APPENDIX A EXERCISES

In Exercises 1–12, list the all of the elements of the given set.

1. The set of all prime numbers less than 20
2. The set of all positive integers whose square roots are less than or equal to 3
3. The set of all positive integers less than 5
4. The set of all nonnegative integers less than 5
5. The set of all positive rational numbers (in lowest terms) whose denominator is less than or equal to 5
6. The set of all positive integers less than 100 that are both a perfect square and a perfect cube
7. \( \{ n : n = 3k + 1, k \in \mathbb{Z}, 0 \leq k \leq 5 \} \)
8. \( \left\{ \frac{m}{2^n} : m, n \in \mathbb{Z}^+, 1 \leq n \leq 3, m < 2^n \right\} \)
9. \( \{ \sin(\pi/n) : n \in \mathbb{Z}, 1 \leq n \leq 4 \} \)
10. \( \{ x \in \mathbb{R} : x = 2^i \} \)
11. \( \{ x \in \mathbb{Q} : 2x^2 - x - 1 = 0 \} \)
12. \( \{ x \in \mathbb{Z} : 2x^2 - x - 1 = 0 \} \)

In Exercises 13–24, write the given set using set builder notation.

13. Exercise 1
14. Exercise 2
15. Exercise 3
16. Exercise 4
17. Exercise 5
18. Exercise 6
In Exercises 25–32, list all of the elements of \( A \cap B \) and \( A \cup B \).

25. \( A = \{1, 2, 4, 8, 16, 32\}, \ B = \{0, 4, 8, 12, \ldots, 32\} \)

26. \( A = \{3, 6, 9, \ldots, 99\}, \ B = \{1, 4, 9, 16, \ldots, 169\} \)

27. \( A = \{n^2 : n \in \mathbb{Z}, 1 \leq n \leq 8\}, \ B = \{4k + 1 : k \in \mathbb{Z}, 0 \leq k \leq 10\} \)

28. \( A = \{2^n : n \in \mathbb{Z}, 1 \leq n \leq 6\}, \ B = \{4k + 1 : k \in \mathbb{Z}, 0 \leq k \leq 10\} \)

29. List all of the subsets of \( \{1, 2\} \).

30. List all of the subsets of \( \{1, 2, 3\} \).

31. List all of the subsets of \( \{1, 2, 3, 4\} \).

32. Based on your answers to Exercises 29–31, how many subsets will an \( n \)-element set have? (If you are familiar with mathematical induction (Appendix B), try to prove your conjecture using induction.)

In Exercises 33–35, show that \( A \subseteq B \) as in Example A.2(c) or A.3.

33. \( A \) is the set of all positive integers whose last (ones) digit is 6,
\( B = \{n : n = 2k, k \in \mathbb{Z}^+\} \)
34. $A$ is the set of all positive integers whose last (ones) digit is even, 
   \[ B = \{ n : n = 2k, k \in \mathbb{Z}^+ \} \]

35. $A = \{(x, y, z) : x = a^2 - b^2, y = 2ab, z = a^2 + b^2, a, b, c \in \mathbb{Z}^+, a > b \}$, 
   \[ B = \{(x, y, z) : x, y, z \in \mathbb{Z}^+, x^2 + y^2 = z^2 \} \]

36. Show that we also have $B \subseteq A$ in Exercise 34. Deduce that $A = B$.

In Exercises 37–42, expand the summations (i.e., rewrite the sums without sigma notation). Then evaluate each sum.

37. $\sum_{i=1}^{4} (i^2 + 1)$

38. $\sum_{k=1}^{3} (3^k - 2^k)$

39. $\sum_{k=1}^{5} \frac{2k}{k+1}$

40. $\sum_{i=1}^{10} 1$

41. $\sum_{j=1}^{6} \sin \left( \frac{j\pi}{6} \right)$

42. $\sum_{n=1}^{5} \left( \frac{1}{n} - \frac{1}{n+1} \right)$

In Exercises 43–46, rewrite each sum using summation notation so that the index of summation starts at $i = 1$.

