An introduction to financial mathematics

Interest can be either simple or compound. Simple interest is calculated only on the original principal. If $1000 is deposited in a bank at a rate of 10 per cent simple interest per annum for three years, the interest per year is $100, i.e. 10 per cent of $1000. So, after three years, the interest earned is $300.

Compound interest

Compound interest is calculated both on the principal and on any interest previously earned. So, if the bank in the above example pays 10 per cent interest compounded annually, how much interest will be earned in three years? The following table shows one way of calculating this.

<table>
<thead>
<tr>
<th>Year</th>
<th>Balance at start</th>
<th>Interest</th>
<th>Balance at end</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1 000</td>
<td>$100</td>
<td>$1 100</td>
</tr>
<tr>
<td>2</td>
<td>$1 100</td>
<td>$110</td>
<td>$1 210</td>
</tr>
<tr>
<td>3</td>
<td>$1 210</td>
<td>$121</td>
<td>$1 331</td>
</tr>
</tbody>
</table>

Therefore the compound interest is $331 compared to $300 simple interest.

Future value

Fortunately, we do not need to work through such calculations as the balance, or future value, at the end of the time period can be determined by using the formula:

\[ FV = A (1 + R)^n \]

where
- \( A \) = amount at the start
- \( R \) = interest rate per annum ÷ 100
- \( n \) = number of years
- \( FV \) = future value after \( n \) years

**Key Concept 1**

**Future value of $1**

The future value of $1 is the value of $1 after \( n \) years at \( r \) per cent compound interest per annum. This is given by \((1 + R)^n\), where \( R = r ÷ 100 \)

Returning to our example, we see that \( A = 1000 \), \( R = 10 ÷ 100 = 0.1 \) and \( n = 3 \). Therefore:

\[ FV = 1000 (1 + 0.1)^3 \]
\[ = 1000 (1.1)^3 \]
\[ = 1000 \times 1.331 \]
\[ = $1331 \]

Calculators and computers make the task of carrying out these calculations far easier.

Note that there are several ways to calculate that \( (1.1)^3 = 1.331 \). Firstly, \( (1.1)^3 = 1.1 \times 1.1 \times 1.1 \). Many calculators have an \( x^y \) key to perform this calculation quickly. Also, tables have been compiled which show the future value of $1. Such a table is provided in Appendix 3 (pp. 555–6).
Example 1
Use Appendix 3 to find the future value of $1, and hence calculate how much $10,000 invested at 12 per cent per annum compounded annually will amount to in ten years.

\[ FV = A (1+R)^n \]

\[ = A \times \text{future value of $1} \]

From Table 1 in Appendix 3, we see that the future value of $1 after ten years at 12 per cent is 3.1058. Hence:

\[ FV = 10,000 \times 3.1058 \]

\[ = $31,058 \]

The simple interest would only have been 12 per cent of $10,000 \times 10 \text{ years} = $12,000, and the future value would be $22,000. This example illustrates the benefit of the compounding of interest compared with simple interest. However, note that we have not considered inflation or other charges.

Present value
In the previous section we looked at how much an amount would grow to in time. We often want to reverse the situation and ask how much needs to be invested now in order to receive a certain amount in the future. This is particularly important in the calculation of capital projects which provide cash flows over several years. This is known as present value. To calculate present value, we use the formula:

\[ PV = A \left[ \frac{1}{(1+R)^n} \right] \]

where

- \( A \) = future amount
- \( R \) = interest rate per annum ÷ 100
- \( n \) = number of years
- \( PV \) = present value, i.e. the amount which needs to be invested

To show that the above formula works, we will reverse the first compound interest question we considered.

Key Concept 2

Present value of $1

The present value of $1 is the amount which needs to be invested now, at \( r \) per cent per annum compound interest, in order to compound to $1 after \( n \) years. This is given by

\[ \frac{1}{(1+R)^n}, \text{ where } R = r \div 100 \]
Example 2

What is the present value of $1331 received in three years’ time, at 10 per cent per annum compound interest?

\[
PV = A \left( \frac{1}{(1+R)^n} \right)
\]

\[
= 1331 \left( \frac{1}{(1.10)^3} \right)
\]

\[
= 1331 \times 0.7513
\]

\[
= $1000
\]

This confirms that we need to invest $1000 today to accumulate $1331 in three years at the compound interest rate of 10 per cent. Therefore, $1000 is the present value of $1331 received in three years at a discount rate of 10 per cent.

There are tables which give the present value of $1 for a range of interest rates and years. Table 2 in Appendix 3 (pp. 557–8) provides these values.

Example 3

Use Table 2 to find the present value of $1 and hence the present value of $31 058 received after ten years at 12 per cent compound interest per annum.

\[
PV = A \left( \frac{1}{(1+R)^n} \right)
\]

\[
= A \times \text{present value of $1}
\]

From Table 2, we see that the present value of $1 after ten years at 12 per cent is 0.32197. Hence:

\[
PV = 31 058 \times 0.32197
\]

\[
= $9999.74
\]

If we refer back to Example A2.1 then the answer we would expect is $10 000. The difference of 26 cents is what we describe as a rounding error. It occurs because the interest factors in the tables are rounded to four or five decimal places.

Annuities

Future value

In the previous examples we either calculated the future value of a single amount invested today, or we determined the present value of a sum to be received in the future. In many instances, money will be invested or withdrawn regularly over the investment period. Where the same amount is invested or withdrawn on a regular basis, we have an annuity. There are different types of annuities. An ordinary annuity is where each of the equal payments is made at the end of each compounding period. An annuity due is where each of the equal payments is made at the beginning of each compounding period.
Example 4

Troy wants to invest $1000 at the end of each year for five years at a compound interest rate of 10 per cent per annum. How much will Troy’s investment be worth in five years?

