time \( B \), and the length of a vertical line segment, such as \( BC \), represents the speed of the object at time \( B \). These lengths are increasing in a uniform way, so Oresme’s graph was of an object whose speed is increasing in a uniform way. Oresme somehow realized that the area of triangle \( ABC \) gives the distance traveled by the object from time \( A \) until time \( B \), although he never explained why this would be the case.

Oresme used this triangle to show that if an object’s speed is increasing uniformly, then that speed is proportional to the time in motion. His reasoning, illustrated in Figure 13.23, went like this:

\[
\triangle ABC \text{ is similar to } \triangle ADE \quad \text{the triangles have equal angles}
\]

\[
\frac{BC}{DE} = \frac{AB}{AD} \quad \text{ratios of corresponding sides are equal}
\]

\[
\frac{\text{speed at time } B}{\text{speed at time } D} = \frac{\text{time in motion until time } B}{\text{time in motion until time } D}
\]

The speed of the object is proportional to its time in motion.

If the algebra of Oresme’s time had been based on equations rather than proportions, he might have reasoned further and obtained a formula for speed. However, algebra didn’t reach that point in its development until the time of Descartes and Galileo, around 300 years later.
Oresme never applied his triangle to the motion of a falling object. However, Galileo was familiar with Oresme’s work and applied the triangle to this type of motion, reasoning that a falling object moves in such a way that its speed increases uniformly. This allowed him to conclude, like Oresme, that speed is proportional to time.

Galileo lived in a time when mathematics was rapidly advancing. Descartes’s *La géométrie*, Fermat’s (unpublished) work on conic sections, and Galileo’s work on the motion of a falling object were finished within a few years of each other. Unfortunately, Galileo did not use the algebra or analytic geometry of Descartes and Fermat, presumably because it was too new and he was not familiar with it; instead, he used the 2,000-year-old analytic geometry of Apollonius and an algebra that was verbal and based on proportions rather than equations. Thus, like Oresme, he was unable to express his conclusion that speed is proportional to time as a formula. Instead, he used language such as “the ratio of the velocities of an object falling from rest is the same as the ratio of the time intervals employed in traversing their distances.” This language is cumbersome and makes it difficult to apply the relationship between speed and time in motion discovered by Galileo. A formula of the form

\[
\text{speed} = \frac{\text{distance}}{\text{time}}
\]

would have made it easier to apply that relationship and might have led to new results and discoveries.

In addition to concluding that the speed of a falling object is proportional to its time in motion, Galileo was able to use Oresme’s triangle to make an observation about the distance traveled by a falling object. Oresme had realized that the area of his triangle gives the distance traveled by the object. Galileo observed that if a falling object covered a certain distance in the first interval of time measuring its fall, then it would cover three times that distance in the second interval of time and five times that distance in the third interval of time (and similarly with following intervals of time). Figures 13.24 and 13.25 illustrate his observations.

Galileo had established the following pattern in Figure 13.26 on the next page.

Notice that in each case, the total distance traveled is the square of the amount of time in motion. Galileo concluded that the distance traveled by a falling object was proportional to the square of its time in motion, or, as he phrased it, “the spaces described by a body falling from rest with a uniformly accelerated motion are as to each other as the squares of the time intervals employed in traversing these distances.”
In a roundabout way, Galileo verified his conclusions regarding falling bodies by timing a ball rolling down a ramp. He was unable to experiment effectively with falling objects, because things fell too fast to be timed with the water clock with which he was obliged to measure intervals. His water clock consisted of a vessel of water that was allowed to drain through a pipe while the ball was rolling. The drained water was weighed at the end of the trip. If a ball was timed during two different trips, he found that

\[
\frac{\text{distance from first trip}}{\text{distance from second trip}} = \frac{(\text{weight of water from first trip})^2}{(\text{weight of water from second trip})^2}
\]

Galileo was never able to actually determine how far a ball would fall (or roll down a ramp) for a given period of time because of his use of proportional, verbal algebra and the quality of his clock.

**Problem 2: Find the Trajectory of a Cannonball—Galileo and Apollonius**

Galileo thought of the motion of a cannonball as having two separate components—one due to the firing of the cannon, in the direction in which the cannon is pointed, and the other due to gravity, in a downward direction. He used his result regarding the distance traveled by a falling object and Apollonius’s description of a parabola to show that the path of the cannonball is actually a parabola.

Galileo reasoned that if a cannon was at the top of a cliff and pointed horizontally, then the cannonball’s motion could be resolved into a horizontal motion due to the firing of the cannon and a vertical motion due to gravity. Consider the cannonball’s position at times \( t_1 \) and \( t_2 \) and let the horizontal distances traveled be \( x_1 \) and \( x_2 \) and the vertical distances be \( y_1 \) and \( y_2 \), as shown in Figure 13.27.

The horizontal motion has constant speed, because there is nothing to slow it down (except a very small amount of air resistance, which Galileo chose to ignore). In twice the time, the cannonball would cover twice the horizontal distance. Thus, the horizontal distance traveled is proportional to the time in motion. In other words,

\[
\frac{x_1}{x_2} = \frac{t_1}{t_2}
\]

The vertical motion is due to gravity, so Galileo’s conclusion that “the spaces described by a body falling from rest with a uniformly accelerated motion are as
to each other as the squares of the time-intervals employed in traversing these distances” applies—or, using modern algebra,

\[
\frac{y_1}{y_2} = \frac{t_1^2}{t_2^2}
\]

Combining these two results yields

\[
\frac{y_1}{y_2} = \frac{t_1^2}{t_2^2} = \left(\frac{t_1}{t_2}\right)^2 = \left(\frac{x_1}{x_2}\right)^2
\]

substituting

\[
= \frac{x_1^2}{x_2^2}
\]

Note: Symbolic algebra is used for the convenience of the reader. Galileo used verbal algebra.

This told Galileo that the path of a cannonball must be parabolic, because Galileo knew that Apollonius had shown the following to be true for any parabola:

\[
\frac{\text{length of line segment 1}}{\text{length of line segment 2}} = \frac{\text{area of square 1}}{\text{area of square 2}}
\]

That is,

\[
\frac{y_1}{y_2} = \frac{x_1^2}{x_2^2}
\]

as shown in Figure 13.28.

Thus, Galileo used his own work on a falling body and Apollonius’s description of a parabola to show that a horizontally fired cannonball follows a parabolic path. He was also able to show that if a cannonball is fired at an angle, it will still follow a parabolic path. Naturally, this conclusion applies to all other projectiles, including balls that are thrown rather than fired.
Unfortunately, Galileo was hampered by his use of ancient algebra, so he was unable to translate this discovery into an equation for the path of a cannonball, and he was unable to determine how to aim a cannon so that the ball would hit the target.

Scientists were able to use Galileo's discovery that a cannonball follows a parabolic path to determine how to aim a cannon, as shown in this engraving. Notice the use of angles in determining the ball’s path; this will be discussed in Section 13.5.

Problem 3: Find the Line Tangent to a Given Curve—Barrow and Fermat

The development of analytic geometry by Fermat made the solution of this tangent problem much more likely, because mathematicians could apply algebra to what had been a strictly geometrical problem.

About 30 years after Fermat’s work became well known, an English mathematician named Isaac Barrow published a method of finding the slope of the tangent line in his book *Lectiones opticae et geometricae* (*Optical and Geometrical Lectures*). His method was very similar to an earlier one due to Fermat and might have been based on that method. Barrow was the first holder of the Lucas chair in mathematics at Cambridge, a prestigious professorship funded by member of parliament Henry Lucas. Isaac Newton was the second holder of the Lucas chair; Stephen Hawking, author of *A Brief History of Time*, held the Lucas chair until he retired in 2009.

Barrow’s method involved two points on the curve, \( P \) and \( Q \), with \( P \) the point at which he wanted to find the tangent line and \( Q \) a point very close to \( P \), as shown in Figure 13.29. The line \( PQ \) is called a secant line because it goes through two points on the curve, \( P \) and \( Q \). The secant line \( PQ \) is very close to the desired
If \( a \) and \( e \) are the lengths of the vertical and horizontal sides of triangle \( PQR \), respectively, then the slope of the secant line \( PQ \) is

\[
\frac{\text{rise}}{\text{run}} = \frac{a}{e}
\]

The tangent line has almost the same slope as the secant line \( PQ \). As we can see in Figure 13.29, both have a rise of \( a \), but the tangent’s run is not quite the same as that of the secant.

If we let point \( P \) be the ordered pair \((x, y)\), then point \( Q \) is \((x - e, y - a)\). (Note: Barrow did not use ordered pairs; they are used here for the convenience of the reader.) Because point \( Q \) is a point on the curve, \((x - e, y - a)\) would successfully substitute into the equation of the curve.

After making this substitution and simplifying the result, Barrow would “omit all terms containing a power of \( a \) or \( e \), or products of these (for these terms have no value).” If the result were solved for \( a/e \), the slope of the tangent line would be obtained. Why these terms should be omitted, or why their omission changes \( a/e \) from the slope of the secant to the slope of the tangent, is not clear, and Barrow did not attempt an explanation. However, the method does work.

**EXAMPLE 1 USING BARROW’S METHOD** Find the slope of the line tangent to the parabolic curve \( y = x^2 \) at the point \((2, 4)\), as shown in Figure 13.30.

\( P \) is at \((2, 4)\), so \( Q \) is at \((2 - e, 4 - a)\).

- the curve’s equation
- substituting \( 2 - e \) for \( x \) and \( 4 - a \) for \( y \)
- multiplying
- collecting like terms
- omitting all terms containing a power of \( a \) or \( e \)
- solving for \( a/e \)

The slope of the tangent line at \((2, 4)\) is 4. This result fits with the sketch of the tangent line; its slope could reasonably be 4.
Once the slope of the tangent line has been found, finding the equation of the tangent line is easy. To do so, use the point-slope formula from intermediate algebra.

**POINT-SLOPE FORMULA**

A line through \((x_1, y_1)\) with slope \(m\) has equation

\[ y - y_1 = m(x - x_1) \]
of torture. It is said that after his forced denial of the earth’s motion, he muttered, “e pur si muove” (“nevertheless it does move”). He was imprisoned in his home and died nine years later. As a result of Galileo’s experience, most scientists left Italy and went to Holland, where new scientific views were viewed with a more tolerant attitude.

A note in Galileo’s handwriting in the margin of his personal copy of the Dialogue states, “In the matter of introducing novelties. And who can doubt that it will lead to the worst disorders when minds created free by God are compelled to submit slavishly to an outside will? When we are told to deny our senses and subject them to the whim of others? When people devoid of whatsoever competence are made judges over experts and are granted authority to treat them as they please? These are the novelties which are apt to bring about the ruin of common-wealths and the subversion of the state.”

It is interesting to note that in 1992, the Roman Catholic Church admitted that it had been wrong in having condemned Galileo and opposed the Copernican system.

Galileo is called the father of modern science because of his emphasis on experimentation and his interest in determining how things work rather than what causes them to work as they do. He insisted that a theory was unsound, no matter how logical it seemed, if observation did not support it. He invented the pendulum clock and the thermometer; he constructed one of the first compound microscopes; and he greatly improved the design of telescopes. His last work, Dialogues Concerning Two New Sciences, contains many of his contributions to science, including those concerning motion and the strength of materials. In that work, he showed that a projectile follows a parabolic path and came to conclusions that foreshadowed Newton’s laws of motion. He also held that motion does not require a force to maintain it (as Aristotle claimed), but rather the “creation or destruction of motion” (that is, acceleration or deceleration) requires the application of force. Thus, he was the first to appreciate the importance of the concept of acceleration.

---

EXAMPLE 2

FINDING THE EQUATION OF A TANGENT LINE

Find the equation of the line tangent to the curve $y = x^2$ at the point (2, 4).

We are given $(x_1, y_1) = (2, 4)$. In Example 1, we found that the slope of this line is $m = 4$.

$$y - y_1 = m(x - x_1) \quad \text{point-slope Formula}$$

$$y - 4 = 4(x - 2) \quad \text{substituting}$$

$$y = 4x - 4 \quad \text{distributing}$$

---

Problem 4: Find the Area of Any Shape—Kepler

Some of the ancient Greek geometers developed clever ways of finding areas and volumes. Eudoxus and Archimedes found areas of curved shapes by filling in the region with a sequence of successively smaller triangles and finding the sum of the areas of those triangles. See Figure 13.31.
Around 1450, European mathematicians learned of these methods when a manuscript was found at Constantinople. In the early 1600s, the astronomer and mathematician Johann Kepler adapted Archimedes’ methods to find areas and volumes. While his main use for these techniques was in his studies of the planets’ orbits, he also used them in a paper on the volumes of wine barrels. We will illustrate Kepler’s method by showing that the area of a circle is \( \pi r^2 \).

Kepler envisioned the circumference of a circle as being composed of an infinite number of very short straight lines, as illustrated in Figure 13.32. If each of these short lines is taken as the base of a triangle with vertex at the center of the circle and if the radius of the circle is \( r \), then the area of the circle would be as follows:

\[
\text{(area of first triangle)} + \text{(area of second triangle)} + \cdots \\
= \left( \frac{1}{2} \cdot \text{base} \cdot \text{height} \right) + \left( \frac{1}{2} \cdot \text{base} \cdot \text{height} \right) + \cdots \\
= \frac{1}{2} \cdot \text{height} \left( \text{base} + \text{base} + \cdots \right) \quad \text{because the heights are all equal} \\
= \frac{1}{2} r (\text{base} + \text{base} + \cdots) \quad \text{because the height goes from the circumference to the center} \\
= \frac{1}{2} r (\text{circumference of the circle}) \\
= \frac{1}{2} r (2\pi r) \\
= \pi r^2
\]

Twenty years after Kepler published his method of computing areas, a student of Galileo named Bonaventura Cavalieri published a popular book that showed how to compute areas by thinking of a region as being composed of an infinite number of rectangles rather than an infinite number of triangles. See Figure 13.33. Galileo himself had used a similar approach in analyzing Oresme’s triangle.
13.2 Exercises

Reread the discussion of Oresme's triangle at the beginning of this section. Use that triangle to answer the questions in Exercises 1–4 about an object whose speed increases uniformly from the moment motion begins.

1. After 10 seconds, an object is moving at a rate of 4 feet per second.
   a. Find the distance it has traveled in those 10 seconds.
      \(HINT: \) According to Oresme, distance = area of triangle.
   b. Find its speed after 15 seconds.
      \(HINT: \) Use similar triangles.
   c. Find the distance it has traveled in 15 seconds.
   d. Find its average speed during its first 15 seconds of travel.
      \(HINT: \) Average speed = distance/time

2. After 15 minutes, an object is moving at a rate of 6 feet per minute.
   a. Find the distance it has traveled in those 15 minutes.
   b. Find its speed after 10 minutes.
   c. Find the distance it has traveled in 10 minutes.
   d. Find its average speed during its first 10 minutes of travel.

3. After 5 minutes, an object has traveled 100 feet.
   a. Find its speed at the moment it has traveled 100 feet.
   b. Find its speed after 7 minutes of motion.
   c. Find the distance it has traveled in 7 minutes.
   d. Find its average speed for those 7 minutes.

4. After 3 seconds, an object has traveled 2 feet.
   a. Find its speed at the moment it has traveled 2 feet.
   b. Find its speed after 10 seconds of motion.
   c. Find the distance it has traveled in 10 seconds.
   d. Find its average speed for those 10 seconds.

5. Galileo used Oresme’s triangle (refer back to Figure 13.24) to observe the following pattern in the distance traveled by a falling object. Expand the illustration in Figure 13.24 and the chart in Figure 13.34 to include the fourth through sixth intervals of time.

<table>
<thead>
<tr>
<th>Interval of Time</th>
<th>Distance Traveled During That Interval</th>
<th>Total Distance Traveled</th>
</tr>
</thead>
<tbody>
<tr>
<td>first</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>second</td>
<td>3</td>
<td>1 + 3 = 4</td>
</tr>
<tr>
<td>third</td>
<td>5</td>
<td>1 + 3 + 5 = 9</td>
</tr>
</tbody>
</table>

Figure 13.34 A chart for Exercise 5.

In Exercises 6–9, use the following information: Using a water clock to time a ball rolling down a ramp, Galileo found that if a ball was timed during two different trips, the following relationship held:

\[
\frac{\text{distance from first trip}}{\text{distance from second trip}} = \left(\frac{\text{weight of water from first trip}}{\text{weight of water from second trip}}\right)^2
\]

6. If a ball would roll 25 feet while 8 ounces of water flowed out of a water clock, how far would it roll while 4 ounces of water flowed from the clock?