43. $1 + x + x^2 + x^3 + x^4$

44. $1 - 2 + 3 - 4 + \cdots + 9 - 10$

45. $1 + 1 + \frac{3}{4} + \frac{1}{2} + \frac{5}{16} + \cdots + \frac{9}{256}$

46. $100 + \frac{81}{2} + \cdots + \frac{9}{8} + \frac{4}{9} + \frac{1}{10}$
47. Write out \( \sum_{i,j=1}^{3} (2i + j) \) using both possible orders of summation.

48. Write out \( \sum_{i,j=1}^{3} \frac{i}{j} \) using both possible orders of summation.

In Exercises 49–54, give a direct proof of the given assertion.

49. If \( a \) and \( b \) are odd integers, then \( ab \) is an odd integer.

50. If \( a \) is an odd integer, then \( a^3 \) is odd.

51. If \( a \) is an integer, then \( a^2 - a + 1 \) is always odd.

52. If \( a \) is a positive integer whose ones digit is either 0 or 5, then \( a \) is a multiple of 5.

53. Let \( n \) be a positive two-digit integer and let \( n' \) be the positive integer obtained by reversing the digits of \( n \). (For example, if \( n = 73 \) then \( n' = 37 \).) Prove that \( n - n' \) is a multiple of 9.

54. Let \( a \) and \( b \) be integers such that \( a + b + ab = 0 \). Prove that either \( a = b = 0 \) or \( a = b = -2 \). [Hint: Try adding 1 to both sides of the given equation.]

55. Let \( S = \{ x \in \mathbb{R} : x = a + b\sqrt{2}, a, b \in \mathbb{Z} \} \). Prove that if \( x \) and \( y \) are in \( S \), then so is \( xy \).

56. Let \( T = \{ x \in \mathbb{R} : x = a + b\sqrt{2}, a, b \in \mathbb{Q} \} \). Prove that if \( x \in T \) and \( x \neq 0 \), then \( 1/x \in T \).

In Exercises 57–66, give an indirect proof (by contradiction or contrapositive) of the given assertion.

57. If \( a \) is a rational number and \( b \) is an irrational number, then \( a + b \) is an irrational number.

58. If \( x^2 + y^2 = 0 \) for real numbers \( x \) and \( y \), then \( x = 0 \) and \( y = 0 \).

59. If \( x^2 \geq 1 \) for \( x \in \mathbb{R} \), then \( x \geq 1 \) or \( x \leq -1 \).

60. A mathematics class has \( n \) tutorial sections. Prove that if \( 2n + 1 \) students sign up for tutorials, then some tutorial section has at least 3 students.
61. The grades on a mathematics quiz written by six students turn out to be distinct positive integers. Prove that if the average grade is 20, then at least one of the grades is greater than 22.

62. If \( n = ab \) is a product of positive integers \( a \) and \( b \), then either \( a \leq \sqrt{n} \) or \( b \leq \sqrt{n} \).

63. Prove that \( \sqrt{2} \) is irrational.

64. Prove that in any group of six people there is either a set of three mutual acquaintances (every pair of people in the set is acquainted) or a set of three mutual non-acquaintances (no pair of people in the set is acquainted).

65. Every point in the plane has been coloured with one of two colours. Prove that for any positive real number \( d \) there must be two points of the same colour that are exactly \( d \) units apart.

66. Let \( p(x) = x^2 + ax + b \) be a polynomial with real coefficients \( a \) and \( b \), and let \( r_1 \) and \( r_2 \) be the zeros of \( p(x) \). (See Appendix D.) Prove that if either \( a \) or \( b \) is irrational, then at least one of \( r_1 \) or \( r_2 \) is irrational.

67. Prove that \( a \) is an even integer if and only if \( a + 1 \) is odd.

68. Let \( a \) be a positive integer. Prove that \( a \) is an integer multiple of 5 if and only if the ones digit of \( a \) is either 0 or 5. (See Exercise 52.)

69. Let \( x \) and \( y \) be real numbers. Prove that \( |x| = |y| \) if and only if \( x = y \) or \( x = -y \).

70. A triangular number is an integer \( t \) that can be written in the form \( t = \frac{a^2 + a}{2} \) for some positive integer \( a \). Prove that a positive integer \( n \) is the sum of two triangular numbers if and only if \( 4n + 1 = x^2 + y^2 \) for some integers \( x \) and \( y \).