We could perform five calculations and then add these together. However, we can determine the sum of an annuity by using the formula:

\[ FVA = A \left[ 1 + \left(1 + \frac{R}{100}\right) + \left(1 + \frac{R}{100}\right)^2 + ... + \left(1 + \frac{R}{100}\right)^{n-1} \right] \]

or

\[ FVA = A \left[ \frac{(1 + \frac{R}{100})^n - 1}{\frac{R}{100}} \right] \]

where

- \( A \) = amount of annuity
- \( R \) = interest rate per annum ÷ 100
- \( n \) = number of years
- \( FVA \) = future value of annuity

Once again, there are tables which provide future value of ordinary annuity factors. Table 3 in Appendix 3 (pp. 559–60) shows the future value of $1 per period.

Key Concept 3

**Future value of $1 per period**

The future value of $1 per period is the value of an investment after \( n \) time periods, given that $1 is invested at the end of each time period, at a compound interest rate of \( r \) per cent. The future value of $1 per period is

\[ \frac{(1 + \frac{R}{100})^n - 1}{\frac{R}{100}}, \text{ where } R = \frac{r}{100} \]

For Example A2.4 we have \( A = 1000 \), \( R = 0.1 \) and \( n = 5 \).

\[ FVA = 1000 \left[ \frac{(1.10)^5 - 1}{0.10} \right] \]

\[ = 1000 \times \text{future value per $1 period} \]

\[ = 1000 \times 6.1051 \]

\[ = 6105.10 \]

Example 5

If Stacey invests $500 at the end of each year at 8 per cent per annum compounded annually, how much will she have at the end of five years? (Use Table 3.)

\[ FVA = A \left[ \frac{(1 + \frac{R}{100})^n - 1}{\frac{R}{100}} \right] \]

\[ = A \times \text{future value of $1 per period} \]

From Table 3, the future value of $1 per period after five years at 8 per cent is 5.8666. Hence:

\[ FVA = 500 \times 5.8666 \]

\[ = 2933.30 \]
Present value

We can also reverse Example A2.5 and imagine that Stacey wishes to provide her daughter with an annuity of $500 for five years. How much should Stacey invest now, at 8 per cent per annum, compounded annually, so that the daughter can withdraw $500 at the end of each year for the next five years? This amount is called the present value of an annuity.

The formula is:

\[ PVA = A \left[ \frac{1}{1+R} + \frac{1}{(1+R)^2} + \ldots + \frac{1}{(1+R)^n} \right] \]

or

\[ PVA = A \left[ \frac{1 - \frac{1}{(1 + R)^n}}{R} \right] \]

where

- \( A \) = amount paid out per year
- \( R \) = interest rate ÷ 100
- \( n \) = number of years
- \( PVA \) = present value of annuity, i.e. amount to be invested now

Table 4 in Appendix 3 shows the present value of $1 per period.

Key Concept 4

**Present value of $1 per period**

The present value of $1 per period is the amount which needs to be invested now, at \( r \) per cent per annum compound interest, so that a sum of $1 can be withdrawn at the end of each year, for \( n \) years. The present value of $1 per period is

\[ PVA = \frac{1}{1 - (1+R)^n} \]

where \( R = r \div 100 \)

\[ PVA = 500 \left[ \frac{1}{(1 + 0.08)^5} \right] \]

\[ = 500 \times \text{present value of $1 per period} \]

\[ = 500 \times 3.9927 \]

\[ = $1996.35 \]

Therefore, Stacey needs to invest $1996.35 now at 8 per cent compound interest per annum, for five years, in order for her daughter to receive $500 a year for five years.

More frequent compounding

The tables make the solution to financial mathematical problems reasonably easy. However, what do we do when the interest is compounded quarterly, monthly or daily? We illustrate what can be done by using the equations for future value and present value.
Future value

\[ FV = A \left(1 + \frac{R}{m}\right)^{m \times n} \]

where
- \( A \) = amount
- \( R \) = interest rate per annum ÷ 100
- \( n \) = number of years
- \( m \) = number of times per year that interest is compounded
- \( FV \) = future value

Example 6

How much will $1000 amount to in five years at 12 per cent per annum, compounded quarterly?

\[
0.12 FV = 1000 \left(1 + \frac{0.12}{4}\right)^{5 \times 4}
\]
\[
= 1000 (1.03)^{20}
\]
\[
= 1000 \times 1.8061
\]
\[
= $1806.10
\]

Note that, using Table 1, we look up a 3 per cent interest rate with \( n = 20 \).

Present value

\[
PV = A \left[ \frac{1}{1 + \frac{R}{m}} \right]^{m \times n}
\]

where
- \( A \) = amount
- \( R \) = interest rate per annum ÷ 100
- \( n \) = number of years
- \( m \) = number of times interest is compounded per year
- \( PV \) = present value

Using the data from Example A2.6, what is the present value of $1000 to be received in five years’ time?

\[
PV = 1000 \times \left[ \frac{1}{1 + \frac{0.12}{4}} \right]^{5 \times 4}
\]
\[
= 1000 \left[ \frac{1}{(1.03)^{20}} \right]
\]
\[
= 1000 \times 0.55367
\]
\[
= $553.67
\]

(Use 3% and \( n = 20 \) in Table 2.)