7. If a ball would roll 32 feet while 6 ounces of water flowed out of a water clock, how far would it roll while 12 ounces of water flowed from the clock?

8. If a ball would roll 25 feet while 8 ounces of water flowed out of a water clock, how much water would flow out of the clock while the ball rolled 50 feet?

9. If a ball would roll 40 feet while 7 ounces of water flowed out of a water clock, how much water would flow out of the clock while the ball rolled 10 feet?

In Exercises 10–13, use the following information: Galileo found that “the spaces described by a body falling from rest with a uniformly accelerated motion are as to each other as the squares of the time-intervals employed in traversing these distances.” In other words,

\[
\frac{\text{distance from first trip}}{\text{distance from second trip}} = \left(\frac{\text{time of first trip}}{\text{time of second trip}}\right)^2
\]

Galileo was unable to time a falling object, owing to the inaccuracy of his water clock. Suppose Galileo had a stopwatch and used it to find that an object falls 1,600 feet in 10 seconds. How far would that object fall in the times given?

10. 3 seconds

11. 12 seconds

12. 20 seconds

13. 1 second

14. A cannon is 140 feet away from a stone wall that is 25 feet tall. If a cannonball is fired so that it barely passes over the wall and begins to descend immediately after passing over the wall, at approximately what distance beyond the wall will the cannonball hit the ground?

\(HINT: \) Draw a picture and use Galileo’s conclusion regarding the path of a projectile.

15. A woman throws a ball over an 8-foot fence that is 30 feet away. The ball barely passes over the fence and begins to descend immediately after passing over the fence. Approximately how far from the fence should her friend stand in order to catch the ball?

\(HINT: \) Draw a picture and use Galileo’s conclusion regarding the path of a projectile.
In Exercises 16–25, (a) use Barrow’s method to calculate the slope of the line tangent to the given curve at the given point. Also (b) find the equation of the tangent line, and (c) include an accurate sketch of the curve and the tangent line.

16. \( y = 2x^2 \) at \((1, 2)\)
17. \( y = 2x^2 \) at \((3, 18)\)
18. \( y = x^3 \) at \((2, 8)\)
19. \( y = x^3 \) at \((3, 27)\)
20. \( y = x^2 - 2x + 1 \) at \((1, 0)\)
21. \( y = x^2 + 2x + 2 \) at \((3, 4)\)
22. \( y = x^2 + 4x + 7 \) at \(x = -1\)
23. \( y = x^3 + 6x + 11 \) at \(x = -1\)
24. \( y = x^2 + 6x + 11 \) at \(x = 0\)
25. \( y = x^3 + 4x + 6 \) at \(x = 0\)

26. Oresme thought that the area of his triangle equaled the distance the object travels, but he never explained why this was so. This exercise outlines a possible explanation of the relationship.

a. Why would Oresme use a rectangle instead of a triangle to describe the motion of an object whose speed is constant?

b. If an object moves with constant speed, then the distance traveled = speed \( \times \) time

Why is the area of Oresme’s rectangle equal to the distance the object travels?

c. Oresme described the motion of an object whose speed is increasing uniformly with a triangle. Such a triangle has the same area as a rectangle whose height is half that of the triangle, as shown in Figure 13.35. What does this imply about the average speed of an object whose speed is increasing uniformly?

d. Why is the area of Oresme’s triangle equal to the distance the object travels?

27. Kepler showed that the area of a circle is \( \pi r^2 \). Use similar reasoning to show that the volume of a sphere is \( \frac{4}{3} \pi r^3 \). You will need to envision the sphere as being composed of an infinite number of very thin cones. See Figure 13.36. You will need the following two formulas: The volume of a cone is \( \frac{1}{3} \pi r^2 h \), and the surface area of a sphere is \( 4 \pi r^2 \).

• CONCEPT QUESTIONS

28. Why did Galileo conclude that the distance traveled by a falling object is proportional to the square of its time in motion, even though the chart in Figure 13.26 shows that the total distance traveled is equal to the square of the amount of time in motion?

29. Explain why “in twice the time, the cannonball would cover twice the horizontal distance” implies that \( \frac{x_1}{x_2} = \frac{t_1}{t_2} \) as discussed on page 13-26.

30. In using Kepler’s method to find the area of a circle, “circumference of the circle” was substituted for “base + base + . . . .” Explain why these two quantities are equal.

31. How did Galileo conclude that the distance traveled by a falling object is proportional to the square of its time in motion?

• HISTORY QUESTIONS

32. What role did Oresme play in the “distance traveled by a falling object” problem? What role did Galileo play?

33. Why was Galileo unable to verify experimentally his conclusion that the distance traveled by a falling object is proportional to the square of its time in motion? How did he partially verify it?

34. How did Galileo conclude that a cannonball fired horizontally from the top of a cliff would have a parabolic trajectory?

35. Why was Galileo unable to actually find the equation of the trajectory of a cannonball?

36. What was Galileo studying at the university when he changed his major to science and mathematics? What two events prompted that change?
37. What reaction did his colleagues have to Galileo’s experiments concerning motion? How did he respond?
38. What reaction did his colleagues have to Galileo’s studies of the heavens with his telescope? How did he respond?
39. What reaction did the Church have to Galileo’s studies of the heavens with his telescope? How did he respond?
40. Why is Galileo called the father of modern science?
41. Why was Kepler interested in computing areas and volumes?

**WEB PROJECTS**

42. The four problems discussed in this section were not the only problems that motivated the invention of calculus. A fifth problem involved finding the orbits of the planets. Write a research paper on the history of this problem. Start your research by reading about Copernicus and Kepler.
43. Write a research paper on a historical topic referred to in this section, or a related topic. Following is a partial list of topics from this section. (The topics in italics were not mentioned in the section.)

- Archimedes, Greek inventor and mathematician
- Isaac Barrow, English scientist and mathematician
- Bonaventura Cavalieri, Italian mathematician
- Nicholas Copernicus and the Copernican system
- Eudoxus, Greek astronomer, mathematician, and physicist
- the impact of gunpowder on European warfare
- the Inquisition and the punishment of heretics
- Johann Kepler, German astronomer and mathematician
- the Roman Catholic church, Galileo, and the Copernican system
- Evangelista Torricelli, Italian physicist and mathematician
- John Wallis, English mathematician
- the water clock and other ancient timekeeping devices

Some useful links for these web projects are listed on the companion text web site. Go to **www.cengage.com/math/johnson** to access the web site.

13.3 Newton and Tangent Lines

**OBJECTIVES**

- Understand Newton’s method of finding the slope of a tangent line
- Understand Cauchy’s reformulation of Newton’s method

Mathematics had made tremendous advances by the late 1600s. Algebra had become much more powerful due to Viète, Descartes, and many others. In particular, algebra was based more on equations and less on ratios and proportions, and it had shed its verbal style for a simpler, easier-to-use symbolic notation. Analytic geometry had been developed by Fermat and Descartes. Galileo had accomplished much in his study of the speed and motion of both a falling object and a cannonball. Several mathematicians—including Descartes, Fermat, and Barrow—had developed methods for finding the slope of a tangent line. Isaac Newton, a student of Barrow at Cambridge University, built on this foundation and created calculus.

Newton improved Barrow’s method of finding the slope of a tangent line and in doing so found the key that unlocked all four of the problems. In creating a more logical method of finding the slope of a tangent line, Newton also created a method of finding the speed of any moving object (including a falling object and a cannonball), as well as a method of finding the area of any shape. We will explore Newton’s ideas about tangent lines in this section and his ideas on speed and area in the following sections.

Newton was not only a student of Barrow but also a close friend. He helped Barrow to prepare *Optical and Geometrical Lectures* for publication. In the preface of the book, Barrow acknowledges indebtedness to Newton for some of the material. It should not be surprising, then, that Newton’s method of finding the slope of the tangent line is very similar to Barrow’s. The method given here is a later version
CHAPTER 13 The Concepts and History of Calculus

Newton’s method involves Barrow’s two points \( P \) and \( Q \), except that \( Q \) is thought of as being to the right of \( P \) by a small amount \( o \), as shown in Figure 13.37. (Barrow’s point \( Q \) was to the left of \( P \) by an amount \( e \).) If point \( P \) has an \( x \)-coordinate of \( x \), then point \( Q \) would have an \( x \)-coordinate of \( x + o \) (\( o \) is the letter “oh,” not the number “zero”). Newton called \( o \) an **evanescent increment**—that is, an imperceptibly small increase in \( x \). Newton’s method consisted of computing the slope of the secant line \( PQ \) and then replacing the evanescent increment \( o \) with the number zero. He called the changes in \( x \) and \( y \) **fluxions** and the ratio of their changes (that is, the slope) the **ultimate ratio of fluxions**.

**EXAMPLE 1 USING NEWTON’S METHOD** Use Newton’s method to find the slope of the line tangent to the curve \( y = x^2 \) at the point (2, 4), as shown in Figure 13.38.

**SOLUTION**

\( P \) is at \( x = 2 \), with a \( y \)-coordinate of \( x^2 = 2^2 = 4 \).

\( Q \) is at \( x = 2 + o \) with a \( y \)-coordinate of \( (2 + o)^2 \).

Slope of \( PQ \):

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(2 + o)^2 - 4}{(2 + o) - 2} = \frac{4 + 4 \cdot o + o^2 - 4}{o} = \frac{4 \cdot o + o^2}{o} = \frac{o(4 + o)}{o} = 4 + o
\]

Factoring and canceling since \( o \) is imperceptibly small.

The slope of the tangent line at (2, 4) is 4.

---

**NEWTON’S METHOD OF FINDING THE SLOPE OF THE TANGENT LINE**

1. **Find** \( Q \). Get \( Q \)'s \( x \)-coordinate by adding \( o \) to \( P \)'s \( x \)-coordinate. Use the equation of the curve to get \( Q \)'s \( y \)-coordinate.
2. **Find the slope of** \( PQ \). Simplify the result by factoring and canceling.
3. **To find the slope of the tangent line**, replace \( o \) with 0.

Compare this method with Barrow’s method as illustrated in Example 1 of Section 13.2. Newton’s method is very similar to Barrow’s, except that Newton filled some of the logical holes left by Barrow. In particular, Newton’s method included nothing like Barrow’s unexplained instructions to “omit all terms containing a power of \( a \) or \( e \), or products of these (for these terms have no value).” Instead, Newton’s procedure involved a normal factoring and canceling followed by equating the evanescent increment \( o \) with zero.
Unfortunately, this attempted explanation creates as many problems as it solves. If the evanescent increment \( o \) is so small that it can be replaced with the number zero in the last step of Example 1, then why not replace \( o \) with zero a few steps earlier and obtain

\[
\frac{4 \cdot o + o^2}{o} = \frac{4 \cdot 0 + 0^2}{0} = \frac{0}{0}
\]

which is undefined? Furthermore, if \( P \) is at \( x = 2 \) and \( Q \) is at \( x = 2 + o \) and if \( o \) can be replaced with zero, then aren’t \( P \) and \( Q \) the same point? How can one find the slope of a line between two points when the points are the same?

Gottfried Wilhelm Leibniz, working independently in Europe, had also developed a calculus based on evanescent increments (which he called infinitesimals), and his theory had the same difficulties. (It is interesting to note that Leibniz had previously obtained a copy of Barrow’s *Optical and Geometrical Lectures*.) Both Leibniz and Newton repeatedly changed their explanations of the concept of the infinitely small and repeatedly failed to offer an explanation that had no logical holes. It is probably most accurate to say that each man understood the concept but not well enough to explain it in a logically valid manner.

Newton accepted his theory because it fit with his intuition and because it worked. His contemporaries, however, were far from unanimous in accepting it. Some of the strongest criticisms came from religious leaders, perhaps because mathematics was the principal avenue by which science was invading the physical universe, which had been the exclusive domain of the Church.

The most famous criticism came from Bishop George Berkeley. In his essay “The Analyst: A Discourse Addressed to an Infidel Mathematician,” Berkeley states, “And what are these same evanescent increments? They are neither finite quantities, or quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities? Certainly . . . he who can digest a second or third fluxion . . . need not, methinks, be squeamish about any point in Divinity.”

## Cauchy’s Reformulation of Newton’s Method

A reformulation of Newton’s method that avoids the difficulties caused by evanescent increments or infinitesimals was developed about 150 years later by the French mathematician Augustin Louis Cauchy. This formulation uses the symbol \( \Delta x \) in place of \( o \), where \( \Delta x \) means “change in \( x \)” and is not meant to be infinitely small. With this notation, point \( Q \) would have an \( x \)-coordinate of \( x + \Delta x \) rather than \( x + o \). The smaller the value of \( \Delta x \), the closer the line \( PQ \) is to the desired tangent line, and the closer the slope of the line \( PQ \) is to the slope of the tangent line (Figure 13.39). However, we cannot let \( \Delta x \) equal zero, or we will encounter all the
ISAAC NEWTON, 1642–1727

Where the statue stood
Of Newton, with his prism and silent face,
The marble index of a mind forever
Voyaging through strange seas of thought alone.

WILLIAM WORDSWORTH

Isaac Newton, considered by many to be a creative genius and the greatest scientist who ever lived, was born in 1642, the year of Galileo's death, to a family of farmers in Woolsthorpe, England. His father had died before his birth, and his mother soon remarried and moved to a neighboring town, leaving young Isaac in the care of his grandmother. Although he was not an especially good student in his youth, he did show a real mechanical aptitude. He constructed perfectly functioning mechanical toys, including a water-powered wooden clock and a miniature wheat mill with a fat mouse acting as both power source and product consumer. Perhaps owing to this mechanical skill and the fact that he was a better student than farmer, it was decided to send him to Cambridge University when he was nineteen. Before leaving Woolsthorpe, he became engaged to a local girl. However, even though he remembered her affectionately all his life, he withdrew from her and never married.

At Cambridge Newton became interested in mathematics. He read Euclid's Elements, Descartes's La géométrie, and Viète's work on algebra. He received his bachelor's degree in 1664, after an undistinguished four years of study. Later that same year, the university was closed for two years because of the Great Plague (a bubonic plague), and Newton returned to the family farm in Woolsthorpe to avoid exposure to the plague. In those two years, he invented calculus (which he called "the method of fluxions"), proved experimentally that white light is composed of all colors, and discovered the law of universal gravitation, in which he provided a single explanation of both falling bodies on earth and the motion of planets and comets. In his later years, Newton talked about discovering the law of universal gravitation while sitting under an apple tree at the farm. He said that he was wondering what force could hold the moon in its path when the fall of an apple made him think that it might be the same gravitational force, diminished by distance, that acted on the apple. Unfortunately, he chose not to publish any of his work from this period for many years.

Newton returned to Cambridge when the danger from the plague was over and studied optics and mathematics under Isaac Barrow, holder of Cambridge's Lucas chair in mathematics. Newton communicated some of his discoveries to Barrow, including part of his method of fluxions, and helped him prepare his book Optical Lectures for publication. After several years, Barrow retired and recommended Newton as his successor to the Lucas chair. Newton sent a paper on optics to the Royal Society (a professional scientific organization), where some found his ideas interesting and others attacked them vehemently. Newton disliked the ensuing argument so much that he vowed never to publish again.

Newton became quite neurotic, paralyzed by fear of exposing his discoveries and beliefs to the world. A colleague said that he was "of the most fearful, cautious, and suspicious temper that I ever knew." Early in life, Newton abandoned the orthodox Christian
Newton had used calculus extensively in the development of this work, but he wrote it without using any calculus because he wished to keep his calculus a secret.

Newton wrote three major papers on his calculus between 1669 and 1676. He did not publish these works but merely circulated them among his friends. In 1693, Newton learned that calculus was becoming well known on the continent and that it was being attributed to Leibniz. At the insistence of his friends, Newton slowly began to publish his three papers, which appeared between 1711 and 1736.

By 1712, the question of who really invented calculus and whether either mathematician had plagiarized the other had become matters of consuming public interest, and Newton, Leibniz, and their backers began to attack each other. Leibniz and his followers went on to perfect and expand Leibniz’s calculus. England disregarded this and all other work from the continent out of loyalty to Newton and, as a result, failed to progress mathematically for 100 years.

