Following are derivations for annuity formulas. It is actually easier to start with the formula for a perpetuity. First, consider the following geometric progression, where $A$ is a positive constant that is less than 1, and $X$ is the sum of the geometric progression:

$$X = \sum_{t=1}^{\infty} A^t \quad (28A-1)$$

At first blush, it might seem that $X$ must equal infinity, since the sum goes to infinity. But notice that as $t$ gets large, the term $A^t$ gets very small, because $A$ is less than 1. For example, suppose that $A = 1/2$. Then the sum is

$$X = 1/2 + 1/4 + 1/8 + 1/16 + \cdots$$

Notice that as we continue to add terms, $X$ approaches but never exceeds 1. Perhaps from an algebra or calculus course you recall that the sum of a geometric progression actually has a closed-form solution:

$$X = \sum_{t=1}^{\infty} A^t = \frac{A}{1 - A} \quad (28A-2)$$

For example, in the case of $A = 1/2$, the sum is

$$X = \sum_{t=1}^{\infty} (1/2)^t = \frac{1/2}{1 - (1/2)} = 1 \quad (28A-3)$$
Now consider a perpetuity with a constant payment of PMT and an interest rate of I. The present value of this perpetuity is

\[
PV = \sum_{t=1}^{\infty} \frac{PMT}{(1 + I)^t}
\]  
(28A-4)

This can be written as a geometric progression:

\[
PV = PMT \sum_{t=1}^{\infty} \frac{1}{(1 + I)^t} = PMT \sum_{t=1}^{\infty} \left( \frac{1}{1 + I} \right)^t
\]  
(28A-5)

Because \(1 + I\) is positive and greater than 1 for reasonable values of I, the summation in Equation 28A-5 is a geometric progression with \(A = 1/(1 + I)\). Therefore, using Equation 28A-2, we can write the summation in Equation 28A-5 as

\[
\sum_{t=1}^{\infty} \left( \frac{1}{1 + I} \right)^t = \frac{\left( \frac{1}{1 + I} \right)}{1 - \left( \frac{1}{1 + I} \right)} = \frac{1}{I}
\]  
(28A-6)

Substituting this result into Equation 28A-4 gives us the present value of a perpetuity:

\[
PV = \frac{PMT}{I}
\]  
(28A-7)

Now consider the time lines for a perpetuity that starts at time 1 and a perpetuity that starts at time \(N + 1\):

\[
\begin{array}{cccccccc}
0 & 1 & 2 & \ldots & N & N + 1 & N + 2 & N + 3 \\
\hline
PMT & PMT & \ldots & PMT & PMT & PMT & PMT & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 1 & 2 & \ldots & N & N + 1 & N + 2 & N + 3 \\
\hline
PMT & PMT & \ldots & PMT & PMT & PMT & \\
\end{array}
\]

Notice that if we subtract the second time line from the first, we get the time line for an ordinary annuity with \(N\) payments:

\[
\begin{array}{cccc}
0 & 1 & 2 & \ldots & N \\
\hline
PMT & PMT & \ldots & \\
\end{array}
\]
Therefore, the present value of an ordinary annuity is equal to the present value of the first time line minus the present value of the second time line. The present value of the first time line, which is a perpetuity, is given by Equation 28A-7

\[
PV_{\text{of first time line}} = \frac{PMT}{I} \quad (28A-8)
\]

If we apply Equation 28A-7 to the second time line, it gives the value of the payments discounted back to time \( N \) (because if we just look at the time line from \( N \) on, it is an ordinary annuity that starts at time \( N + 1 \)). To find the present value of the second time line, we just discount this perpetuity value back to time 0:

\[
PV_{\text{of second time line}} = \left( \frac{PMT}{I} \right) \frac{1}{(1 + I)^N} \quad (28A-9)
\]

Subtracting Equation 28A-9 from 28A-8 gives the present value of an ordinary annuity, \( PVA \):

\[
PVA = \left( \frac{PMT}{I} \right) - \left( \frac{PMT}{I} \right) \frac{1}{(1 + I)^N} \quad (28A-10)
\]

This can be rewritten as

\[
PVA = PMT \left[ \left( \frac{1}{I} \right) - \left( \frac{1}{I(1 + I)^N} \right) \right] \quad (28A-11)
\]

The future value of an ordinary annuity is equal to the present value compounded out to \( N \) periods:

\[
FVA = PVA(1 + I)^N = PMT \left[ \left( \frac{1}{I} \right) - \left( \frac{1}{I(1 + I)^N} \right) \right](1 + I)^N \quad (28A-12)
\]

This can be rewritten as

\[
FVA = PMT \left[ \left( \frac{(1 + I)^N}{I} \right) - \left( \frac{1}{I} \right) \right] \quad (28A-13)
\]