7.9 Compounding and Its Applications†

The univariate probability distributions of Chapters 5 and 6 depend on one or more parameters; once the parameters are known, the distributions are completely specified. However, these parameters are frequently unknown and, as in Example 7.20, may sometimes be regarded as random quantities. The process of assigning distributions to these parameters and then finding the marginal distributions of the original random variable is known as compounding. This process has theoretical as well as practical uses, as we illustrate next.

Example 7.21

Suppose that $Y$ denotes the number of bacteria per cubic centimeter in a certain liquid and that, for a given location, $Y$ has a Poisson distribution with mean $\lambda$. Also, assume that $\lambda$ varies from location to location and, for a location chosen at random, that $\lambda$ has a gamma distribution with parameters $\alpha$ and $\beta$, where $\alpha$ is a positive integer. Find the probability distribution for the bacteria count $Y$ at a randomly selected location.

Solution

Because $\lambda$ is random, the Poisson assumption applies to the conditional distribution of $Y$ for fixed $\lambda$. Thus,

$$p(y \mid \lambda) = \frac{\lambda^y e^{-\lambda}}{y!} \quad y = 0,1,2,\cdots$$

Also,

$$f(\lambda) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} & \lambda > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Then the joint distribution of $\lambda$ and $Y$ is given by

$$g(y,\lambda) = p(y \mid \lambda)f(\lambda)$$

$$= \frac{1}{y! \Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda[1+(1/\beta)]}$$

The marginal distribution of $Y$ is found by integration over $\lambda$ and yields

$$p(y) = \frac{1}{y! \Gamma(\alpha)\beta^\alpha} \int_0^\infty \lambda^{\alpha-1} e^{-\lambda[1+(1/\beta)]} d\lambda$$

$$= \frac{1}{y! \Gamma(\alpha)\beta^\alpha} \Gamma(y + \alpha) \left(1 + \frac{1}{\beta}\right)^{-(y+\alpha)}$$

†Optional section.
Because \( \alpha \) is an integer,

\[
p(y) = \frac{(y + \alpha - 1)!}{(\alpha - 1)!y!} \left( \frac{1}{\beta} \right)^\alpha \left( \frac{\beta}{1 + \beta} \right)^{y+\alpha}
\]

\[
= \frac{(y + \alpha - 1)}{\alpha - 1} \left( \frac{1}{1 + \beta} \right)^\alpha \left( \frac{\beta}{1 + \beta} \right)^y
\]

If we let \( y + \alpha = n \) and \( 1/(1 + \beta) = p \), then \( p(y) \) has the form of a negative binomial distribution. Hence, the negative binomial distribution is a reasonable model for counts in which the mean count may be random.

**Example 7.22**

Suppose a customer arrives at a checkout counter in a store just as the counter is opening. A random number \( N \) of customers will be ahead of him because some customers may arrive before the counter opens. Suppose this number has the probability distribution

\[
p(n) = P(N = n) = pq^n \quad n = 0, 1, 2, \cdots
\]

where \( 0 < p < 1 \) and \( q = 1 - p \). (This is a form of the geometric distribution.)

Customer service times are assumed to be independent and identically distributed exponential random variables with mean \( \theta \). Find the expected time for this customer to complete his check out. (This is a general model that could apply to telephone calls and other typical “waiting lines.”)

**Solution**

For a given value of \( n \), the waiting time \( W \) is the sum of \( n + 1 \) independent exponential random variables and thus has a gamma distribution with \( \alpha = n + 1 \) and \( \beta = \theta \). That is,

\[
f(w | n) = \frac{1}{\Gamma(n + 1)\theta^{n+1}} w^n e^{-w/\theta}
\]

Hence,

\[
f(w, n) = \frac{p}{\Gamma(n + 1)\theta^{n+1}} (qw)^n e^{-w/\theta}
\]

and

\[
f(w) = \frac{p}{\theta} e^{-w/\theta} \sum_{n=0}^{\infty} \left( \frac{qw}{\theta} \right)^n \frac{1}{n!} = \frac{p}{\theta} e^{-w/\theta} e^{qw/\theta}
\]

\[
= \frac{p}{\theta} e^{-(w/\theta)(1-q)} = \frac{p}{\theta} e^{-w(p/\theta)}
\]

The waiting time \( W \) is still exponential, but the mean is \( (\theta/p) \).
Supplementary Exercises

7.63 The number of defects per yard, denoted by \( X \), for a certain fabric is known to have a Poisson distribution with parameter \( \lambda \). However, \( \lambda \) is not known and is assumed to be random with a probability density function given by

\[
f(\lambda) = \begin{cases} e^{-\lambda} & \lambda \geq 0 \\ 0 & \text{elsewhere} \end{cases}
\]

Find the unconditional probability function for \( X \).

7.64 Let \( X_1, X_2, \) and \( X_3 \) be random variables, either continuous or discrete. The joint moment-generating function of \( X_1, X_2, \) and \( X_3 \) is defined by

\[
M(t_1, t_2, t_3) = E(e^{t_1 X_1 + t_2 X_2 + t_3 X_3})
\]

a) Show that \( M(t, t, t) \) gives the moment-generating function of \( X_1 + X_2 + X_3 \).

b) Show that \( M(t, t, 0) \) gives the moment-generating function of \( X_1 + X_2 \).

c) Show that

\[
\frac{\partial^{k_1+k_2+k_3} M(t_1, t_2, t_3)}{\partial t_1^{k_1} \partial t_2^{k_2} \partial t_3^{k_3}} \bigg|_{t_1=t_2=t_3=0} = E(X_1^{k_1} X_2^{k_2} X_3^{k_3})
\]

7.65 Let \( X_1, X_2, \) and \( X_3 \) have a multinomial distribution with the probability function

\[
p(X_1, X_2, X_3) = \frac{n!}{X_1! X_2! X_3!} p_1^{X_1} p_2^{X_2} p_3^{X_3} \sum_{i=1}^{n} x_i = n
\]

Use the results of Exercise 7.64 to answer the following:

a) Find the joint moment-generating function of \( X_1, X_2, \) and \( X_3 \).

b) Use the joint moment-generating function to find \( \text{Cov}(X_1, X_2) \).