In his later years, Newton retired from mathematics and physics and turned his attention to alchemy, chemistry, theology, and history. He became Cambridge’s representative in Parliament and then Master of the Mint. Occasionally, though, Newton returned to mathematics. Johann Bernoulli (a follower of Leibniz) once posed a challenging problem to all the mathematicians of Europe. Newton heard of the problem about six months after it had been posed, during which time no one had solved it. Newton solved the problem after dinner that same day and sent the solution to Bernoulli anonymously. Despite the anonymity, Bernoulli knew its source, saying, “I recognize the lion by his claw.”

Newton was knighted for his work at the mint and for his scientific discoveries. Near the end of his life, he appraised his efforts, saying, “I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.” He was buried with great ceremony at Westminster Abbey.
logical holes of Newton’s method. Rather, the slope of the tangent line is obtained by calculating the slope of the secant line and observing the effect of allowing $\Delta x$ to approach zero without reaching zero.

**EXAMPLE 2**

**USING CAUCHY’S METHOD** Use Cauchy’s method to find the slope of the line tangent to the curve $y = x^2$ at the point $x = 2$, as shown in Figure 13.40.

**SOLUTION**

$P$ is at $x = 2$, with a $y$-coordinate of $2^2 = 4$. $Q$ is at $x = 2 + \Delta x$, with a $y$-coordinate of $(2 + \Delta x)^2$:

$$\text{Slope of secant } PQ = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(2 + \Delta x)^2 - 4}{(2 + \Delta x) - 2} = \frac{4 + 4 \cdot \Delta x + \Delta x^2 - 4}{\Delta x} = \frac{4 \cdot \Delta x + \Delta x^2}{\Delta x} = \frac{\Delta x(4 + \Delta x)}{\Delta x} \quad \text{factoring}$$

Next, observe the effect of allowing $\Delta x$ to approach zero without reaching zero. Reading from top to bottom in Figure 13.41, we can see that

<table>
<thead>
<tr>
<th>If $P$ is at $x =$</th>
<th>and $Q$ is at $x =$</th>
<th>then $\Delta x =$</th>
<th>and the slope of $PQ$ is $4 + \Delta x =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>$3 - 2 = 1$</td>
<td>$4 + \Delta x = 4 + 1 = 5$</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>$2.5 - 2 = 0.5$</td>
<td>$4 + \Delta x = 4 + 0.5 = 4.5$</td>
</tr>
<tr>
<td>2</td>
<td>2.1</td>
<td>$2.1 - 2 = 0.1$</td>
<td>$4 + \Delta x = 4 + 0.1 = 4.1$</td>
</tr>
</tbody>
</table>

**FIGURE 13.41** Allowing $\Delta x$ to approach zero without reaching zero.
Q’s x-coordinate gets closer to 2, without reaching 2.
• Δx gets closer to 0, without reaching 0.
• The slope of PQ gets closer to 4, without reaching 4.

This is illustrated in Figure 13.42.

The closer Δx is to zero, the closer Q is to P, and the closer the slope of the secant line PQ is to the slope of the tangent line. If Δx is allowed to approach zero without reaching zero, then 4 + Δx will approach 4 without reaching 4. The slope of the tangent line at (2, 4) is 4.

You should review Newton’s original method in Example 1 and compare it to Cauchy’s version in Example 2. Notice that in finding the slope of line PQ in Example 2, the Δx’s can be canceled, because Δx does not equal zero. In finding the slope of the tangent line in Example 2, we are still not allowing Δx to equal zero; we are observing the result of allowing Δx to approach zero. Realize that this is a very sophisticated concept that is difficult to grasp. Newton and Leibniz each had a great deal of difficulty explaining it, and it was 150 years before mathematicians were able to make it logically correct. So if you do not achieve understanding overnight, you’re not alone!

13.3 Exercises

In Exercises 1–10, do the following.

a. Use Newton’s method to find the slope of the line tangent to the given curve at the given point.
b. Use Cauchy’s method to find the slope of the line tangent to the given curve at the given point.
c. Find the equation of the tangent line.

1. \( y = 3x^2 \) at (4, 48)  
2. \( y = 4x^2 \) at (1, 4)  
3. \( y = x^2 + 2 \) at \( x = 3 \)  
4. \( y = x^2 + 5 \) at \( x = 4 \)  
5. \( y = 2x^2 - 5x + 1 \) at \( x = 7 \)  
6. \( y = 3x^2 + 4x - 3 \) at \( x = 5 \)  
7. \( y = x^3 \) at \( x = 2 \)  
8. \( y = x^3 \) at \( x = 5 \)  
9. \( y = x^3 - x^2 \) at \( x = 1 \)  
10. \( y = x^3 - x - 1 \) at \( x = 3 \)
13.4 Newton on Falling Objects and the Derivative

**OBJECTIVES**

- Understand the difference between instantaneous speed and average speed
- Learn of Newton’s discoveries regarding the speed of a falling object and the distance it travels
- Explore Newton’s discovery of the derivative

**Average Speed and Instantaneous Speed**

Before further investigating Newton’s solutions to the four problems (area of any shape, tangent of a curve, speed of a falling object, and path of a cannonball), we need to discuss instantaneous speed. Suppose you’re driving in your car and after
one hour have gone 45 miles; then, after three hours, you have gone 155 miles. Your **average speed** in the last two hours of driving would be as follows:

\[
\text{Average speed} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta d}{\Delta t}
\]

\[
= \frac{155 \text{ mi} - 45 \text{ mi}}{3 \text{ hr} - 1 \text{ hr}}
\]

\[
= \frac{110 \text{ mi}}{2 \text{ hr}}
\]

\[
= 55 \text{ mi/hr}
\]

This does not mean that your speedometer would show a speed of 55 miles/hour at every moment during those two hours of driving; the speedometer shows not average speed but instantaneous speed. **Instantaneous speed** means the speed during an instant, rather than during two hours. Clearly, instantaneous speed is more useful to the driver than average speed; few cars have average speedometers, but all have instantaneous speedometers.

**Newton on Gravity**

Newton’s method of fluxions and evanescent increments (which involved using \(o\) and then replacing \(o\) with 0) was motivated as much by the question of the speed of a falling object as by the tangent line problem. In fact, his word *fluxion* has the same Latin root as the word *flux*, which means continual change.

Recall that Galileo had determined that the distance traveled by a falling object is proportional to the square of the time in motion. After Galileo’s death, others were able to use the new algebra to phrase these facts as equations rather than proportions.

\[
\frac{d_1}{d_2} = \frac{t_1^2}{t_2^2}
\]

\[
d_1t_2^2 = d_2t_1^2 \quad \text{cross-multiplying}
\]

\[
\frac{d_1}{t_1} = \frac{d_2}{t_2} \quad \text{dividing by } t_1^2 t_2^2
\]

This means that dividing \(d\) by \(t^2\) will always result in the same number, regardless of how long or how far an object has been falling.

The pendulum clock, which was much more accurate than Galileo’s water clock, was invented by Galileo and perfected by the Dutch scientist Christiaan Huygens. It allowed scientists to time falling objects accurately enough to determine that dividing \(d\) by \(t^2\) will always result in the number 16 (if \(d\) is measured in feet, \(t\) is measured in seconds, and the object is shaped so that it meets little wind resistance). Thus,

\[
\frac{d}{t^2} = 16
\]

\[
d = 16t^2 \quad \text{multiplying by } t^2
\]

**EXAMPLE 1**

**Finding a Falling Object’s Distance Traveled and Average Speed**

a. Find the distance a falling object has traveled after 2 seconds.
b. Find the distance a falling object has traveled after 3 seconds.
c. Find the falling object’s average speed during the time interval from 2 to 3 seconds after the motion starts.
a. \( d = 16t^2 \)
   \( = 16 \cdot 2^2 \)
   \( = 64 \text{ ft} \)

b. \( d = 16t^2 \)
   \( = 16 \cdot 3^2 \)
   \( = 144 \text{ ft} \)

c. Averages speed = \( \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta d}{\Delta t} \)

\[ = \frac{144 \text{ ft} - 64 \text{ ft}}{3 \text{ sec} - 2 \text{ sec}} \]
\[ = 80 \text{ ft/sec} \]

### Example 2

**FROM AVERAGE SPEED TO INSTANTANEOUS SPEED**

a. Use the formula \( d = 16t^2 \) to find a formula for the average speed of a falling object from an earlier time \( t \) to a later time \( t + \Delta t \).

b. Use the result of part (a) to find the average speed of a falling object from an earlier time \( t = 2 \) to a later time \( t + \Delta t = 3 \).

c. Use the result of part (a) to find the instantaneous speed of a falling object at time \( t \).

**SOLUTION**

a. The distance at the earlier time \( t \) is \( 16t^2 \), and the distance at the later time \( t + \Delta t \) is \( 16(t + \Delta t)^2 \).

Average speed = \( \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta d}{\Delta t} \)

\[ = \frac{\text{distance at time} (t + \Delta t) - \text{distance at time} t}{(t + \Delta t) - t} \]
\[ = \frac{16(t + \Delta t)^2 - 16t^2}{\Delta t} \]
\[ = \frac{16(t^2 + 2t\Delta t + \Delta t^2) - 16t^2}{\Delta t} \]
\[ = \frac{32t\Delta t + 16\Delta t^2}{\Delta t} \]
\[ = \frac{\Delta t(32t + 16\Delta t)}{\Delta t} \]
\[ = 32t + 16\Delta t \]

b. The earlier time is \( t = 2 \), the later time is \( t + \Delta t = 3 \), and \( \Delta t = 3 - 2 = 1 \). Using the new formula, we have

average speed = \( 32t + 16\Delta t \)
\[ = 32 \cdot 2 + 16 \cdot 1 \]
\[ = 80 \text{ ft/sec} \]

This answer agrees with that obtained in Example 1.

c. The speed in which Newton and Galileo were most interested was not average speed but rather instantaneous speed. Instantaneous speed can be thought of as the average speed during an instant, or, as Newton would have said, during an “evanescent increment” of time. Thus,
Instantaneous speed = average speed during an instant
= $32t + 16\Delta t$
= $32t + 16 \cdot 0$
= $32t$

To use Cauchy’s language, if $\Delta t$ is allowed to approach zero without reaching zero, then average speed approaches instantaneous speed, and $32t + 16\Delta t$ approaches $32t$.

Thus, Newton found that the instantaneous speed of a falling object is given by $s = 32t$. Newton called this the “fluxion of gravity.”

**FALLING OBJECT FORMULAS**

If an object falls for $t$ seconds, then its distance $d$ (in feet) and its speed $s$ (in feet per second) are given by the equations

\[
\begin{align*}
    d &= 16t^2 \\
    s &= 32t
\end{align*}
\]

Technically, these falling object formulas have limited accuracy. At a certain speed (that depends on the shape of the object), air resistance will become so great that a falling object will stop accelerating, and the speed will no longer be given by $s = 32t$. We will ignore this, as did Galileo and Newton.

**EXAMPLE 3**

**FINDING INSTANTANEOUS SPEED AND AVERAGE SPEED**

a. Find the instantaneous speed of a falling object after 3 seconds.

b. Find the average speed of a falling object during its first 3 seconds of descent.

**SOLUTION**

a. The instantaneous speed is

\[
\begin{align*}
    s &= 32t \\
    &= 32 \cdot 3 \\
    &= 96 \text{ ft/sec}
\end{align*}
\]

b. The average speed is

\[
\begin{align*}
    \text{Average speed} &= \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta d}{\Delta t} \\
    &= \frac{\text{distance at time 3} - \text{distance at time 0}}{3 - 0} \\
    &= \frac{16 \cdot 3^2 - 16 \cdot 0^2}{3} \\
    &= \frac{144}{3} \\
    &= 48 \text{ ft/sec}
\end{align*}
\]

Alternatively, the earlier time is $t = 0$, the later time is $t + \Delta t = 3$, and $\Delta t = 3 - 0 = 3$; thus,

\[
\begin{align*}
    \text{Average speed} &= 32t + 16\Delta t \\
    &= 32 \cdot 0 + 16 \cdot 3 \\
    &= 48 \text{ ft/sec}
\end{align*}
\]

This example is illustrated in Figure 13.43.
The Derivative

The procedure used in Examples 2 and 3 to find the instantaneous speed of a falling object is extremely similar to that used in Section 13.3 to find the slope of a tangent line. To see this important similarity, compare the two procedures’ steps, as shown in Figure 13.44.

<table>
<thead>
<tr>
<th>To Find the Instantaneous Speed of a Falling Object (as discussed in Example 2)</th>
<th>To Find the Slope of the Tangent Line (as discussed in Section 13.3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Find the later time and the distance at that later time. The later time is ( t + \Delta t ). Use the equation ( d = 16t^2 ) to get the distance at that later time.</td>
<td>1. Find ( Q )'s ( x )-coordinate and ( y )-coordinate. The ( x )-coordinate is ( x + \Delta x ). Use the equation of the curve to get ( Q )'s ( y )-coordinate.</td>
</tr>
<tr>
<td>2. Find the average speed ( \Delta d/\Delta t ).</td>
<td>2. Find the slope ( \Delta y/\Delta x ).</td>
</tr>
<tr>
<td>3. To find the instantaneous speed, allow ( \Delta t ) to approach zero.</td>
<td>3. To find the slope of the tangent line, allow ( \Delta x ) to approach zero.</td>
</tr>
</tbody>
</table>

Newton’s concept of fluxions and evanescent increments is the main idea of calculus. It can be used to find the slope of a tangent line or the speed of a moving object. What Newton called the ultimate ratio of fluxions is now called the derivative. To find the derivative of a function \( f(x) \), we use the procedure that was used both in Section 13.3 to find the slope of a tangent line and in this section to find the instantaneous speed of a falling object.
If point $P$ has an $x$-coordinate of $x$ and a $y$-coordinate of $f(x)$, then point $Q$ has an $x$-coordinate of $x + \Delta x$ and a $y$-coordinate of $f(x + \Delta x)$, as shown in Figure 13.45. The slope of line $PQ$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - (x)} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The slope of the tangent line is the limiting result of allowing $\Delta x$ to approach zero without reaching zero. This is written as

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

where

$$\lim_{\Delta x \to 0}$$

is read “the limit as $\Delta x$ approaches zero.”

The derivative of a function can be used to find the speed of a moving object as well as to find the slope of a tangent line, so it would be inappropriate to always refer to it as “slope of tangent,” as we did above. Instead, it is referred to as either $f'(x)$ or $df/dx$. The former notation is Newton’s, and the latter notation is Leibniz’s. Leibniz’s notation $df/dx$ is derived from

$$\frac{\Delta f}{\Delta x} = \frac{\text{change in } f}{\text{change in } x}$$

(The $\Delta$s are merely changed to $d$’s.)

**THE DERIVATIVE**

The derivative of a function $f(x)$ is the function

$$f'(x) \text{ or } \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
EXAMPLE 4

**FINDING THE DERIVATIVE OF A LINEAR FUNCTION**

Find the derivative of $f(x) = mx + b$, the line with slope $m$ and $y$-intercept $b$.

**SOLUTION**

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{m(x + \Delta x) + b - (mx + b)}{\Delta x} = \lim_{\Delta x \to 0} \frac{mx + m\Delta x + b - mx - b}{\Delta x} = \lim_{\Delta x \to 0} \frac{m\Delta x}{\Delta x} = \frac{m\Delta x}{\Delta x} = m
\]

Thus, the slope of the line tangent to $f(x) = mx + b$ is $m$. This result should make sense to you, because $f(x) = mx + b$ is itself a line with slope $m$.

EXAMPLE 5

**FINDING A DERIVATIVE OF A SPECIFIC LINEAR FUNCTION**

In Example 4 we found that the derivative of $f(x) = mx + b$ is $\frac{df}{dx} = m$. Use that result to find the derivative of $f(x) = 7x + 2$.

**SOLUTION**

the derivative of $f(x) = mx + b$ is $\frac{df}{dx} = m$ the derivative of $f(x) = 7x + 2$ is $\frac{df}{dx} = 7$ since $m$ matches 7

EXAMPLE 6

**FINDING A DERIVATIVE**

a. Find the derivative of $f(x) = x^2$.

b. Use the derivative of $f(x)$ to find the slope of the tangent line at $x = 1$ and at $x = 2$.

c. Graph $f(x)$ and the tangent line at $x = 1$ and $x = 2$. Show the slopes on the tangent lines’ graphs.

**SOLUTION**

a. The derivative of $f(x)$ is

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x + 0 = 2x
\]

The derivative of $f(x) = x^2$ is $\frac{df}{dx} = 2x$. 
b. To find the slope of the tangent line at \( x = 1 \), substitute 1 for \( x \) in \( \frac{df}{dx} \):

\[
\frac{df}{dx} = 2x = 2 \cdot 1 = 2
\]

To find the slope of the tangent line at \( x = 2 \), substitute 2 for \( x \) in \( \frac{df}{dx} \):

\[
\frac{df}{dx} = 2x = 2 \cdot 2 = 4
\]

c. The graph of \( f(x) \) and the tangent lines is shown in Figure 13.46. Notice that the calculated slopes perfectly match the tangent lines' graphs; that is, the tangent lines do in fact seem to be of slopes 2 and 4.

---

**EXAMPLE 7**

**FINDING A DERIVATIVE** If \( f(x) = 5x^3 \), find \( \frac{df}{dx} \)

**SOLUTION**

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[5(x + \Delta x)^3] - [5x^3]}{\Delta x}
\]

since \( f(x + \Delta x) = 5(x + \Delta x)^3 \)

Before continuing, we’ll multiply out \( (x + \Delta x)^3 \).

\[
(x + \Delta x)^3 = (x + \Delta x)^2(x + \Delta x) = (x^2 + 2x\Delta x + (\Delta x)^2)(x + \Delta x) = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3
\]

Now, we’ll return to finding \( \frac{df}{dx} \).

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{[5(x + \Delta x)^3] - [5x^3]}{\Delta x}
\]
CHAPTER 13 The Concepts and History of Calculus

In Example 4, we found that the derivative of \( f(x) = mx + b \) is \( \frac{df}{dx} = m \). This result is called the linear function rule. It shows us how to take the derivative of any specific linear function, as we did in Example 5.

In Example 6, we found that the derivative of \( f(x) = x^2 \) is \( \frac{df}{dx} = 2x^2-1 = 2x^1 \).

In Example 7, we found that the derivative of \( f(x) = 5x^3 \) is \( \frac{df}{dx} = 5 \cdot 3x^{2-1} = 15x^2 \). Generalizing from these, we could show that the derivative of \( f(x) = ax^n \) is \( \frac{df}{dx} = n \cdot ax^{n-1} \). This result is called the power rule.

### Rules of Differentiation

In Example 4, we found that the derivative of \( f(x) = mx + b \) is \( \frac{df}{dx} = m \). This result is called the linear function rule. It shows us how to take the derivative of any specific linear function, as we did in Example 5.

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### Rules of Differentiation

The linear function rule: If \( f(x) = mx + b \) then \( \frac{df}{dx} = m \)

The power rule: If \( f(x) = ax^n \) is \( \frac{df}{dx} = n \cdot ax^{n-1} \)

### Example 8

**Using the Rules of Differentiation** If \( f(x) = 4x^3 - 3x + 17 \) find \( \frac{df}{dx} \).

**SOLUTION**

We’ll use the power rule to take the derivative of \( 4x^3 \), and the linear function rule to take the derivative of \( -3x + 17 \).

- The derivative of \( ax^n \) is \( n \cdot ax^{n-1} \) the power rule
- The derivative of \( 4x^3 \) is \( 3 \cdot 4x^{3-1} = 12x^2 \) substituting 4 for \( a \) and 3 for \( n \)
- The derivative of \( mx + b \) is \( m \) the linear function rule
- The derivative of \( -3x + 17 \) is \( -3 \) substituting \( -3 \) for \( m \)
- The derivative of \( 4x^3 - 3x + 17 \) is \( 12x^2 - 3 \) using the above two results

\[
\frac{df}{dx} = 12x^2 - 3
\]
Interpreting a Derivative

In any situation, \( \Delta x \) means the change in \( x \), and \( \Delta f = f(x + \Delta x) - f(x) \) means the change in \( f(x) \). A rate of change refers to one change divided by another change, so

\[
\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

is a rate of change; it is called the \textbf{average rate of change of} \( f \). If we allow \( \Delta x \) to approach zero, then the result,

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

is called the \textbf{instantaneous rate of change of} \( f \). The word \textit{instantaneous} is used here because \( x \) often measures time, so \( \Delta t \) measures the change in time; if \( \Delta x \) approaches zero, then the amount of time becomes an instant.

We’ve already seen that if \( x \) measures time and \( f(x) \) measures distance traveled, then

\[
\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

measures the average rate of change of distance, or average speed, and

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

measures the instantaneous rate of change of distance, or instantaneous speed. Furthermore, if time is measured in seconds and distance in feet, then \( \Delta t \) would also be in seconds and \( \Delta f \) would also be in feet; therefore, both \( \Delta f/\Delta x \) and \( df/dx \) would be in feet/second, or feet per second.

What do

\[
\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

and

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

measure under different circumstances? We can answer this question by replacing the symbols \( \Delta f/\Delta x \) with the words

\[
\frac{\text{change in (whatever } f \text{ measures)}}{\text{change in (whatever } x \text{ measures)}}
\]

For example, if \( x \) measures time and \( f(x) \) measures speed, then

\[
\frac{\Delta f}{\Delta x} \quad \text{means} \quad \frac{\text{change in speed}}{\text{change in time}}
\]

It measures the average rate of change of speed, which is called average acceleration \( (\text{acceleration means a change in speed}) \). Also,

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

measures the \textit{instantaneous} rate of change of speed, or \textit{instantaneous} acceleration, because \( \Delta x \) means “change in time,” and if \( \Delta x \) approaches zero, then the amount of time becomes an instant.

If \( x \) measures time and \( f(x) \) measures the population of a state, then \( \Delta f/\Delta x \) means (change in population)/(change in time); it measures the average rate of change of population, which is called \textbf{average growth rate}. Also, \( df/dx \) measures the instantaneous rate of change of population, or \textit{instantaneous growth rate}. Growth rates are commonly used in biology and demographics.
If $x$ measures the number of items manufactured by a factory each day and $f(x)$ measures the profit obtained from selling those items, then $\frac{\Delta f}{\Delta x}$ means (change in profit)/(change in number of items); it measures the average rate of change of profit, which is called average marginal profit. Also, $\frac{df}{dx}$ measures the instantaneous rate of change of profit, or instantaneous marginal profit, and is used by manufacturers to determine whether production should be increased. The meanings of $\frac{\Delta f}{\Delta x}$ and $\frac{df}{dx}$ under these different circumstances are summarized in Figure 13.47.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$</th>
<th>$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>the $x$-value of a point</td>
<td>the $y$-value of a point</td>
<td>the slope of the secant line through $x$ and $x + \Delta x$</td>
<td>the slope of the tangent line at $x$</td>
</tr>
<tr>
<td>time in seconds</td>
<td>distance in feet</td>
<td>average speed in feet per second</td>
<td>instantaneous speed in feet per second</td>
</tr>
<tr>
<td>time</td>
<td>instantaneous speed</td>
<td>average acceleration</td>
<td>instantaneous acceleration</td>
</tr>
<tr>
<td>time in years</td>
<td>population of a town in thousands</td>
<td>average growth rate in thousands of people per year</td>
<td>instantaneous growth rate in thousands of people per year</td>
</tr>
<tr>
<td>number of items manufactured by a factory per day</td>
<td>profit from selling those items</td>
<td>average marginal profit</td>
<td>instantaneous marginal profit</td>
</tr>
</tbody>
</table>

**FIGURE 13.47** Interpreting a derivative.

Given a function that measures some physical quantity, the derivative of that function measures the rate at which that quantity changes. Calculus is the mathematical tool that is used to study how things change. Calculus has given society, for better or worse, the power to control and predict the physical universe. For this reason, the invention of calculus may well be the single most important intellectual achievement of the Renaissance.

### 13.4 Exercises

In Exercises 1–4, find (a) the distance an object will fall in the given amount of time, (b) the instantaneous speed of an object after it has fallen for the given amount of time, and (c) the average speed of an object that falls for the given amount of time.

1. 1 second
2. 10 seconds
3. 5 seconds
4. 15 seconds
5. A parachutist falls for 8 seconds before she opens her parachute.
   a. How far does she fall in those 8 seconds?
   b. At what speed is she falling when she opens the parachute?
   c. What is her average speed while she is in free fall?
6. A parachutist falls for 20 seconds before he opens his parachute.
   a. How far does he fall in those 20 seconds?
   b. At what speed will the rock be falling when it hits the ground?
   c. What is the average speed of the rock during its journey?

**HINT:** distance = 64, time = ?

Selected exercises available online at [www.webassign.net/brookscole](http://www.webassign.net/brookscole)
8. A tourist drops a penny from the observation platform on the top of the Empire State Building in New York City. It falls 1,248 feet.
   a. How long will it take the penny to hit the ground?
   b. At what speed will the penny be falling when it hits the ground?
   c. What is the average speed of the penny during its journey?

9. Newton took the derivative of \( d = 16t^2 \) to find \( s = 32t \), which he called the “fluxion of gravity.” Find Newton’s “second fluxion of gravity” by taking the derivative of \( s = 32t \). Determine what this second fluxion measures.

In Exercises 10–14, use the linear function rule to find the derivative of the given function.

10. \( f(x) = 9x - 49 \)
11. \( f(x) = 72x - 37 \)
12. \( f(x) = -18x - 52 \)
13. \( f(x) = \frac{3}{4}x - \frac{7}{26} \)
14. \( f(x) = \frac{4}{7}x^4 \)

In Exercises 15–18, use the power rule to find the derivative of the given function.

15. \( f(x) = 19x^{13} \)
16. \( f(x) = -32x^{18} \)
17. \( f(x) = \frac{4}{7}x^4 \)
18. \( f(x) = \frac{2}{9}x^{11} \)

In Exercises 19–24, use the rules of differentiation to find the derivative of the given function.

19. \( f(x) = 2x^3 + 3x^2 - 4x + 22 \)
20. \( f(x) = -3x^8 - 2x^2 + 9 \)
21. \( f(x) = -5x + x^3 - x^2 + 2 \)
22. \( f(x) = -3x - 2x^2 + 9x + 15 \)
23. \( f(x) = \frac{2}{7}x^3 - 7x^2 + 5x - 9 \)
24. \( f(x) = \frac{2}{9}x^{11} - \frac{4}{7}x^4 + 9x^2 - 5 \)
25. a. Rewrite \( \sqrt{x} \) in the form \( x^{1/2} \)
   b. Use the power rule and the result of part (a) to find the derivative of \( f(x) = \sqrt{x} \). Write your answer as a radical.
26. a. Rewrite \( \sqrt{x} \) in the form \( x^{1/2} \)
   b. Use the power rule and the result of part (a) to find the derivative of \( f(x) = 3\sqrt{x} \). Write your answer as a radical.
27. a. Use exponent rules to simplify \( x^{3/2}/x^3 \).
   b. Use the result of part (a) to find the derivative of \( f(x) = x^{3/2} \).
   c. Use the result of part (a) to find the derivative of \( g(x) = 4x^{3/2}/7x^3 \).
28. a. Use exponent rules to simplify \( x^2/x^3 \)
   b. Use the result of part (a) to find the derivative of \( f(x) = x^2/x^3 \)
   c. Use the result of part (a) to find the derivative of \( g(x) = -11x^2/12x^3 \).

In Exercises 29–32, do the following.
29. \( f(x) = x^2 - 2x + 2 \)
30. \( f(x) = x^4 + 4x + 5 \)
31. \( f(x) = x^2 - 4x + 2 \)
32. \( f(x) = x^3 - 2x + 3 \)

In Exercises 33–36, find the derivative of the given function by calculating
\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]
33. \( f(x) = x^3 \)
34. \( f(x) = x^3 + 3 \)
35. \( f(x) = 2x^3 + x^2 \)
36. \( f(x) = 3x^3 + 4x - 1 \)
37. \( f(x) = 3x^2 - 2 \)
   a. Find the slope of the secant line from a point \((x, 3x^2 - 2)\) to a point \((x + \Delta x, 3(x + \Delta x)^2 - 2\).
   b. Use the results of part (a) to find the slope of the secant line from the point on the graph of \( f \) where \( x = 1 \) to the point where \( x = 3 \).
   c. Use the results of part (a) and limits to find \( df/dx \).
   d. Use the results of part (c) to find the slope of the tangent line at the point on the graph of \( f \) where \( x = 1 \).
   e. Use the result of part (a) to find the derivative of \( g(x) = -32x^{18} \).
38. \( f(x) = -2x^2 + 4x \)
   a. Find the slope of the secant line from a point \((x, -2x^2 + 4x)\) to a point \((x + \Delta x, -2(x + \Delta x)^2 + 4(x + \Delta x))\).
   b. Use the results of part (a) to find the slope of the secant line from the point on the graph of \( f \) where \( x = 2 \) to the point where \( x = 2.1 \).
   c. Use the results of part (a) and limits to find \( df/dx \).
   d. Use the results of part (c) to find the slope of the tangent line at the point on the graph of \( f \) where \( x = 2 \).
39. \( f(x) = 5x - 7x^3 \)
   a. Find the slope of the secant line from a point \((5x - 7x^3, 5(x + \Delta x) - 7(x + \Delta x)^3)\).
   b. Use the results of part (a) to find the slope of the secant line from the point on the graph of \( f \) where \( x = 9 \) to the point where \( x = 8 \).
   c. Use the results of part (a) and limits to find \( df/dx \).
   d. Use the results of part (c) to find the slope of the tangent line at the point on the graph of \( f \) where \( x = 9 \).
40. \( f(x) = 5/x^2 \)
   a. Find the slope of the secant line from a point \((x, 5/x^2)\) to a point \((x + \Delta x, 5/(x + \Delta x)^2)\).
b. Use the results of part (a) to find the slope of the secant line from the point on the graph of \( f \) where \( x = 1 \) to the point where \( x = 3 \).

c. Use the results of part (a) and limits to find \( \frac{df}{dx} \).

d. Use the results of part (c) to find the slope of the tangent line at the point on the graph of \( f \) where \( x = 1 \).

**CONCEPT QUESTIONS**

In Exercises 41–47, determine what is measured by each of the following. (Include units with each answer.)

\[
\begin{align*}
\text{a.} & \quad \frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
\text{b.} & \quad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\end{align*}
\]

41. \( x \) measures time in seconds, and \( f(x) \) measures the distance from a NASA rocket to its launching pad in miles.

42. \( x \) measures time in minutes, and \( f(x) \) measures the volume of water flowing out of a reservoir in gallons.

43. \( x \) measures time in hours, and \( f(x) \) measures the volume of water flowing through a river in gallons.

44. \( x \) measures time in years, and \( f(x) \) measures the height of a certain tree in inches.

45. \( x \) measures time in years, and \( f(x) \) measures the length of a certain shark in centimeters.

46. \( x \) measures time in days, and \( f(x) \) measures the population of a certain beehive.

47. \( x \) measures time in years, and \( f(x) \) measures the population of Metropolis in thousands.

48. At the end of Example 4, we stated that \( \lim_{\Delta x \to 0} m = m \). Explain why this is true.

49. If \( x \) measures time after launching, in seconds, and \( f(x) = 72x - 6x^2 + 95 \) measures the distance from a NASA rocket to its launching pad, in feet, then find:

\[
\begin{align*}
a. & \quad \frac{df}{dx} \\
b. & \quad \text{The speed of the rocket 4 seconds after launching. Include units.}
\end{align*}
\]

50. If \( x \) measures time in minutes, and \( f(x) = 32 - 1.2x + 0.9x^2 \) measures the volume of water flowing out of a reservoir, in gallons, then find:

\[
\begin{align*}
a. & \quad \frac{df}{dx} \\
b. & \quad \text{The speed of the water after 20 minutes. Include units.}
\end{align*}
\]

51. If \( x \) measures time after a tree was 100 feet high, in years, and \( f(x) = 0.6x^2 - 2x \) measures the height of the tree, in feet, then find:

\[
\begin{align*}
a. & \quad \frac{df}{dx} \\
b. & \quad \text{The rate at which the tree is growing, 3 years after it reached 100 feet in height. Include units.}
\end{align*}
\]

52. If \( x \) measures the number of years after the year 2000, and \( f(x) = 72x - 16x^2 + 2,986 \) measures the population of Metropolis in thousands, then find:

\[
\begin{align*}
a. & \quad \frac{df}{dx} \\
b. & \quad \text{The rate at which Metropolis will be growing, in 2019. Include units.}
\end{align*}
\]

---

**13.5 The Trajectory of a Cannonball**

**OBJECTIVES**

- Learn how calculus is used to analyze the motion of a cannonball
- Discover how to tell where a cannonball will strike the ground
- Find whether a cannonball will clear a wall

Galileo showed that the motion of a cannonball could be resolved into two simultaneous motions, one due to the explosion and the other due to gravity. The explosion causes a motion in the direction in which the cannon is pointed; this motion has constant speed, because there is nothing to slow it down (ignoring air resistance). Gravity causes a vertical motion, with speed changing in a uniform way like that of a falling body. Galileo used this idea, along with Apollonius’s description of a parabola, to show that a cannonball follows a parabolic path. He was unable to get an equation for the path (and thus unable to determine how to aim a cannon so that the ball would hit the target) because he used ancient, proportion-based algebra rather than modern, equation-based algebra.
After Galileo died, other mathematicians used the modern analytic geometry of Fermat and Descartes, the modern algebra of Viète and Descartes, and Newton’s work on gravity to find an equation for the path of a cannonball. We will find such an equation for the situation in which the cannon is pointed at a 30° angle, a 45° angle, or a 60° angle.

Our goal is to obtain an equation for the cannonball’s motion. Since an equation is algebraic and the motion it describes is geometric, we must use analytic geometry to obtain that equation. To use analytic geometry, we will place an \(x\)-axis and a \(y\)-axis such that the \(x\)-axis runs along the ground and the \(y\)-axis runs through the mouth of the cannon (as illustrated in Figure 13.48). This means that the position of the cannonball at a time \(t\) is \((x, y)\), where \(x\) measures the horizontal distance from the cannon to the cannonball and \(y\) measures the vertical distance from the cannonball to the ground. We will be able to obtain our equation if we can find these horizontal and vertical distances.

Galileo resolved a cannonball’s motion into a diagonal motion due to the explosion and a downward motion due to gravity. We will follow Galileo’s lead and separately consider the motion due to the explosion and the motion due to gravity. In considering the motion due to the explosion, we will not need to take gravity into account, and in considering the motion due to gravity, we will not need to take the explosion into account. Later, we will combine the results of our two analyses and get one equation that describes the cannonball’s motion.

The motion due to gravity fits naturally into an analytic geometry perspective, since it is in the \(y\)-direction. The motion due to the explosion fits less naturally, since it is both forward and upward—that is, in both the \(x\)- and \(y\)-directions. We can break this diagonal motion down into an \(x\)-component and a \(y\)-component by using similar triangles.

The Motion Due to the Explosion

Think of a cannonball that’s fired from a cannon pointed at a 45° angle. That cannonball does not move in a straight line at a 45° angle, because gravity pulls it downward. For the time being, however, we are considering only the motion due to the explosion, without the effects of gravity, so we must view the cannonball as moving along a straight line at a 45° angle, as shown in Figure 13.49. If the cannonball’s speed is \(s\) feet per second, then in 1 second, the ball would move \(s\) feet along that line. The horizontal component \(h\) of that motion is represented by the horizontal side of the triangle in Figure 13.50; the vertical component \(v\) of that motion is represented by the vertical side of the triangle. The upper angle of that triangle must be 45°, since the three angles of a triangle always add to 180°.
The triangle in Figure 13.50 is similar to any other triangle with the same angles. In particular, it is similar to a triangle with a horizontal side of length \( a = 1 \), as shown in Figure 13.51. That triangle’s vertical side must be of the same length, since the horizontal and vertical sides are across from angles of the same size. The triangle is a right triangle, so we can apply the Pythagorean Theorem to find the length of the diagonal side.

\[
\begin{align*}
    a^2 + b^2 &= c^2 \\
    1^2 + 1^2 &= c^2 \\
    1 + 1 &= c^2 \\
    c^2 &= 2 \\
    c &= \sqrt{2}
\end{align*}
\]

Since the two triangles are similar, the ratios of their corresponding sides are equal, as shown in Figure 13.52. In particular,

\[
\begin{align*}
    \frac{h}{1} &= \frac{s}{\sqrt{2}} \\
    \frac{v}{1} &= \frac{s}{\sqrt{2}}
\end{align*}
\]

Recall that \( s \) is the cannonball’s speed, \( h \) is its horizontal component, and \( v \) is its vertical component. Thus, the horizontal speed (or rate) due to the explosion is \( h = s/\sqrt{2} \), and the vertical speed (or rate) due to the explosion is \( v = s/\sqrt{2} \). And we can use the formula “distance = rate \cdot time” to find the horizontal and vertical distances traveled due to the explosion. See Figure 13.53.

<table>
<thead>
<tr>
<th>Direction</th>
<th>Rate</th>
<th>Time</th>
<th>Distance = Rate \cdot Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>horizontal</td>
<td>( \frac{s}{\sqrt{2}} )</td>
<td>( t )</td>
<td>( \frac{s}{\sqrt{2}} \cdot t )</td>
</tr>
<tr>
<td>vertical</td>
<td>( \frac{s}{\sqrt{2}} )</td>
<td>( t )</td>
<td>( \frac{s}{\sqrt{2}} \cdot t )</td>
</tr>
</tbody>
</table>

**FIGURE 13.53** The motion due to the explosion.
13.5 The Trajectory of a Cannonball

The Motion Due to Gravity

As Galileo discovered, a cannonball moves along a parabola, unlike a falling object. However, we are now considering only the motion due to gravity, without the effects of the explosion. If there were no explosion, the cannonball would fall straight down, just like any other falling object; in \( t \) seconds, it would fall \( 16t^2 \) feet. And since this motion is due to gravity, it is all vertical motion.

The Equation for the Cannonball’s Motion

Figure 13.54 summarizes our work so far.

<table>
<thead>
<tr>
<th>Motion Due to</th>
<th>Direction</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>the explosion</td>
<td>forward (x)</td>
<td>( \frac{s}{\sqrt{2}} \cdot t )</td>
</tr>
<tr>
<td>the explosion</td>
<td>upward (y)</td>
<td>( \frac{s}{\sqrt{2}} \cdot t )</td>
</tr>
<tr>
<td>gravity</td>
<td>downward (y)</td>
<td>( 16t^2 )</td>
</tr>
</tbody>
</table>

Recall that we placed the \( x \)- and \( y \)-axes such that the position of the cannonball at a time \( t \) is \((x, y)\), where \( x \) measures the horizontal distance from the cannonball to the cannon and \( y \) measures the vertical distance from the cannonball to the ground, as shown in Figure 13.48 on page 13-55. The horizontal motion is due solely to the explosion, and we found that distance to be \((s/\sqrt{2})t\). Thus, \( x = (s/\sqrt{2})t \).

The vertical motion is due both to the explosion and to gravity. Initially, the \( y \)-coordinate is just the distance from the cannon to the origin—that is, the height of the cannon’s muzzle. Call this distance \( c \). The cannonball’s height is increased by the explosion and decreased by gravity, so to the initial height, we add \((s/\sqrt{2})t\) and subtract \(16t^2\). Thus,

\[
y = c + \frac{s}{\sqrt{2}} \cdot t - 16t^2
\]

To get the equation of the cannonball’s path, we combine the formula for the cannonball’s \( x \)-coordinate \([x = (s/\sqrt{2})t]\) and the formula for the cannonball’s \( y \)-coordinate \([y = c + (s/\sqrt{2})t - 16t^2]\).

\[
x = \frac{s}{\sqrt{2}} \cdot t
\]

\[
x \cdot \frac{\sqrt{2}}{s} = t
\]

\[
y = c + \frac{s}{\sqrt{2}} \cdot \frac{x \cdot \sqrt{2}}{s} - 16 \left( \frac{x \cdot \sqrt{2}}{s} \right)^2
\]

\[
y = c + x - 16 \left( \frac{x^2 \cdot 2}{s^2} \right)
\]

\[
y = \frac{32x^2}{s^2} + x + c
\]

This is the equation of the trajectory of a cannonball that is fired at a 45° angle.
Similar work would allow us to find the equation of the trajectory of a cannonball that is fired at a 30° angle and one that is fired at a 60° angle. Those equations are given below.

Although the following equations were discovered in an intensive study of cannons, they also apply to other projectiles, such as a ball, a rock, an arrow from a bow or crossbow, and a shell from a rifle or a musket. Thus, while the following example involves a cannonball shot with a certain initial speed at a certain angle, it could just as well involve a football kicked at the same speed and angle. The problem would be solved in the same manner, and the solutions would be the same.

**EXAMPLE 1**

**FINDING AND USING A TRAJECTORY EQUATION** A cannonball is shot with an initial speed of 200 feet per second from a cannon whose muzzle is 3 feet above the ground and at a 30° angle.

a. Find the equation of the cannonball’s trajectory.

b. How far from the cannon will the ball strike the ground?

c. Will the ball clear an 85-foot-high wall that is 1,000 feet from the cannon?

d. Sketch the cannonball’s path.

**SOLUTION**

a. \[ y = \frac{-64x^2}{3s^2} + \frac{x}{\sqrt{3}} + c \] using the 30° angle formula

\[ = \frac{-64x^2}{3(200)^2} + \frac{x}{\sqrt{3}} + 3 \] substituting 200 for \( s \) and 3 for \( c \)

\[ = \frac{-64x^2}{120,000} + \frac{x}{\sqrt{3}} + 3 \]

\[ = -0.0005x^2 + 0.5774x + 3 \] rounding to four decimal places

b. Finding how far the ball will travel means finding the value of \( x \) that will make \( y = 0 \), because \( y \) measures the distance above the ground.

\[ y = -0.0005x^2 + 0.5774x + 3 \]

\[ 0 = -0.0005x^2 + 0.5774x + 3 \] substituting 0 for \( y \)
To solve this equation, we need to use the Quadratic Formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

First, we calculate the radical and put it into the calculator’s memory (usually with a [STO] or \([x \rightarrow M]\) button; see Appendix A or B for more information on using a calculator’s memory).

Our last equation for \(x\) becomes

\[
x = \frac{-0.5774 \pm \text{memory}}{-0.001}
\]

\[
= \frac{-0.5774 + \text{memory}}{-0.001} \quad \text{or} \quad \frac{-0.5774 - \text{memory}}{-0.001}
\]

We want to combine \(-0.5774\) with the number that is in the calculator’s memory. On most scientific calculators, this is done with a [RCL] or [RM] button.

This gives \(x = -5.172536 \ldots \) or \(1.159972536 \ldots \approx -5\) or \(1.160\). The parabola intersects the \(x\)-axis at approximately \(x = -5\) and at \(x = 1.160\). The negative value for \(x\) does not fit our situation. The cannonball will travel approximately \(1,160\) feet.

c. The ball will clear an 85-foot-high wall that is 1,000 feet away if \(y > 85\) when \(x = 1,000\).
Thus, the ball will be about 80 feet high when it is 1,000 feet from the cannon; it will not clear an 85-foot-high wall. The cannon’s angle of elevation would have to be increased a little.

d. The path is sketched in Figure 13.55.

**EXAMPLE 2**

**FINDING AND USING ANOTHER TRAJECTORY EQUATION** A cannonball is shot with an initial speed of 200 feet per second from a cannon whose muzzle is 4 feet above the ground and at a 45° angle.

a. Find the equation of the cannonball’s trajectory.

b. What is the maximum height the ball will attain?

**SOLUTION**

a. \[ y = -\frac{32}{200}x^2 + x + 4 \] \textbf{using the 45° angle formula}

\[ = -0.0008x^2 + x + 4 \] \textbf{substituting}

b. Finding the ball’s maximum height means finding the vertical distance from the ground at the highest point; that is, it asks for the value of \( y \) at the vertex. In Section 13.0, we were able to find vertices of parabolas by plotting a handful of points in the region where the slope of the secant line is close to 0. In our current situation, such an approach would involve finding hundreds of points. Instead, observe that the cannonball is at its highest point when the slope of the tangent line is 0, as shown in Figure 13.56.
Thus, we will first find the slope of the tangent line at an arbitrary point \( P \), by taking the derivative of the equation of the path. Next, we will set that slope equal to 0 and solve for \( x \). This will give us the location of the point \( P \) at which the slope of the tangent line is 0.

**Step 1**

*Find the slope of the tangent line at an arbitrary point \( P \) by taking the derivative.*

Rather than taking the derivative of \( f(x) \) with Newton’s or Cauchy’s method, we will use the rules of differentiation from page 13-50.

The derivative of \( ax^n \) is \( n \cdot ax^{n-1} \)  
the power rule
The derivative of \(-0.0008x^2\)  
using the power rule
is \( 2 \cdot (-0.0008)x^{2-1} \)
The derivative of \( mx + b \) is \( m \)  
the linear function rule
The derivative of \( 1x + 4 \) is 1  
using the linear function rule
The derivative of \(-0.0008x^2 + 1x + 4\)  
combining the above results
is \( 2 \cdot (-0.0008)x^{2-1} + 1 \)
\[ = -0.0016x + 1 \]  
simplifying

**Step 2**

*The cannonball is at its highest point when the slope of the tangent line is 0.*

\[ -0.0016x + 1 = 0 \]
\[ -0.0016x = -1 \]
\[ x = \frac{-1}{-0.0016} = 625 \]

This is the \( x \)-coordinate of the point \( P \), where the ball is at its highest point. The question asked for the vertical distance from the ground at the highest point; that is, it asks for the \( y \)-coordinate of \( P \).

\[ y = -0.0008x^2 + x + 4 \]
\[ = -0.0008(625)^2 + 625 + 4 \]  
substituting 625 for \( x \)
\[ = 316.5 = 317 \]  
rounding to the nearest foot

The cannonball will travel to a maximum height of 317 feet (rounded off), as shown in Figure 13.57.
The Generalized Equation for the Cannonball’s Motion (for those who have read Section 8.5: Trigonometry)

Earlier in this section, we derived the equation of the trajectory of a cannonball shot from a cannon whose angle of elevation is $45^\circ$. In the exercises you will derive equations for two other angles of elevation: $30^\circ$ and $60^\circ$. There are quite a few angles other than these three; clearly, it’s impractical to have separate equations for $1^\circ$, $2^\circ$, and so on. Trigonometry allows us to generate one equation that will work for any angle.

If a cannon’s angle of elevation is $\theta$ (the Greek letter “theta”), its speed is $s$ feet per second, and that speed’s horizontal and vertical components are $h$ and $v$, respectively, then we can draw the triangle shown in Figure 13.58, a generalized version of the specific triangle drawn for a $45^\circ$ angle of elevation in Figure 13.52 on page 13-56.

Since the opposite side is $v$, the adjacent side is $h$, and the hypotenuse is $s$, we have

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{v}{s} \quad \text{and} \quad \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{h}{s}$$

We can solve the first equation for $v$ and the second equation for $h$ by multiplying by $s$:

$$s \cdot \sin \theta = s \cdot \frac{v}{s} = v \quad \text{and} \quad s \cdot \cos \theta = s \cdot \frac{h}{s} = h$$

Since $h$ is the horizontal speed (or rate), and distance = rate • time, the horizontal distance is rate • time = $s \cos \theta \cdot t = st \cos \theta$. Similar work gives us the vertical distance, as shown in Figure 13.59.

<table>
<thead>
<tr>
<th>Direction</th>
<th>Rate</th>
<th>Time</th>
<th>Distance = Rate • Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>horizontal</td>
<td>$h = s \cos \theta$</td>
<td>$t$</td>
<td>$(s \cos \theta) \cdot t = st \cos \theta$</td>
</tr>
<tr>
<td>vertical</td>
<td>$h = s \sin \theta$</td>
<td>$t$</td>
<td>$(s \sin \theta) \cdot t = st \sin \theta$</td>
</tr>
</tbody>
</table>

The motion due to gravity is not affected by the angle of elevation; the cannonball still falls $16t^2$ feet in $t$ seconds. Naturally, this motion is all vertical.

Figure 13.60 summarizes our work so far.

<table>
<thead>
<tr>
<th>Motion Due to</th>
<th>Direction</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>the explosion</td>
<td>forward ($x$)</td>
<td>$st \cos \theta$</td>
</tr>
<tr>
<td>the explosion</td>
<td>upward ($y$)</td>
<td>$st \sin \theta$</td>
</tr>
<tr>
<td>gravity</td>
<td>downward ($y$)</td>
<td>$16t^2$</td>
</tr>
</tbody>
</table>

A summary of our work so far.
The horizontal motion is due solely to the explosion, and we found that distance to be $st \cos \theta$; thus, $x = st \cos \theta$. The vertical motion is due both to the explosion and to gravity. Initially, the $y$-coordinate is just the distance $c$ from the cannon to the origin—that is, the height of the cannon’s muzzle. The cannonball’s height is increased by the explosion and decreased by gravity, so to the initial height, we must add $st \sin \theta$ and subtract $16t^2$. Thus,

$$y = c + st \sin \theta - 16t^2$$

To get the equation of the cannonball’s path, we solve the $x$-equation for $t$ and substitute it into the $y$-equation, as we did for the $45^\circ$ equation earlier in this section:

$$x = st \cos \theta$$

solving for $t$

$$\frac{x}{s \cos \theta} = t$$

$$y = c + s \cdot t \sin \theta - 16t^2$$

the $y$-equation

$$y = c + s \cdot \frac{x}{s \cos \theta} \sin \theta - 16 \left(\frac{x}{s \cos \theta}\right)^2$$

substituting $\frac{x}{s \cos \theta}$ for $t$

$$y = c + \frac{x \sin \theta}{\cos \theta} - 16 \frac{x^2}{(s \cos \theta)^2}$$

canceling and simplifying

This is the equation of the trajectory of a cannonball shot from a cannon whose angle of elevation is $\theta$, but it can be further simplified. Recall that $\sin \theta = \text{opp}/\text{hyp}$, $\cos \theta = \text{adj}/\text{hyp}$, and $\tan \theta = \text{opp}/\text{adj}$. Thus,

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{\text{adj}}{\text{hyp}} \cdot \frac{\text{hyp}}{\text{adj}} = \frac{\text{opp}}{\text{adj}} = \tan \theta$$

This means that we can replace the $\sin \theta/\cos \theta$ in the above equation with $\tan \theta$. Thus, the equation is

$$y = c + x \tan \theta - 16 \frac{x^2}{(s \cos \theta)^2}$$

$$y = -16 \frac{x^2}{(s \cos \theta)^2} + x \tan \theta + c$$
EXAMPLE 3

FINDING A TRAJECTORY EQUATION

A cannonball is shot with an initial speed of 250 feet per second from a cannon whose muzzle is 4 feet above the ground and at a 40° angle.

a. Find the equation of the cannonball’s trajectory.

b. What is the maximum height the ball will attain?

SOLUTION

a. 

\[
y = -16 \frac{x^2}{(s \cos \theta)^2} + x \tan \theta + c
\]

if the cannon’s angle of elevation is \( \theta \)

\[
= -16 \frac{x^2}{(250 \cos 40°)^2} + x \tan 40° + 4
\]

\[
= -0.0004362465769x^2 + 0.8390996312x + 4
\]

To compute the coefficient of \( x^2 \), type

\[
16 \div 16 \div (250 \times \cos 40°) \times x^2 =
\]

\[
\approx -0.0004x^2 + 0.8391x + 4 \quad \text{rounding to four decimal places}
\]

b. We will follow the steps used in Example 2 to find the ball’s maximum height.

Step 1

Find the slope of the tangent line at an arbitrary point \( P \) by taking the derivative.

The derivative of \(-0.0004x^2 + 0.8391x + 4\) is

\[
2 \cdot (-0.0004)x^{2-1} + 0.8391
\]

\[
= -0.0008x + 0.8391
\]
Step 2

The cannonball is at its highest point when the slope of the tangent line is 0.

\[-0.0008x + 0.8391 = 0\]
\[-0.0008x = -0.8391\]
\[x = -0.8391 / -0.0008 = 1048.875 \approx 1049\]

This is the \(x\)-coordinate of the point \(P\), where the ball is at its highest point. The question asked for the vertical distance from the ground at the highest point; that is, it asks for the \(y\)-coordinate of \(P\).

\[y = -0.0004x^2 + 0.8391x + 4\]
\[= -0.0004(1048.875)^2 + 0.8391(1048.875) + 4\]
\[= 444.055063 \approx 444\text{ feet}\]

The cannonball will travel to a maximum height of 444 feet, as shown in Figure 13.61.

13.5 Exercises

In Exercises 1–6, answer the following questions.

a. Find the equation of the trajectory of a cannonball shot with the given initial speed from a cannon whose muzzle is at the given orientation.

b. How far will the ball travel?

c. What is the maximum height the ball will attain?

d. Will the ball clear an 80-foot wall that is 1,900 feet from the cannon?

e. Sketch the cannonball’s path. In your sketch, include everything that is known about the path.

(Round all decimals to four decimal places, and round distance answers to the nearest foot, as in the examples.)

1. initial speed: 250 feet per second
   muzzle height: 4 feet
   angle of elevation: 30°

2. initial speed: 300 feet per second
   muzzle height: 4 feet
   angle of elevation: 30°

3. initial speed: 300 feet per second
   muzzle height: 4 feet
   angle of elevation: 45°

4. initial speed: 250 feet per second
   muzzle height: 4 feet
   angle of elevation: 45°

5. initial speed: 325 feet per second
   muzzle height: 5 feet
   angle of elevation: 60°

6. initial speed: 275 feet per second
   muzzle height: 4 feet
   angle of elevation: 60°

Selected exercises available online at www.webassign.net/brookscole
In Exercises 7–12, answer the following questions.

a. Find the equation of the trajectory of a cannonball shot with the given initial speed from a cannon whose muzzle has the given orientation.
b. How far will the ball travel?
c. What is the maximum height the ball will attain?
d. Will the ball clear a 70-foot wall that is 1,100 feet from the cannon?
e. Sketch the cannonball’s path. In your sketch, include everything that is known about the path.

(Round all decimals to four decimal places, and round distance answers to the nearest foot, as in the examples.)

7. initial speed: 200 feet per second
   muzzle height: 5 feet
   angle of elevation: 30°

8. initial speed: 225 feet per second
   muzzle height: 4 feet
   angle of elevation: 30°

9. initial speed: 250 feet per second
   muzzle height: 4 feet
   angle of elevation: 45°

10. initial speed: 300 feet per second
    muzzle height: 5 feet
    angle of elevation: 45°

11. initial speed: 310 feet per second
    muzzle height: 3 feet
    angle of elevation: 60°

12. initial speed: 360 feet per second
    muzzle height: 5 feet
    angle of elevation: 60°

In Exercises 13–16, answer the following questions.

a. Find the equation of the trajectory of a cannonball shot with the given initial speed from a cannon whose muzzle is at the given orientation.
b. How far from the target must this cannon be placed, if the target is placed as indicated? (Choose the answer that allows the cannon crew to be as far as possible from the enemy.)
c. Sketch the cannonball’s path. In your sketch, include everything that is known about the path.

13. initial speed: 240 feet per second
    muzzle height: 4 feet
    angle of elevation: 45°
    target: 20 feet above the ground

14. initial speed: 200 feet per second
    muzzle height: 3 feet
    angle of elevation: 30°
    target: 50 feet above the ground

15. initial speed: 300 feet per second
    muzzle height: 4 feet
    angle of elevation: 30°
    target: 20 feet above the ground

16. initial speed: 200 feet per second
    muzzle height: 5 feet
    angle of elevation: 45°
    target: 40 feet above the ground

17. A baseball player hits a home run that just clears the 10-foot-high fence 400 feet from home plate. He hit the ball at a 30° angle of elevation, 3 feet above the ground.
   a. Find the ball’s initial speed. (Round to the nearest foot per second.)
   HINT: Substitute 400 for x and 10 for y in the 30° equation.
   b. Find the equation of the ball’s trajectory. (Round all decimals to four decimal places.)
   c. What is the maximum height the ball attained? (Round to the nearest foot.)
   d. Find the ball’s initial speed if the angle of elevation were 45°. (Round to the nearest foot per second.)

18. A football player kicks a field goal at a 45° angle of elevation from a distance of 50 yards from the goal posts.
   a. Find the ball’s initial speed if the ball hits the goal post’s crossbar, 3 yards above ground. (Round to the nearest foot per second.)
   HINT: See Exercise 17.
   b. Find the equation of the ball’s trajectory. (Round all decimals to four decimal places.)
   c. What is the maximum height the ball attained? (Round to the nearest foot.)
   d. What would the football player have to do differently to get the ball over the crossbar?

19. A 6-foot-tall hiker is walking on a 100-foot-high cliff overlooking the ocean. He throws a rock at a 45° angle of elevation with an initial speed of 90 feet per second.
   a. Find the equation of the trajectory of the rock.
   b. How far does the rock travel?
   c. How long does it take for the rock to land in the water below?
   HINT: Use the answer to part (b) and the “x-equation” that was obtained when we derived the equation of a projectile with a 45° angle of elevation.

20. A firecracker is launched at a 45° angle of elevation from ground level with an initial speed of 120 feet per second.
   a. Find the equation of the trajectory of the firecracker.
   b. Find the firecracker’s maximum height and the horizontal distance between the launch point and the point of maximum height.
   c. Find the amount of time that the fuse should burn if the firecracker is to detonate at the peak of its trajectory.
13.5 Exercises

13-67

HINT: Use the answer to part (b) and the “x-equation” that was obtained when we derived the equation of a projectile with a 45° angle of elevation.

21. In finding the equation of a cannonball’s trajectory, we used a right triangle with a 45° angle, because our equation was for a cannon pointed at a 45° angle. That triangle had sides of length 1, 1, and $\sqrt{2}$. If the cannon is pointed at a 30° angle, we use a right triangle with a 30° angle. In this exercise, we will find the lengths of such a triangle’s sides. Explain the reasoning that you use to answer each of the following questions.

a. What is the size of the triangle’s third angle?
b. Make a larger triangle by attaching a duplicate of our 30° triangle underneath it, as shown in Figure 13.62. What are the sizes of that larger triangle’s three angles?
c. If the original 30° triangle’s vertical side is of length 1, what is the length of the larger triangle’s vertical side?
d. What are the lengths of the larger triangle’s other two sides?
e. You now know the length of two of the 30° triangle’s sides. Use the Pythagorean Theorem to find the length of its third side.

22. Use Exercise 21 and a procedure similar to that used in this section (in finding the trajectory equation for a cannon whose angle of elevation is 45°) to find the trajectory equation for a cannon whose angle of elevation is 30°.

23. Use Exercise 21 and a procedure similar to that used in this section (in finding the trajectory equation for a cannon whose angle of elevation is 45°) to find the trajectory equation for a cannon whose angle of elevation is 60°.

24. Use the “x-equation” that was obtained in Exercise 22 to find how long it takes for the cannonball in Exercise 2 to hit the ground.

25. Use the “x-equation” that was obtained in Exercise 22 to find how long it takes for the baseball in Exercise 17 to hit the ground.

26. Find how long it takes for the football in Exercise 18 to hit the crossbar.

27. Use the “x-equation” that was obtained in Exercise 23 to find how long it takes for the cannonball in Exercise 5 to hit the ground.

Exercises 28–31 are for those who have read Section 8.5: Trigonometry. In Exercises 28–31, answer the following questions.

a. Find the equation of the trajectory of a cannonball shot with the given initial speed from a cannon whose muzzle has the given orientation.
b. How far will the ball travel?
c. What is the maximum height that the ball will attain?
d. Sketch the cannonball’s path. In your sketch, include everything that is known about the path.

(Round all decimals to four decimal places, and round distance answers to the nearest foot, as in the examples.)

28. initial speed: 200 feet per second
   muzzle height: 5 feet
   angle of elevation: 10°

29. initial speed: 200 feet per second
   muzzle height: 5 feet
   angle of elevation: 20°

30. initial speed: 200 feet per second
   muzzle height: 5 feet
   angle of elevation: 80°

31. initial speed: 200 feet per second
   muzzle height: 5 feet
   angle of elevation: 70°

Answer the following questions using complete sentences and your own words.

- CONCEPT QUESTIONS

32. Why do the equations of the trajectory of a projectile apply equally well to cannonballs, baseballs hit by bats, and thrown objects?

33. Would the equations of the trajectory of a projectile apply to missiles? Why or why not?
13.6 Newton and Areas

OBJECTIVES
- Investigate how calculus is used to find areas
- Understand what an antiderivative is

Find the Area of Any Shape

Recall that the Greek geometers had found areas of specific shapes such as circles, rectangles, and triangles and that Archimedes and Kepler had found areas of regions by filling those regions with an infinite number of triangles or rectangles. Newton used his calculus to find areas in a different manner. We will use modern notation in explaining Newton’s concept.

The Area Function $A(x)$

Consider the region bounded below by the $x$-axis, above by the horizontal line $y = 5$, on the left by the $y$-axis, and on the right by a vertical line, as illustrated in Figure 13.63. Different locations of a right-hand boundary produce different regions with different areas.

Let $A(x)$ be a function whose input $x$ is the specific location of the right-hand boundary and whose output is the area of the resulting region. Thus, as shown in Figure 13.64, $A(2)$ would be the area of a rectangle with height 5 and base 2: $A(2) = 5 \cdot 2 = 10$. Similarly, $A(8)$ would be the area of a rectangle with height 5 and base 8; $A(8) = 5 \cdot 8 = 40$. More generally, $A(x)$ would be the area of a rectangle with height 5 and base $x$; $A(x) = 5x$, as shown in Figure 13.65. This is the area function for a rectangle of height 5; that is, it is the area function associated with $f(x) = 5$.

Every positive function $f(x)$ has an area function $A(x)$ associated with it. The area is that of the region bounded below by the $x$-axis, above by the curve or line given by $f(x)$, on the left by the $y$-axis, and on the right by a vertical line at $x$, as shown in Figure 13.66. We just found that the function $f(x) = 5$ (or $y = 5$) has an area function $A(x) = 5x$ associated with it.
EXAMPLE 1

FINDING AND USING AN AREA FUNCTION

a. Find the area function associated with \( f(x) = 3x \).

b. Sketch the region whose area is found by that area function.

c. Find \( A(6) \) and describe the region whose area is \( A(6) \).

d. Find \( A(4) \) and describe the region whose area is \( A(4) \).

e. Find \( A(6) - A(4) \) and describe the region whose area is \( A(6) - A(4) \).

SOLUTION

a. \( f(x) = 3x \) (or \( y = 3x + 0 \)) is a line whose slope is 3 and whose intercept is 0. Its area function \( A(x) \) gives the area of the region bounded below by the \( x \)-axis, above by the line \( f(x) = 3x \), on the left by the \( y \)-axis, and on the right by a vertical line at \( x \), as shown in Figure 13.67. That region is a triangle, so its area is given by the formula “area = \( \frac{1}{2} \cdot \text{base} \cdot \text{height} \).” Since the base is \( x \) and the height is \( y \), the area is

\[
A(x) = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot x \cdot y = \frac{1}{2} \cdot x \cdot 3x \quad \text{since } y = 3x
\]

Thus, the area function associated with \( f(x) = 3x \) is \( A(x) = \frac{3x^2}{2} \).

b. The region is shown in Figure 13.67.

c. \( A(x) = \frac{3x^2}{2} \) so \( A(6) = \frac{3 \cdot 6^2}{2} = 54 \)

This is the area of the region bounded below by the \( x \)-axis, above by the line \( f(x) = 3x \), on the left by the \( y \)-axis, and on the right by a vertical line at \( x = 6 \).

d. \( A(x) = \frac{3x^2}{2} \) so \( A(4) = \frac{3 \cdot 4^2}{2} = 24 \)

This is the area of the region bounded below by the \( x \)-axis, above by the line \( f(x) = 3x \), on the left by the \( y \)-axis, and on the right by a vertical line at \( x = 4 \).

e. \( A(6) - A(4) = 54 - 24 = 30 \)

This is the area of the region bounded below by the \( x \)-axis, above by the line \( f(x) = 3x \), on the left by a vertical line at \( x = 4 \), and on the right by a vertical line at \( x = 6 \), as shown in Figure 13.68.
CHAPTER 13 The Concepts and History of Calculus

Newton and the Area Function

We can find areas of rectangles, triangles, and other basic shapes quite easily, because the ancient Greeks found formulas for those areas centuries ago (as was discussed in Chapter 8). These formulas are of little value in finding the area bounded by curves (unless, like Kepler and Archimedes, you fill the region with an infinite number of triangles).

Newton’s goal was to find the area of a region bounded below by the \( x \)-axis, above by a curve \( y = f(x) \), on the left by the \( y \)-axis, and on the right by a vertical line at \( x \), as shown in Figure 13.67. He wanted to find the area function associated with the curve \( y = f(x) \).

The key to the area problem is to find the derivative of the area function \( A(x) \):

\[
\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x}
\]

We can interpret this derivative geometrically by analyzing it piece by piece. \( A(x + \Delta x) - A(x) \) refers to the area of the region bounded on the right by a vertical line at \( x + \Delta x \), slightly to the right of \( x \), as shown in Figure 13.69. \( A(x + \Delta x) - A(x) \) then refers to the difference between this area and the area given by \( A(x) \), as shown in Figure 13.70. This is the area of the narrow strip bounded on the left by a vertical line at \( x \) and on the right by a vertical line at \( x + \Delta x \). Because \( \Delta x \) is allowed to approach zero, the strip is quite narrow; it is so narrow that we can think of it as a rectangle whose height is \( y \), or \( f(x) \), and whose width is \( \Delta x \). Thus, the numerator can be rewritten as

\[
A(x + \Delta x) - A(x) = \text{area of thin rectangle} = \text{base} \cdot \text{height} = \Delta x \cdot f(x)
\]

and the entire fraction can be rewritten as

\[
\frac{A(x + \Delta x) - A(x)}{\Delta x} = \frac{\Delta x \cdot f(x)}{\Delta x} = f(x)
\]

This means that the derivative of the area function is \( dA/dx = f(x) \) or, to put it another way, that the area function \( A(x) \) is that function whose derivative is \( f(x) \). For this reason, \( A(x) \) is called the antiderivative of \( f(x) \); it is denoted by \( A(x) = \int f(x) \, dx \).

The antiderivative symbol \( \int \) has its origins in Archimedes’ and Kepler’s methods of finding areas. Each of these methods involves breaking down a region into very small pieces and finding the sum of the areas of those pieces. The symbol \( \int \) is a stylized \( S \) and stands for sum.
13.6 Newton and Areas

Before we continue with the area problem, we must pause and discuss antiderivatives.

**Antiderivatives**

Derivative

Antiderivative

**ANTIDERIVATIVE DEFINITION**

If the derivative of \( f(x) \) is \( g(x) \), then the antiderivative of \( g(x) \) is \( f(x) \).

\[
\frac{df}{dx} = g(x), \text{ then } \int g(x) \, dx = f(x).
\]

**DERIVATIVES AND ANTIDERIVATIVES**

**a.** Find the derivative of \( f(x) = 3x + c \) (where \( c \) is any constant).

**b.** Rewrite the derivative found in part (a) as an antiderivative.

**a.** Although we could find the derivative of \( f(x) = 3x + c \) by computing

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

it is much easier to use the linear function rule.

If \( f(x) = mx + b \) then \( \frac{df}{dx} = m \) **the linear function rule**

If \( f(x) = 3x + c \) then \( \frac{df}{dx} = 3 \) **our problem**

**b.** In part (a), we found that the derivative of \( f(x) = 3x + c \) is \( df/dx = 3 \). This means that the antiderivative of 3 is \( 3x + c \). Using the antiderivative symbol, we have

\[
\int 3 \, dx = 3x + c
\]

The relationship between the above derivative and antiderivative statements is illustrated in Figure 13.71.

**EXAMPLE 2**

**SOLUTION**

**EXAMPLE 3**

**SOLUTION**

**FINDING AN ANTIDERIVATIVE**

Find \( \int 8x^3 \, dx \).

To find the antiderivative of \( 8x^3 \) is to find the function whose derivative is \( 8x^3 \). What could we take the derivative of and get \( 8x^3 \)? Finding it involves a series of successive guesses.

**First Guess** It would have to involve an \( x^4 \), because the derivative of \( x^4 \) is \( 4x^3 \) (using the power rule). However, we don’t want the derivative to be \( 4x^3 \); we want it to be \( 8x^3 \). We’ve got the correct power but an incorrect coefficient.

**Second Guess** It must be \( 2x^4 \), because the derivative of \( 2x^4 \) is \( 2 \cdot 4x^3 = 8x^3 \) (using the power rule). This seems to be the right answer. However, the derivative of \( 2x^4 + 4 \) is \( 8x^3 \), and the derivative of \( 2x^4 + 19 \) is \( 8x^3 \), because the derivative of any constant is 0.

**Answer** It must be \( 2x^4 + c \), because the derivative of \( 2x^4 + c \) is \( 8x^3 + 0 = 8x^3 \). Thus, \( \int 8x^3 \, dx = 2x^4 + c \).
EXAMPLE

FINDING ANOTHER ANTIDERIVATIVE

Find \( \int (5x^7 + 3x + 7) \, dx \).

To find the antiderivative of \( 5x^7 + 3x + 7 \) is to find the function whose derivative is \( 5x^7 + 3x + 7 \). What could we take the derivative of and get \( 5x^7 + 3x + 7 \)?

**SOLUTION**

**First Guess** The goal of the first guess is to get the powers correct. Since taking a derivative decreases the powers by 1, our antiderivative would have to involve \( x^8 + x^2 + x^1 \), because the derivative of \( x^8 + x^2 + x^1 \) is

\[
8x^7 + 2x^1 + 1x^0 = 8x^7 + 2x + 1 \quad \text{using the power rule}
\]

However, we don’t want the derivative to be

\[
8x^7 + 2x + 1
\]

we want it to be

\[
5x^7 + 3x + 7
\]

We’ve got the correct powers but incorrect coefficients.

**Second Guess** The goal of the second guess is to make the coefficients correct. If we put \( \frac{5}{8} \) in front of the \( x^8 \), then the eights will cancel and the 5 will remain: The derivative of \( \frac{5}{8}x^8 \) is \( \frac{5}{8} \cdot 8x^7 = 5x^7 \). Similarly, we must put \( \frac{3}{2} \) in front of the \( x^2 \) and a 7 in front of the \( x^1 \). Thus, the antiderivative of \( 5x^7 + 3x + 7 \) must be \( \frac{5}{8}x^8 + \frac{3}{2}x^2 + 7x \), since the derivative of \( \frac{5}{8}x^8 + \frac{3}{2}x^2 + 7x \) is

\[
\frac{5}{8} \cdot 8x^7 + \frac{3}{2} \cdot 2x + 7 = 5x^7 + 3x + 7
\]

**Answer** The antiderivative must be \( \frac{5}{8}x^8 + \frac{3}{2}x^2 + 7x + c \), because the derivative of \( \frac{5}{8}x^8 + \frac{3}{2}x^2 + 7x + c \) is \( 5x^7 + 3x + 7 + 0 = 5x^7 + 3x + 7 \). Thus,

\[
\int (5x^7 + 3x + 7) \, dx = \frac{5}{8}x^8 + \frac{3}{2}x^2 + 7x + c
\]

**Antiderivatives and Areas**

Newton found that the derivative of the area function \( A(x) \) is \( f(x) \). In other words, he found that \( A(x) = \int f(x) \, dx \) [that the area function is the antiderivative of \( f(x) \)].

EXAMPLE

USING AN ANTIDERIVATIVE TO FIND AN AREA

a. Use antiderivatives to find the area function associated with \( f(x) = x + 2 \).

b. Find the area of a region bounded below by the \( x \)-axis, above by \( f(x) = x + 2 \), on the left by a vertical line at \( x = 1 \), and on the right by a vertical line at \( x = 2 \).

**SOLUTION**

a. \( A(x) = \int (x + 2) \, dx = ? \) What can we take the derivative of and get \( x + 2 \)?

**First Guess** Since taking a derivative decreases the power by 1, and since \( x + 2 = x^1 + 2x^0 \), we would have to take the derivative of \( x^1 + x^0 \). However, the derivative of \( x^2 + x^1 \) is \( 2x^1 + 1x^0 = 2x + 1 \).
Second Guess To make the coefficients right, we need a $\frac{1}{2}$ in front of the $x^2$ and a 2 in front of the $x^1$. The derivative of $\frac{1}{2}x^2 + 2x^1$ is

$$\frac{1}{2} \cdot 2x + 2 \cdot 1x^0 = x + 2$$

Answer The antiderivative of $f(x) = x + 2$ is $\frac{1}{2}x^2 + 2x + c$:

$$A(x) = \int (x + 2) \, dx = \frac{1}{2}x^2 + 2x + c$$

b. The area of a region bounded below by the $x$-axis, above by $f(x) = x + 2$, on the left by a vertical line at $x = 1$, and on the right by a vertical line at $x = 2$, is $A(2) - A(1)$, as shown in Figure 13.72.

We could find the area of the region in Example 5 without calculus, since the region is a rectangle topped with a triangle, as shown in Figure 13.73 on page 13-74. The rectangle’s base is $2 - 1 = 1$. Its height is the length of the vertical line at $x = 1$; that is, its height is $f(1)$. And since $f(x) = x + 2, f(1) = 1 + 2 = 3$. Thus, the rectangle’s area is

$$\text{Rectangle’s area} = b \cdot h = 1 \cdot 3 = 3$$

The triangle’s base is $2 - 1 = 1$. Its height is the length of the top part of the vertical line at $x = 2$; that is, its height is $f(2) - f(1) = 4 - 3 = 1$. Thus, the triangle’s area is

$$\text{Triangle’s area} = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

The area of our region is then

$$\text{Rectangle’s area + triangle’s area} = 3 + \frac{1}{2} = 3\frac{1}{2}$$

In the following example, $f(x)$ is a curve rather than a line, and we would be unable to find the area without calculus.
EXAMPLE 6

FINDING AREA Find the area of the region bounded above by the curve \( f(x) = 3x^2 + 1 \), below by the \( x \)-axis, and on the sides by

a. a vertical line at \( x = 1 \) and a vertical line at \( x = 2 \)

b. the \( y \)-axis and a vertical line at \( x = 2 \)

SOLUTION

a. The curve is a parabola; the region is illustrated in Figure 13.74. To find the area, we must first find the area function \( A(x) \) associated with \( f(x) = 3x^2 + 1 \). This antiderivative is \( x^3 + x + c \):

\[
A(x) = \int (3x^2 + 1) \, dx = x^3 + x + c
\]

Our region has a right-hand boundary at \( x = 2 \) and a left-hand boundary at \( x = 1 \), so we want \( A(2) - A(1) \), as shown in Figure 13.75.

\[
A(2) - A(1) = (2^3 + 2 + c) - (1^3 + 1 + c) = (10 + c) - (2 + c) = 8
\]

\[\text{FIGURE 13.74} \quad \text{Area} = A(2) - A(1) = 10 - 2 = 8.\]

\[\text{FIGURE 13.75} \quad \text{Finding the area in part (a).}\]
b. Our region has a right-hand boundary at $x = 2$ and a left-hand boundary at the $y$-axis, as shown in Figure 13.76. Its area is $A(2)$:

$$A(x) = x^3 + x + c$$
$$A(2) = 2^3 + 2 + c = 10 + c$$

This doesn’t do us any good because we don’t know what $c$ is! Rather than trying to find the value of $c$, we realize that we can view our left-hand boundary as the $y$ axis or as a line at $x = 0$, so we want $A(2) - A(0)$:

$$A(x) = x^3 + x + c$$
$$A(2) - A(0) = (2^3 + 2 + c) - (0^3 + 0 + c)$$
$$= 8 + 2 + c - c$$
$$= 10$$

**Area Steps**

To find the area of a region bounded below by the $x$-axis, above by the curve $y = f(x)$, and on the left and right by vertical lines at $a$ and at $b$:

1. Find $A(x) = \int f(x) \, dx$, the antiderivative of $f(x)$.
2. Find $A(b) - A(a) = A(\text{right boundary}) - A(\text{left boundary})$.

**13.6 Exercises**

In Exercises 1–8, do the following.

a. Find the area function associated with the given function.

b. Sketch the region whose area is found by that area function.

c. Find $A(3)$, and describe the region whose area is $A(3)$.

d. Find $A(4)$, and describe the region whose area is $A(4)$.

e. Find $A(4) - A(3)$, and describe the region whose area is $A(4) - A(3)$.

1. $f(x) = 2$  
2. $f(x) = 6$  
3. $f(x) = 2x$  
4. $f(x) = 4x$  
5. $f(x) = x + 2$  
6. $f(x) = x + 1$  
7. $f(x) = 2x + 3$  
8. $f(x) = 3x + 1$

In Exercises 9–12, find the antiderivative of the given function.

9. $f(x) = 5$  
10. $f(x) = 9$  
11. $f(x) = 6x + 4$  
12. $f(x) = 10x - 13$

► Selected exercises available online at [www.webassign.net/brookscole](http://www.webassign.net/brookscole)
In Exercises 13–16, do the following.

a. Use the results of Exercises 9–12 to find the area of a region bounded below by the x-axis, above by the line \( y = f(x) \), and on each side by the given vertical lines.
b. Use triangles and rectangles to find the area of the same region.

13. \( f(x) = 5 \), vertical lines at 2 and 7.
14. \( f(x) = 9 \), vertical lines at 3 and 11.
15. \( f(x) = 6x + 4 \), vertical lines at 0 and 5.
16. \( f(x) = 10x - 13 \), vertical lines at 4 and 20.

In Exercises 17–20, find the antiderivative of the given function.

17. \( g(x) = 8x + 7 \)
18. \( g(x) = 9x - 3 \)
19. \( g(x) = 3x^2 - 2x + 5 \)
20. \( g(x) = 4x^2 + 2x + 1 \)

In Exercises 21–24, use the results of Exercises 17–20 to find the area of a region bounded below by the x-axis, above by the curve \( y = g(x) \), and on each side by the given vertical lines.

21. \( g(x) = 8x + 7 \), vertical lines at 8 and 11.
22. \( g(x) = 9x - 3 \), vertical lines at 2 and 5.
23. \( g(x) = 3x^2 - 2x + 5 \), vertical lines at 0 and 7.
24. \( g(x) = 4x^2 + 2x + 1 \), vertical lines at 1 and 12.

In Exercises 25–32, find (a) the derivative of the given function, and (b) the antiderivative of the given function.

25. \( f(x) = x \)
26. \( g(x) = 9 \)
27. \( h(x) = 5 \)
28. \( k(x) = 2x \)
29. \( m(x) = 8x \)
30. \( n(x) = 9x^3 - 7 \)
31. \( p(x) = 3x^3 + 4 \)
32. \( q(x) = 4x^3 - 3x \)

In Exercises 33–39, find the area of the region bounded below by the x-axis, above by the given function, and on each side by the given vertical lines.

33. \( f(x) = 3x^2 \), vertical lines at
   a. 1 and 4
   b. 3 and 100
34. \( f(x) = 3x^2 + 5 \), vertical lines at
   a. 0 and 6
   b. 2 and 12
35. \( f(x) = 6x^2 + 4 \), vertical lines at
   a. 2 and 6
   b. 5 and 17
36. \( f(x) = 9x^2 + 7 \), vertical lines at
   a. 1 and 2
   b. 3 and 20
37. \( f(x) = 5x^6 + 3 \), vertical lines at
   a. 4 and 7
   b. 3 and 10
38. \( f(x) = 4x^7 + 2 \), vertical lines at
   a. 3 and 11
   b. 4 and 10
39. \( f(x) = 3x^4 - \frac{1}{2}x^3 - 5x + 11 \),
   a. vertical lines at 2 and 19
   b. the y-axis and a vertical line at 5
40. a. Use the procedure from Example 3 to determine a formula for \( \int x^n \, dx \).
    HINT: Follow the steps used in Example 3, except use an \( n \) in place of 3.
b. Check your answer to part (a) by taking its derivative.
c. Use the procedure from Example 3 to determine a formula for \( \int x^a \, dx \).
d. Check your answer to part (c) by taking its derivative.

41. Use the formula developed in Exercise 40 to find the antiderivative of the function in Exercise 17.
42. Use the formula developed in Exercise 40 to find the antiderivative of the function in Exercise 18.
43. Use the formula developed in Exercise 40 to find the antiderivative of the function in Exercise 19.
44. Use the formula developed in Exercise 40 to find the antiderivative of the function in Exercise 20.
45. Use the formula developed in Exercise 40 to find the antiderivative of \( f(x) = 5x^6 - 3x^2 + 13 \).
46. Use the formula developed in Exercise 40 to find the antiderivative of \( g(x) = 9x^7 + 13x^4 + 7 \).
47. Use the formula developed in Exercise 40 to find the antiderivative of \( h(x) = 18/x^2 \).
   HINT: Use exponent rules to rewrite the function without a quotient.
48. Try to use the formula developed in Exercise 40 to find the antiderivative of \( k(x) = \frac{1}{x} + 3 \). Why does the formula fail?

Answer the following question using complete sentences and your own words.

**CONCEPT QUESTION**

49. Of Exercises 21–24, which could be done with triangles and rectangles rather than antiderivatives? Which could not be done with triangles and rectangles? Why?

### 13.7 Conclusion

We have briefly investigated several of the more common applications of calculus. Each of the applications we explored represents only the tip of an iceberg.
**Rates of Change**

Newton’s method of fluxions and evanescent increments was motivated in part by the problem of finding the speed of a falling object. One use of calculus is to determine the rate at which things change, and “speed” is the rate at which distance changes.

Calculus can be used to determine many other rates of change, including the rate at which a population grows (“growth rate”), the rate at which a manufacturing firm’s profit changes as its production increases (“marginal profit”), and the rate at which water flows through a river (“flow rate”).

**Sketching Curves**

Newton’s method of fluxions was also motivated by the problem of finding the slope of a tangent line. The ability to find this slope leads to a powerful method of sketching curves that involves plotting certain key points.

In Section 13.5, we sketched parabolas to answer questions concerning the trajectory of a cannonball. We did this by plotting key points that were important in analyzing the trajectory. In particular, we plotted the points at which the curve intersects the x- and y-axes and the vertex, which we found by determining where the tangent line has a slope of zero (see Figure 13.77).

By expanding on this procedure, we can sketch the graph of almost any curve—for example, the one in Figure 13.78. This is an important skill in many fields, including business and economics, in which graphs are sketched to analyze a firm’s profit and to analyze the relationship between the sales price of a product and its success in the marketplace.

**Example 1**

**Sketching a Graph**

a. Find where the graph of $f(x) = 2x^3 + 3x^2 - 12x$ intersects the y-axis by substituting 0 for $x$.

b. Find where the graph of $f(x) = 2x^3 + 3x^2 - 12x$ intersects the x-axis by substituting 0 for $f(x)$ and solving for $x$.

c. Find where the graph of $f(x) = 2x^3 + 3x^2 - 12x$ has a vertex by determining where the tangent line has a slope of zero.

d. Use the results of parts (a), (b), and (c) to sketch the graph of $f(x) = 2x^3 + 3x^2 - 12x$. 
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SOLUTION

a. \( f(x) = 2x^3 + 3x^2 - 12x \)
\[ f(0) = 2 \cdot 0^3 + 3 \cdot 0^2 - 12 \cdot 0 = 0 \quad \text{substituting 0 for } x \]
The graph of \( f(x) = 2x^3 + 3x^2 - 12x \) intersects the \( y \)-axis at the point (0, 0), as shown in Figure 13.79.

b. \( f(x) = 2x^3 + 3x^2 - 12x \)
\[ 0 = 2x^3 + 3x^2 - 12x \]
\[ 0 = x(2x^2 + 3x - 12) \]
either \( x = 0 \) or \( 2x^2 + 3x - 12 = 0 \)
either \( x = 0 \) or \( x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot -12}}{2 \cdot 2} \) \quad \text{using the quadratic formula as discussed in Section 13.5}
either \( x = 0 \) or \( x = 1.81173769149 \ldots \) or \( x = -3.31173769149 \ldots \)
either \( x = 0 \) or \( x \approx 1.8 \) or \( x \approx -3.3 \)
The graph of \( f(x) = 2x^3 + 3x^2 - 12x \) intersects the \( x \)-axis at the points (0, 0), (1.8, 0), and (-3.3, 0) (approximately), as shown in Figure 13.79.
13.7 Conclusion

c. Rather than taking the derivative of \( f(x) \) with Newton’s or Cauchy’s method, we can use the power rule and the linear function rule.

The derivative of \( 12x^3 + 3x^2 - 12x + 0 \) using the power rule and the linear function rule is \( 3 \cdot 12x^2 + 2 \cdot 3x - 12 \).

By comparing the formula with our problem, we can see that the “a” in our formula matches “2,” the “b” matches “3,” the “c” matches “-12,” and the “d” matches “0.” Thus, the derivative of \( f(x) = 2x^3 + 3x^2 - 12 \) is \( df/dx = 6x^2 + 6x - 12 \).

The graph of \( f(x) = 2x^3 + 3x^2 - 12x \) has a vertex where the tangent line has a slope of zero.

\[
slope \text{ of tangent} = 0
\]
\[
df/dx = 6x^2 + 6x - 12 = 0
\]
\[
x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 6 \cdot -12}}{2 \cdot 6}
\]
\[
= 1 \text{ or } -2
\]

The graph of \( f(x) = 2x^3 + 3x^2 - 12x \) has two vertices: one at \( x = 1 \) and a second at \( x = -2 \). And since \( f(1) = 2(1)^3 + 3(1)^2 - 12(1) = -7 \) and \( f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) = 20 \), the vertices are at the points \( (1, -7) \) and \( (-2, 20) \), as shown in Figure 13.79.

d. The graph of \( f(x) = 2x^3 + 3x^2 - 12x \) is shown in Figure 13.79 on page 13-78.

Finding Maximum and Minimum Values

In Section 13.5, we found the maximum height attained by a cannonball. In many fields, finding where a function has its maximum or minimum value is important. In business, it’s important to determine the circumstances under which a company’s profit will be maximized and its costs minimized. In engineering, it is important to determine the maximum weight a structure can support.

The basic principle involved in finding the maximum or minimum value of a function is to find where the tangent line has a slope of zero, as we did when we found the maximum height attained by a cannonball. When we found that maximum height, we knew that the graph was that of a parabola; in most situations, the shape of the curve is not known, and it will be necessary to do more work than simply finding where the tangent line has a slope of zero.

Motion

Newton’s method of fluxions was motivated in part by the problem of finding the path followed by a cannonball as well as by the problem of finding the paths followed by the planets. Calculus can be used to analyze any type of motion, including that of the space shuttle as it leaves the earth and goes into orbit and that of an automobile negotiating a tight curve. Such analyses are important in designing the space shuttle or the automobile in order to develop a vehicle that can withstand the pressures that will be exerted on it and perform as required.
Areas and Volumes

In Section 13.6, we found the areas of shapes bounded by one curve and some straight lines. We did not explore the techniques involved in finding areas of more complicated shapes. If the region is bounded by more than one function, you compute the antiderivative of each function.

In Figure 13.80, the area of the region is the sum of the area of the left side and the area of the right side. The left side’s area is \( A(3) - A(1), \) where \( A(x) = \int f(x) \, dx. \) The right side’s area is \( A(5) - A(3), \) where \( A(x) = \int g(x) \, dx. \)

![Figure 13.80 Finding the area, with two curves side-by-side.](image)

In Figure 13.81, the area of the region is the area of the top part minus the area of the bottom part. The top part’s area is \( A(8) - A(2), \) where \( A(x) = \int f(x) \, dx. \) The bottom part’s area is \( A(8) - A(2), \) where \( A(x) = \int g(x) \, dx. \)

![Figure 13.81 Finding the area with two curves one above the other.](image)

Summing Up

The point of this chapter is to explore concepts, not to develop a proficiency in calculus. We have investigated the main ideas of this subject, but of course there is much more to calculus than what we have seen here.
13.7 Exercises

• Projects

1. The distance traveled by a falling object is \( d = 16t^2 \), if the distance \( d \) is in feet. Only the United States uses feet to measure distance; the rest of the world uses meters and the metric system. If the distance is measured in meters rather than feet, the formula is almost the same; the only difference is the multiplier is not 16. That is, the distance traveled by a falling object is \( d = k \cdot t^2 \), where \( k \) is some number other than 16, if the distance \( d \) is in meters. In this exercise we will find \( k \).
   a. Complete the chart in Figure 13.82, using dimensional analysis (see Appendix E) and the fact that 1 meter \( \approx 3.28 \) feet.
   b. In the equation \( d = k \cdot t^2 \), substitute 1 for \( t \) and the appropriate distance in meters (from part a) for \( d \), and solve for \( k \).
   c. In the equation \( d = k \cdot t^2 \), substitute 2 for \( t \) and the appropriate distance in meters (from part a) for \( d \), and solve for \( k \).
   d. In the equation \( d = k \cdot t^2 \), substitute 3 for \( t \) and the appropriate distance in meters (from part a) for \( d \), and solve for \( k \).
   e. Use the results of parts (b), (c), and (d) to find the metric system version of the formula \( d = 16t^2 \).
   f. Use derivatives and the result of part (e) to find the metric system equation for the speed \( s \) of a falling object. What units would \( s \) be in?

<table>
<thead>
<tr>
<th>Time in Seconds</th>
<th>Distance Traveled in Feet</th>
<th>Distance Traveled in Meters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 sec</td>
<td>( 16 \cdot 1^2 = 16 ) ft</td>
<td>(convert 16 feet to meters)</td>
</tr>
<tr>
<td>2 sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 sec</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**FIGURE 13.82** A chart for Exercise 1.

2. a. Find where the graph of \( f(x) = -x^4 + 8x^2 \) intersects the \( y \)-axis by substituting 0 for \( x \).
   b. Find where the graph of \( f(x) = -x^4 + 8x^2 \) intersects the \( x \)-axis by substituting 0 for \( f(x) \) and solving for \( x \).
   c. Find where the graph of \( f(x) = -x^4 + 8x^2 \) has a vertex by determining where the tangent line has a slope of zero.

3. We will find this area.
   a. Find the area of the left side, as shown below.
   b. Find the area of the right side, as shown below.
   c. Find the area of the entire region by combining the answers to parts (a) and (b).

In Exercises 3–6, use antiderivatives to find the indicated area.

d. Use the results of parts (a), (b), and (c) to sketch the graph of \( f(x) = -x^4 + 8x^2 \).

e. By inspecting the graph from part (d), determine the maximum value of \( f(x) \).
4. We will find this area.

b. Find the area of the bottom part, as shown below.

\[ f(x) = x^2 - 12x + 54 \]
\[ g(x) = -x^2 + 12x \]

a. Find the area of the left part.

b. Find the area of the middle part.

c. Find the area of the right part.

d. Find the area of the entire region by combining the answers to parts (a), (b), and (c).

5. We will find this area.

b. Find the area of the entire region by combining the answers to parts (a) and (b).

\[ f(x) = x^2 - 8x + 17 \]
\[ g(x) = x + 9 \]

a. Find the area of the part shown below.

\[ f(x) = -x^2 + 6x + 20 \]
\[ g(x) = -x^2 + 12x - 4 \]

6. We will find this area.

a. Find the area of the left part.

b. Find the area of the right part.

c. Find the area of the entire region by combining the answers to parts (a) and (b).

\[ h(x) = x^2 - 7x + 13 \]
**TERMS**
- analytic geometry
- antiderivative
- average versus instantaneous growth rate
- average versus instantaneous marginal profit
- fluxion
- function
- functional notation
- infinitesimal
- line of symmetry of a parabola
- proportion
- rate
- rate of change
- ratio
- secant line
- similar triangles
- slope symmetrical
tangent line
- trajectory
- ultimate ratio of functions
- vertex of a parabola

**Review Exercises**

1. What role did the following people play in the development of calculus?
   - Isaac Barrow
   - George Berkeley
   - Augustin Louis Cauchy
   - René Descartes
   - Pierre de Fermat
   - Galileo Galilei
   - Mohammed ibn Musa al-Khowarizmi
   - Gottfried Leibniz
   - Isaac Newton
   - Nicole Oresme
   - François Viète

2. Find the unknown lengths in Figure 13.83.

3. When Jorge Gonzalez filled his car’s tank with gas, the odometer read 4,215.6 miles. Later, he added 8.2 gallons, and the odometer read 4,465.1 miles.
   a. Find the ratio of miles traveled to gallons consumed.
   b. Predict the number of miles that he could travel on a full tank if his tank holds 15 gallons.

4. \( f(x) = 8x^2 - 7 \)
   a. Find \( f(-3) \)
   b. Find \( f(x + \Delta x) \).

5. An object is moving in such a way that its speed increases uniformly from the moment that motion begins. After 30 seconds, the object is moving at a rate of 8 feet per second. Use Oresme’s triangle to find the following:
   a. the distance it has traveled in those 30 seconds
   b. its speed after 25 seconds
   c. the distance it has traveled in 25 seconds
   d. its average speed during its first 25 seconds of travel

6. Using a water clock to time a ball rolling down a ramp, Galileo found that if a ball was timed during two different trips, the following relationship held:

   \[
   \frac{\text{distance from first trip}}{\text{distance from second trip}} = \frac{(\text{weight of water from first trip})^2}{(\text{weight of water from second trip})^2}
   \]

   If a ball would roll 100 feet while 5 ounces of water flowed out of a water clock, how far would it roll while 10 ounces of water flowed from the clock?

7. Given \( y = 2x^2 \), do the following.
   a. Sketch the parabola given by the equation.
   b. At \( x = 1 \) and \( x = 2 \), compute the length of the line segment and the area of the square described by Apollonius. Sketch these two line segments and two squares in their proper places, superimposed over the parabola.
   c. Compute the ratio of the lengths of the two line segments and the ratio of the areas of the two squares in part (b).
   d. At \( x = 3 \) and \( x = 4 \), compute the length of the line segment and the area of the square described by Apollonius. Sketch these two line segments and two squares in their proper places, superimposed over the parabola.
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e. Compute the ratio of the lengths of the two line segments and the ratio of the areas of the two squares in part (d).
8. A square and 10 roots are equal to 11 units.
   a. Solve the problem using the method of al-Khowarizmi. Do not use modern terminology and notation.
   b. Illustrate your solution to the problem in the manner of al-Khowarizmi.
   c. Give a modern version of your solution from part (a).
9. Given \( y = x^2 + 4x + 11 \), do the following.
   a. Graph the parabola.
   b. Give the equation of its line of symmetry.
   c. Find its vertex.
   d. Find the slope of the secant line that intersects the parabola at \( x = 1 \) and \( x = 2 \).
   e. Find the slope of the tangent line at \( x = 1 \).
   f. Find the equation of the tangent line at \( x = 1 \).
   g. Graph the secant line from part (d) and the tangent line from parts (e) and (f).
10. Calculate the slope of the line tangent to \( y = 3x^2 - 1 \) at the point where \( x = 2 \), using the following methods. Find the equation of the tangent line and sketch the graph of the curve and the tangent line.
   a. Barrow’s method
   b. Newton’s method
   c. Cauchy’s method
11. You drop a rock off a bridge. After 2.5 seconds, you hear it hit the water below.
   a. How far does the rock fall?
   b. At what speed is the rock moving when it hits the water?
   c. What is the average speed of the rock during its fall?

In Exercises 12–14, find the derivative of the given function, by calculating

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

12. \( f(x) = 3 \)
13. \( f(x) = 2x - 15 \)
14. \( f(x) = x^2 - 3x + 5 \)
15. Craig James is dieting. If \( f(x) \) measures Craig’s weight \( x \) weeks after he started to diet, find what is measured by each of the following. (Include units with each answer.)
   a. \( \frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \)
   b. \( \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \)
16. In October 1965, the state Department of Fish and Game started to keep records concerning the number of bass in Ocean Bay. A biologist has used these records to create a function \( f(x) \) that measures the number of bass in the bay \( x \) weeks after October 1, 1965. Find what is measured by each of the following. (Include units with each answer.)
   a. \( \frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \)
   b. \( \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \)
17. A cannonball is shot with an initial speed of 200 feet per second from a cannon whose muzzle is 4.5 feet above the ground at a 30° angle.
   a. Find the equation of the path of the cannonball.
   b. How far will the ball travel?
   c. What is the maximum height the ball will attain?
   d. Will the ball clear a 45-foot-high wall that is 100 feet from the cannon?
   e. Sketch the cannonball’s path. In your sketch, include everything that is known about the path.
18. Find the antiderivative of \( f(x) = 3x - 2 \).
19. Find the area of a region bounded below by the \( x \)-axis, above by the curve \( y = 3x^2 + 7 \), and on the sides by vertical lines at 1 and 5.

**HISTORY QUESTIONS**

20. What were the four problems that led to the invention of calculus? Who worked on these problems prior to Newton? What did they discover?
21. How did Galileo use Oresme’s triangle to investigate the distance traveled by a falling object? How did he use it to investigate the speed of a falling object?
22. Why was Galileo unable to find the equation of the trajectory of a cannonball?
23. What did Apollonius contribute to analytic geometry?
24. What did Oresme contribute to analytic geometry?
25. What did Descartes contribute to analytic geometry?
26. What did Fermat contribute to analytic geometry?
27. What did Viète contribute to algebra?
28. What did Descartes contribute to algebra?
29. What changes did mathematics need to undergo before calculus could be invented? Why were these changes necessary?
30. Why didn’t Newton initially publish his works on calculus? What finally prompted him to publish them?